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Birational geometry of algebraic varieties, by János Kollár and Shigefumi Mori, with the collaboration of C. H. Clemens and A. Corti, Cambridge University Press, 1998, viii + 254 pp., ISBN 0-521-63277-3

There has been a revolutionary change in the field of birational geometry in the last twenty plus years. This is based on the theory of extremal rays initiated by Mori and is central to the investigation of minimal models called the *Minimal Model Program* (MMP) or *Mori program*. But there has been no reasonably accessible textbook for this theory besides more professionally oriented surveys such as [KMM87] and [K^+92] (we use the reference list of the book). In this sense, this book, written by two of the main players in this development, answers a demand for a long awaited introductory textbook for the beginners in this field. The exposition is sufficiently elementary, self-contained and comprehensive, and requires fewer prerequisites, so this book will become a standard reference. A caution is that the proof of the existence theorem of 3-dimensional flips is treated only in the easy semistable case. One has to refer to the original articles in order to study the general case.

Let us briefly recall the main results of the minimal model program. We consider all the algebraic varieties which are birationally equivalent to a given algebraic variety, called birational *models*. The classical approach to the minimal model problem was categorical; one looked for a minimal object among all the proper smooth models. Our new approach is numerical; we look at the numerical property of the canonical divisor on the models. The model is called *minimal* if it is projective and the canonical divisor is *nef* in the sense that the intersection number with an arbitrary curve is nonnegative. In our approach, we have to admit that the model has some mild singularities called $\mathbb{Q}$*-factorial terminal singularities*. So we also need arguments from commutative algebra.

The minimal model is expected to be obtained by an inductive procedure explained as follows. We take any model X which is projective and has only $\mathbb{Q}$-factorial terminal singularities. If the canonical divisor is nef, then it is a minimal model. If not, then the *cone theorem* says that there exists an *extremal ray* on the cone of the numerical classes curves, called the *Mori cone*, in a finite dimensional real vector space. Then the *contraction theorem* guarantees the existence of a surjective morphism $f : X \to Y$ to another projective variety which is associated to the extremal ray.

There are 3 cases. If the dimension of Y is smaller than that of X, then f is called a *Mori fiber space* (or Fano contraction, or Fano fiber space in this book), and the program stops here because there does not exist a minimal model in this case. Indeed, one can prove that X is covered by a family of rational curves [MM86]. Otherwise, f is a birational morphism. If it contracts a prime divisor on X, then it is called a *divisorial contraction*, and Y has again only $\mathbb{Q}$-factorial terminal singularities. Moreover, the Picard number, the dimension of the real vector space, drops by 1. In this case, we replace X by Y and continue the program. Otherwise, f is called a *small contraction* (or flipping contraction in this book).

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In this last case, we need a conjecture, the *existence of the flip*, to continue the program. This conjecture says that there is another small birational morphism $f' : X' \to Y$ from a projective variety with only $\mathbb{Q}$-factorial terminal singularities such that the composite birational map $f'^{-1} \circ f$ is not an isomorphism. We replace X by X' and continue the program. We need another conjecture, the *termination* of the sequence of flips, in order to obtain the final result, a Mori fiber space or a minimal model, after a finite number of steps. These 2 conjectures are proved only in the case where the dimension is 3.

This MMP consisting of 4 conjectures—the cone, the contraction, the flip and the termination conjectures—was formally stated in [Rei83b] after [Mor82] and [6]. It was first a 'problem' from a conservative point of view. After the first 2 conjectures were proved, it may be called a 'program'. But it is still not a 'project'.

Now we explain the contents of the book by chapters.

The book can be divided into two parts. Chapters 1-3 introduce quickly the MMP and explain the general machinery which works in all dimensions. We note that the whole MMP is completed only in dimension 3, besides the classically known cases of dimension 1 and 2.

The goal of the second part, Chapters 4-7, is to prove the 2 flip conjectures to complete the MMP when the given 3-fold arises from a semistable family of surfaces. The general case in dimension 3 is too complicated and left to the references.

Chapter 1. "Rational curves and the canonical class". Mori's famous *bend and break* method is explained. As an application, the cone theorem is obtained for the case of smooth varieties. Historically, this was the starting point of the theory of extremal rays. The contraction morphisms associated to extremal rays for smooth 3-folds are also classified. It is important to note that this method is still the only mathematical way of proof for the existence of rational curves on algebraic varieties. The drawback is that this method applies only for smooth varieties. This chapter follows [Mor82].

Chapter 2. "Introduction to the minimal model program". The MMP naturally introduces singularities on the varieties through the operations of contractions and flips. The concept of $\mathbb{Q}$-*factorial terminal* singularities is defined. The category of varieties with these kinds of singularities is the smallest one on which MMP works. The largest one is the category of varieties with $\mathbb{Q}$-factorial *divisorially log terminal singularities* (called *weak log terminal singularities* in [KMM87]; the equivalence is proved in [Sza95]).

The proof of the Kawamata-Viehweg vanishing theorem ([Kaw82] and [Vie82]), called Kodaira vanishing theorem II in this book, is given. This is a numerical and $\mathbb{Q}$-divisorial version of the Kodaira vanishing theorem and is a fundamental tool for the proof of the cone and contraction theorems given in the next chapter.

Chapter 3. "Cone theorems". The idea to describe birational morphisms between algebraic varieties by using the cones inside some finite dimensional real vector spaces goes back to Hironaka's unpublished thesis [Hir60]. Kleiman's ampleness criterion [Kle66] with respect to this cone is a basic result from this point of view. (Toric geometry is another important development.) On the other hand, the geometry of varieties behave very differently according to the sign of the canonical divisor. This is reflected by the distinction of two categories of varieties which have quite different characteristics: varieties with negative Kodaira dimension and those with nonnegative Kodaira dimension. The former is predicted (according to the

MMP) to be covered by a family of rational curves, and holomorphic differential forms play an important role in the latter.

Mori's key observation is that the cone of curves looks also very different when it is cut into halves by the hyperplane defined by the zero intersection locus with the canonical divisor. Indeed, the half cone on which the canonical divisor is negative has a special shape and is generated by the extremal rays which are locally finite. In particular, if the canonical divisor is not nef, then there exists an extremal ray. This cone theorem and also the contraction theorem are proved according to the idea of the base point free theorem by using the Kawamata-Viehweg vanishing theorem in this chapter.

The MMP is naturally extended to the log and relative versions. It is natural to consider the MMP for log varieties that are pairs consisting of varieties and divisors. Indeed, the Kawamata-Viehweg vanishing theorem may be regarded as a log version of the Kodaira vanishing theorem. The relative MMP is easily obtained as a generalization of the absolute MMP.

Chapter 4. "Surface singularities of the minimal model program". The main reason why the MMP was completed in dimension 3 is that the classification of singularities was possible in low dimensions. The starting point of this classification is the construction of the simultaneous resolution for families of rational double points and the classification of elliptic Gorenstein singularities. They are in turn used for the classification of canonical singularities in dimension 3, because they have finite coverings whose hyperplane sections are either rational double points or elliptic Gorenstein singularities according to Reid.

Chapter 5. "Singularities of the minimal model program". In general, it is proved that log terminal singularities are rational singularities. In the case of dimension 3, a very detailed study on singularities was possible due to the results proved in the previous chapter. The terminal singularities in dimension 3 are completely classified by Mori and Reid.

Chapter 6. "Three-dimensional flops". A flop is a pair of small birational morphisms which is similar to a flip. The difference is that a flop is *crepant* in the sense that the canonical divisor is relatively numerically trivial so that the level of the canonical divisor does not change after the transformation. Flops of three dimensional varieties having canonical singularities are constructed as an important step toward the existence theorem of flips for 3-folds. Flops for varieties with terminal singularities are constructed in [Rei83b] by using the simultaneous resolution for families of rational double points. They are generalized for varieties with canonical singularities in [Kaw88] by the so-called *crepant descent*.

The key observation in [Kaw88] is that the existence problem of the flips may be reduced to the existence theorem of the flops by using a double covering trick. Indeed, a flipping contraction of a 3-fold with *terminal* singularities has a double covering which is a flopping contraction of a 3-fold with *canonical* singularities, if the so-called *general bi-elephant conjecture* [Kaw88], which asserts the existence of a good member in the anti-bicanonical linear system, holds. This conjecture is proved in the semistable case in [Kaw88] and is completely proved in [Mor88], hence the existence theorem of the flips for 3-folds. This conjecture is a weaker form of Reid's *general elephant conjecture* on the existence of a good anti-canonical divisor. This circle of ideas was generalized in [Sho92] as the theory of *complements*.

The termination theorem of the flips for 3-folds was proved by [Sho85] using the numerical invariant of terminal singularities called the *difficulty*.

Chapter 7. "Semi-stable minimal models". The proof of the existence theorem of the minimal models is completed for the case of semistable degeneration of surfaces. The proof presented in this book is according to Corti's version which uses the ideas of [Sho92]. This result is proved in a much easier way compared to the general existence theorem for 3-folds, but already has important applications such as the compactification theorem of the moduli space of surfaces of general type ([KSB88] and [2]).

There are some preceeding proofs of the theorem in the semistable case. The proofs were first announced by Tsunoda and Mori, and the first published proof appeared in [Kaw88] by proving the general bi-elephant conjecture. There is also a proof in [Kaw94] which holds also in positive characteristics except 2 and 3, and Corti's version after [Sho92].

At the end of this chapter, there are remarks on further development of the MMP. Besides the existence conjecture of the minimal models (any algebraic variety is birational to a minimal model or a Mori fiber space), the *abundance conjecture* is the most important step toward the birational classification of algebraic varieties. This conjecture implies that any minimal model has the expected fiber space structure derived from the pluricanonical linear systems. This is a statement which retrieves the geometric property, the semi-ampleness, of the canonical divisor from its numerical property, the nefness.

This conjecture is also completely proved only in dimension 3 in [11], [13] and [14]. It is proved in [9] that, if the *numerical Kodaira dimension* of a minimal model X, denoted by $\nu(X)$, is equal to the Kodaira dimension $\kappa(X)$, then the canonical divisor is semi-ample. This condition, the *goodness* in [9], is called the *abundance* of the canonical divisor in [KMM87] because it asserts the existence of sufficiently many sections for the multiples of the canonical divisor. Thus the semi-ampleness conjecture is now called the abundance conjecture.

As was already seen in [KMM87], it is natural to extend the MMP to the log MMP, that is the generalization of the MMP to the log varieties, which are defined to be the pairs of varieties and divisors. In the case of dimension 3, the log MMP is completely proved; the existence of the log flip is proved in [Sho92] (see also [16] for easier exposition) and the termination in [Kaw92b]. As an alternative proof of the log flip theorem, [K^+92] reduced it to the usual flip theorem [Mor88] by using an argument in [Kaw92b]. Both in [Mor88] and [Sho92], the proof of the existence of the flip relies on some kind of detailed classification, though the proof in [Sho92] is more inductive in nature. So it is complicated and difficult. It is desirable to have a simpler proof of the flip theorem which does not depend on the classification but only on some kind of induction.

The log abundance theorem for 3-folds is proved in [KMM94a].

The investigation of log surfaces, pairs of surfaces and divisors, was another source of the numerical geometry and the MMP [7], [17]. The importance of the numerical property of $\mathbb{Q}$-divisors was realized during the course of the classification of log surfaces. Indeed, the log minimal models in dimension 2 were characterized by the nefness of the log canonical divisors, and the log terminal singularities for surfaces appeared first in this context. This research area provided a good test ground for the new machinery such as the vanishing theorem for $\mathbb{Q}$-divisors.

As explained in the first half of the book, there are two different methods in order to prove the cone theorem. The first one, explained in Chapter 1, uses the deformation theory of morphisms in positive characteristic. This ingenious method of Mori works in arbitrary characteristic. It is based on the spectacular use of the fundamental theory of schemes such as the Hilbert scheme and modulo p reduction of a scheme which is of finite type over Spec $\mathbb{Z}$. This idea was further developed, and one obtained the rational connectedness of Fano varieties ([4] and [12], also [Kol96]).

The second one, explained in Chapters 2-3, is the cohomological method. This alternative method shows another power of algebraic geometry. This is an application of the Kodaira-Kawamata-Viehweg vanishing theorem, which is true only in characteristic 0. The advantage is that it is applicable for singular varieties and well suited for the log generalization, because it is originally log.

This base point free technique first appeared in [Kaw84b] in order to prove the finite generatedness of the canonical ring for minimal 3-folds. Using this idea, [Ben83] completed the result by proving some non-vanishing. In [8] the contraction theorem for a general 3-fold and the cone theorem in the case of the non-negative Kodaira dimension were proved. [Rei83c] proved the cone theorem for general 3-folds. [Sho85] proved a general non-vanishing theorem, hence the contraction theorem, using precisely the base point free technique together with the concentration method which creates singularities artificially. Finally [Kaw84a] proved the cone theorem in arbitrary dimension, and [Kol84] proved the discreteness of the extremal rays.

One of the origins of the MMP was the finite generatedness conjecture of the canonical ring. The approach of the MMP is inductive and decomposes the problem into elementary steps of the MMP algorithm. On the other hand, the conjectural Zariski decomposition achieves the result in one step [10]. One notes that the existence conjecture of the flip can be viewed as a special case of the finite generatedness conjecture.

Another possible way to prove the existence of the flip is to use the theory of the derived categories [3].

There are recent developments of the birational geometry which are outside the MMP ideology. The pluricanonical forms on algebraic varieties are basic objects for their classification. So the invariance of plurigenera of algebraic varieties under smooth deformations is one of the fundamental questions in birational geometry. [15] reduced this question to the existence and the abundance conjectures of minimal models. But [Siu98] proved this statement for arbitrary dimensional algebraic varieties of general type directly without using the MMP. The vanishing theorem was again one of the key ingredients of the proof.

It was a long-standing problem to factorize a given birational map between smooth proper algebraic varieties into blow-ups and blow-downs with smooth centers. After the MMP was established, this problem was replaced by a new one that the birational map should be factorized into divisorial contractions, flips and their inverses. But recent developments ([18], [1] and [19]) revived the old form of the factorization. The key word is *torification* which also appeared in [AK97].

A Japanese edition was simultaneously published: *Souyuuri Kikagaku*, by János Kollár and Shigefumi Mori, with the collaboration of C. H. Clemens and A. Corti, Iwanami Shoten, Tokyo, 1998, ix + 328pp., ISBN 4-00-010653-8. The Japanese edition is more friendly to readers, with slightly more detailed explanations such as the citation to [Har77]. Some of the typos in the English edition are corrected in the Japanese edition.

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