

*Geometry: Euclid and beyond*, by Robin Hartshorne, Springer-Verlag, New York, 2000, xi+526 pp., \$49.95, ISBN 0-387-98650-2

## 1. INTRODUCTION

The first geometers were men and women who reflected on their experiences while doing such activities as building small shelters and bridges, making pots, weaving cloth, building altars, designing decorations, or gazing into the heavens for portentous signs or navigational aides.

Main aspects of geometry emerged from three strands of early human activity that seem to have occurred in most cultures: art/patterns, navigation/stargazing, and building structures. These strands developed more or less independently into varying studies and practices that eventually were woven into what we now call *geometry*.

**Art/Patterns:** To produce decorations for their weaving, pottery, and other objects, early artists experimented with symmetries and repeating patterns. Later the study of symmetries of patterns led to tilings, group theory, crystallography, finite geometries, and in modern times to security codes and digital picture compactifications. Early artists also explored various methods of representing existing objects and living things. These explorations led to the study of perspective and then projective geometry and descriptive geometry, and (in the 20th century) to computer-aided graphics, the study of computer vision in robotics, and computer-generated movies (for example, *Toy Story*).

**Navigation/Stargazing:** For astrological, religious, agricultural, and other purposes, ancient humans attempted to understand the movement of heavenly bodies (stars, planets, Sun, and Moon) in the apparently hemispherical sky. Early humans used the stars and planets as they started navigating over long distances, and they used this understanding to solve problems in navigation and in attempts to understand the shape of the Earth. Ideas of trigonometry apparently were first developed by Babylonians in their studies of the motions of heavenly bodies. Even Euclid wrote an astronomical work, *Phaenomena*, in which he studied properties of curves on a sphere. Navigation and large-scale surveying developed over the centuries around the world and along with it cartography, trigonometry, spherical geometry, differential geometry, Riemannian manifolds, and thence to many modern spatial theories in physics and cosmology.

**Building Structures:** As humans built shelters, altars, bridges, and other structures, they discovered ways to make circles of various radii, and various polygonal/polyhedral structures. In the process they devised systems of measurement and tools for measuring. The (2000-600 BC) *Sulbasutram* [Sul] contains a geometry handbook for altar builders with proofs of some theorems and a clear general statement of the “Pythagorean” Theorem. Building upon geometric knowledge from Babylonian, Egyptian, and early Greek builders and scholars, Euclid (325-265 BC) wrote his *Elements*, which became the most used mathematics textbook in the

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world for the next 2300 years, and codified what we now call Euclidean geometry. Using *Elements* as a basis in the period 300 BC to about 1000 AD, Greek and Islamic mathematicians extended its results, refined its postulates, and developed the study of conic sections and geometric algebra, what we now call “algebra”. Within Euclidean geometry, there later developed analytic geometry, vector geometry (linear algebra and affine geometry), and algebraic geometry. The *Elements* also started what became known as the axiomatic method in mathematics. Attempts by mathematicians for 2000 years to prove Euclid’s Fifth (Parallel) Postulate as a theorem (based on the other postulates) culminated around 1825 with the discovery of hyperbolic geometry. Further developments with the axiomatic methods in geometry led to the axiomatic theories of the real numbers and analysis and to elliptic geometries and axiomatic projective geometry.

## 2. THE BOOK

The book under review, *Geometry: Euclid and beyond*, is situated in this “Building Structures” historical strand of geometry. The author states in his Preface:

In recent years, I have been teaching a junior-senior-level course on the classical geometries. This book has grown out of that teaching experience. I assume only high-school geometry and some abstract algebra. The course begins in Chapter 1 with a critical examination of Euclid’s *Elements*. Students are expected to read concurrently Books I-IV of Euclid’s text, which must be obtained separately. The remainder of the book is an exploration of questions that arise naturally from this reading, together with their modern answers. To shore up the foundations [in Chapter 2] we use Hilbert’s axioms. The Cartesian plane over a field provides an analytic model of the theory [Chapter 3], and conversely, we see that one can introduce coordinates into an abstract geometry [Chapter 4]. The theory of area [Chapter 5] is analyzed by cutting figures into triangles. The algebra of field extensions [Chapter 6] provides a method for deciding which geometrical constructions are possible. The investigation of the parallel postulate leads to the various non-Euclidean geometries [Chapter 7]. And in [Chapter 8] we provide what is missing from Euclid’s treatment of the five Platonic solids in Book XIII of the *Elements*.

There is almost no mention in this book of the other two strands of geometry. This is a shame since the title (without the subtitle) is *Geometry* and most of the mathematical research activity in geometry in recent times is situated in the other two strands. A more accurate title would be *Companion to Euclid*, which happens to be the title of an earlier version of this book that appeared in the Berkeley Mathematics Lecture Notes, volume 9.

## 3. EUCLIDEAN GEOMETRY

Until the 20th century, *Euclidean geometry* was usually understood to be the study of points, lines, angles, planes, and solids based on the 5 propositions and 5 common notions in Euclid’s *Elements*. In the *Elements* there is no concept of distance as a real number in the sense we know it today. There is only the concept of congruence of line segments (thus one can say that two segments are equal) and of proportion (so that we can say that two segments are in certain proportion to each

other), but we cannot say that they have the same (numerical) length. Geometry as studied in this way is usually called *synthetic Euclidean geometry* and is the subject of Chapter 1 of *Geometry: Euclid and beyond*.

Interest in the synthetic geometry of triangles and circles flourished during the late 19th century and early 20th century. One of the best known results of 19th century synthetic geometry is the existence of the *nine point circle*:

Given any triangle in the Euclidean plane, the midpoints of its three sides, the midpoints of the lines joining the orthocenter (the point of intersection of the three altitudes) to its three vertices, and the feet of its three altitudes all lie on the same circle.

This nine point circle and similar synthetic Euclidean results are discussed in the last section of Chapter 1.

It is usual in schools today for “Euclidean geometry” or just “*plane geometry* and *solid geometry*” to not mean synthetic geometry but rather a version of Euclid’s geometry with the addition of the real number measure of distances, angles, and areas. This school geometry is a highbred of synthetic (Euclid’s) geometry and analytic (or Cartesian) geometry.

Though numbers as measure for lengths and areas are not explicit in Euclid’s geometry, they are implicit in his arithmetic of line segments (discussed in Chapter 4) and his “scissors and paste” theory of areas (discussed in Chapter 5). Euclid’s arithmetic of line segments, after defining a unit length, determines an ordered field whose positive elements are the congruence equivalence classes of line segments. Then to develop *analytic* (or *Cartesian*) *geometry* starting from the *synthetic Euclidean Plane* we choose:

1. A point that we call the *origin*,  $O$ ,
2. A segment that we call the *unit* (*length*), and
3. Two lines through  $O$ , which we call the *coordinate axes* (today these are almost always perpendicular, but Descartes did not require them to be so).

We can compare a line segment,  $a$ , with the *unit* and define the length of  $a$  to be equal to the ratio of  $a$  to the *unit*. Then both coordinate axes can be labeled as number lines with  $O$  being the zero. In the usual way we develop the *real Cartesian plane* and can now study geometric properties using algebra.

We can also define a plane geometry over any field by considering its points to be pairs of field elements. This is what Hartshorne does in Chapter 3. In general, different fields give rise to different geometries and these geometries can be used, for example, to study the interdependence of various geometric axioms. The rigorous axiomatic structure that Hartshorne develops in Chapter 2 and further analyzes in Chapter 3 is essentially the axiom system proposed by David Hilbert in 1899 [Hil-a]. It is in this context that Hartshorne defines and discusses the rigid motions: translations, rotations, and reflections. But curiously he does not discuss glide reflections or the classification theorem of rigid motions that states that any rigid motion of the plane is either the identity, a reflection, a translation, a rotation, or a glide reflection.

#### 4. NON-EUCLIDEAN GEOMETRIES

By far the longest chapter in *Geometry: Euclid and beyond* is the seventh, entitled “Non-Euclidean Geometry”. Notice the use of the singular in his title, as opposed to my title of this section of the review. Wholly within the context of

the Building Structures strand it makes sense to talk about the non-Euclidean geometry, but in the other strands there are numerous geometries that are not Euclidean, such as *projective geometry* in the Art/Patterns strand. However, most of the non-Euclidean geometries exist in the Navigation/Stargazing strand, as I will discuss:

**4.1. Spherical Geometry.** Spherical geometry can be said to be the first non-Euclidean geometry. For at least 2000 years humans have known that the Earth is (almost) a sphere and that the shortest distances between two points on the Earth is along great circles (the intersection of the sphere with a plane through the center of the sphere). For example:

... it will readily be seen how much space lies between the two places themselves on the circumference of the large circle which is drawn through them around the earth. ... [W]e grant that it has been demonstrated by mathematics that the surface of the land and water is in its entirety a sphere, ... and that any plane which passes through the center makes at its surface, that is, at the surface of the earth and of the sky, great circles, and that the angles of the planes, which angles are at the center, cut the circumferences of the circles which they intercept proportionately,...

— Claudius Ptolemy, *Geographia* (ca. 150 AD), Book One, Chapter II

Spherical geometry is the geometry of a sphere. The great circles are intrinsically straight on a sphere in the sense that the shortest distances on a sphere are along great circle arcs and because great circles have the same symmetries on a sphere as straight lines have on the Euclidean plane. The geometry on spheres of different radii are different; however, the difference is only one of scale. In Aristotle we can find evidence that spherical non-Euclidean geometry was studied even before Euclid. (See [Hea], page 57, and [Toth].) Even Euclid in his *Phaenomena* [EucPh] (a work on astronomy) discusses propositions of spherical geometry. Menelaus, a Greek of the first century AD, published a book, *Sphaerica*, which contains many theorems about spherical triangles and compares them to triangles on the Euclidean plane. (*Sphaerica* survives only in an Arabic version. For a discussion see [Kli], pages 119-120.)

There is an axiom system for spherical geometry that is in the spirit of Hilbert's axioms (see, for example, [Bor]); in this context spherical geometry is usually called *double-elliptic geometry*. However, the axiomatization has seemed not to be useful. However, there are many popular accounts that attempt to distinguish between Euclidean and spherical geometries on the basis of Euclid's Fifth (or Parallel) Postulate, which states:

If a straight line intersecting two straight lines makes the interior angles on the same side less than two right angles, then the two lines (if extended indefinitely) will meet on that side on which the angles are less than two right angles.

It can be easily checked that this Fifth Postulate is provable on the sphere. This also shows that, contrary to many accounts, Euclid's Fifth Postulate is not equivalent to the Playfair (Parallel) Postulate that is familiar from high school geometry:

Given a line and a point not on the line there is one and only one line through the point which is parallel to the given line.

The book under review does not fall into this trap of trying to distinguish spherical geometry through parallel postulates. Since spherical geometry does not fit easily into the Euclidean (Hilbert) axiomatic structure of this book, the author defines spherical geometry as the geometry of a sphere in Euclidean 3-space. Spherical geometry is relegated to Exercises 34.13 and 45.3-8, where it is claimed (and the reader is asked to show) that most of the first 26 propositions of the *Elements* are valid in spherical geometry if one restricts spherical triangles to be those contained in a (open) hemisphere. However, in Exercise 45.8, the reader will find that in proving Angle-Angle-Side congruence one needs to further restrict spherical triangles to have each side less than  $1/4$  of a great circle.

## 5. HYPERBOLIC GEOMETRY

Starting soon after the *Elements* were written and continuing for the next 2000 years, mathematicians attempted either to prove Euclid's Fifth Postulate as a theorem (based on the other postulates) or to modify it in various ways. These attempts culminated around 1825, when Nicolai Lobatchevsky (Russian, 1792-1850), János Bolyai (Hungarian, 1802-60), and Karl Frederick Gauss (German, 1777-1855) independently discovered a geometry that satisfies all of Euclid's Postulates and Common Notions, except that the Fifth Postulate does not hold. It is this geometry that is called hyperbolic geometry. It is this 2000-year struggle that Hartshorne details in the first part of Chapter 7, though he leaves out any mention of the contributions from the Islamic world to this struggle. In particular, about 1100 AD Omar Khayyam wrote the "Discussion of Difficulties in Euclid" [Kha] and in the process defined and investigated the quadrilaterals called by Hartshorn (and most other books) *Saccheri Quadrilaterals* after Gerolamo Saccheri (Italy, 1667-1733), a common bit of Western chauvinism.

He then followed with a discussion of the analytic models of Poincaré, based on the theory of inversions in circles, and with an axiomatic development based on Hilbert's theory of limiting parallel rays and not using the real numbers. I know of no other discussion of hyperbolic geometry at this level that includes a detailed description of all three of these perspectives.

However, this leaves open the question of whether hyperbolic geometry is the geometry of any surface in Euclidean space, in the same sense that spherical geometry is the geometry of the sphere in Euclidean 3-space. In the mid-19th century beginnings of differential geometry it was shown that hyperbolic surfaces would be precisely surfaces with constant negative curvature. This aspect of hyperbolic geometry belongs in the Navigation/Stargazing strand of geometry.

Mathematicians looked for surfaces that would be the complete hyperbolic geometry in the same sense that a sphere has the complete spherical geometry. In 1868, Beltrami described a surface, called the *pseudosphere*, which locally has hyperbolic geometry but is not complete in that some geodesics (intrinsically straight lines) cannot be continued in definitely. In 1901, David Hilbert [Hil-b] proved that it is impossible to define by (real analytic) equations a complete hyperbolic surface. In those days "surface" normally meant one defined by real analytic equations, and so the search for a complete hyperbolic surface was abandoned, and still today many texts state that a complete hyperbolic surface is impossible. In 1964, N. V. Efimov [Efi] extended Hilbert's result by proving that there is no isometric embedding defined by functions whose first and second derivatives are continuous. However, in

1955, Nicolas Kuiper [Kui] proved, without giving an explicit construction, the existence of complete hyperbolic surfaces defined by continuously differentiable functions. Then in the 1970's William Thurston described the construction of complete hyperbolic surfaces (that can be made out of paper); see [Thu], pages 49 and 50. Directions for constructing Thurston's surface out of paper or by crocheting can be found in [HenEG] or [HenCr]. In these references there is also a description of an easily constructible polyhedral hyperbolic surface, called the "hyperbolic soccer ball", that consists of heptagons (7-sided regular polygons) each surrounded by 7 hexagons (the usual spherical soccer ball consists of pentagons each surrounded by 5 hexagons). The geodesics ("intrinsic straight lines") on a hyperbolic surface can be found by folding the surface (in the same way that folding a sheet of paper will produce a straight line on the paper). This folding also determines a reflection about the geodesic. Hartshorne shows in Chapter 7 (Section 43) how these reflections can be used to generate a "calculus of reflections" which leads to a very different axiomatic approach to hyperbolic planes.

**5.1. Differential Geometry and Manifolds.** Differential geometry is the branch of mathematics that studies the geometry of curves, surfaces, and manifolds (the higher-dimensional analogues of surfaces). Despite its name, differential geometry often uses algebraic [MiPa] and/or purely geometric [HenDG] techniques instead of the differential techniques of calculus. Even though the basic definitions, notations, and descriptions of differential geometry vary widely, the following geometric questions are central: How does one measure the curvature of a curve within a surface (intrinsic) versus within the encompassing space (extrinsic)? How can the curvature of a surface be measured? What is the shortest path within a surface between two points on the surface? How is the shortest path on a surface related to the concept of a straight line? Rigorous answers to these questions involve techniques from geometry, calculus, differential equations, algebra, and other areas. The methods of calculus opened the stage to the investigation of curves and surfaces in space – it is this investigation that was the start of differential geometry. For an in-depth discussion of the connections between Euclid and differential geometry including much historical material, see [McC].

G.F.B. Riemann in his inaugural address at the University of Gottingen introduced the notion of what is now called *Riemannian manifolds*. Riemannian manifolds form a part of differential geometry that considers intrinsic descriptions of manifolds (higher-dimensional versions of surfaces). On any Riemannian manifold (think of a sphere or cylinder), the basic geometric notions (such as straightness, distance, angle, curvature) can be defined intrinsically without reference to any surrounding (extrinsic) space. Because Riemannian geometry is intrinsic it can apply to abstract spaces that are not thought of as existing in an ambient space, such as the geometry of our own physical universe. See [Wee], [Thu], and [CoWe] for discussions of the possible 3-dimensional geometries of physical space and the 8 different local simply-connected 3-dimensional geometries.

*What are the intrinsically straight paths on a surface?* From our outside, or extrinsic, point-of-view no curve is straight on a sphere – they all have (extrinsic) curvature. However, the great circles are intrinsically straight: From the point-of-view of an imaginary 2-dimensional ant crawling along a great circle, there will be no turning or curving with respect to the surface. Ferdinand Minding (1806-85, taught at Dorpat, now Tartu in Estonia) defined in about 1830 a curve on a

surface to be a *geodesic* if it is intrinsically straight, that is, if there is no curvature identifiable from within the surface. It is one of the major tasks of differential geometry to determine what are the geodesics on a Riemannian manifold. The great circles (being circles) do extrinsically have curvature, but the curvature is in the direction of the center of the sphere and thus cannot be experienced intrinsically. The great circles are the geodesics (intrinsically straight paths) on a sphere. This is a theorem that is alluded to in the above quote from Ptolemy and that belongs in the Navigation/Stargazing strand. In the Building Structures strand it is usual to do as Hartshorne does and *define* the straight lines on a sphere to be the great circles.

## 6. CONSTRUCTIONS

One of the things I really like about this book is that Hartshorne starts in Section 2 talking about *ruler and compass* constructions and throughout the book almost all of the geometric diagrams (and there are many - I suspect more than one per page) are drawn by hand. He encourages readers to also draw their own diagrams as they read. According to the rules implicit in Euclid's *Elements*, the ruler may only be used to draw a straight line and it cannot be used to measure distance nor may it have any markings on it. Therefore, as Hartshorne points out, a more accurate term than 'ruler' would be 'straight edge'; however, he uses the more usual term 'ruler', as will I in this review.

Constructions using ruler and compass are paramount in the *Elements*. However, they have also led to many inaccurate statements in the mathematical literature. For example, it is often stated that it is impossible to trisect an arbitrary angle using only a ruler and compass. However, Hartshorne proves (Theorem 30.1) that it is possible to trisect any angle using only compass and *marked* ruler, a ruler with two marks on it. OK, so is it possible to trisect any angle with compass and *unmarked* ruler? The answer is still yes: It has been known for centuries (for a discussion, see [Mar], page 49) that it is possible to use an (unmarked) ruler and compass to construct a tool that then can in turn be used to trisect any angle. Hartshorne avoids these mistakes by reasonably carefully defining what he means by ruler and compass constructions (page 21), but it takes half a page to state. He then (Theorem 28.4) proves, using field extensions, that it is impossible to trisect a  $60^\circ$  angle using only ruler and compass.

But there are other confusions. On page 167, Hartshorne discusses the question of whether Euclid did know about real numbers and wrote their definition. In arguing for the negative, he states: "I see no evidence that [Euclid] conceived of the existence of any other real numbers" [other than "ratios of line segments that might be obtained by ruler and compass constructions"]. And then 2 sentences later: "... there is no evidence that the ancients believed in the existence of such an angle before it was constructed." He is not explicit about what he means by this last 'constructed', but readers could be excused if they took it to mean ruler and compass construction. In fact, these are the only kinds of constructions mentioned in the book up to this point. But later (page 260), he talks about many methods of construction used before and after Euclid that differ from ruler and compass constructions. On pages 221-224 Hartshorne describes some methods used by the ancient Greeks to approximate  $\pi$ . The ancient Greeks certainly accepted the existence of  $\pi$  (as the ratio of the circumference to the diameter of a circle) even

though they had only approximate constructions of a segment of that length. The issue of what it means for a mathematical entity to exist has been the objective of discussion among mathematicians up to the current time. For example, for a modern discussion of why one should only accept existence if there is a construction, see Errett Bishop's "Constructivist Manifesto" in [Bis]. Regardless of one's views on these issues it is important not to confuse "constructions" with "ruler and compass constructions", and it is important to avoid the absurdity that appeared in a calculus book once used at Cornell - in this text the students were told (without clarification) that one could not construct the position of  $\pi$  or  $2^{(1/3)}$  on the real number line.

## 7. TRANSFORMATIONAL VIEW OF GEOMETRY

Felix Klein [1849-1925] (in his inaugural address at the University of Erlangen in 1872) proposed a program to describe a geometry as a space with a group of transformations (of the whole space). Then the notions and propositions of the geometry are those which are preserved by the actions of the transformations. This became known as the *Erlangen Program*. This transformational view of geometry fits mainly in the Art/Patterns strand of geometry.

The transformations of the (*synthetic*) *Euclidean geometry* are the *isometries* (translations, rotations, reflections, and glide reflections) together with the *similarities* (*dilations*). The Euclidean properties are triangles, segments, angles, and the congruencies of these objects. For example, if two triangles are congruent in the Euclidean plane, then they are still congruent if the whole plane is rotated or changed by a dilation. Similarities preserve angles and take two congruent segments to segments that are still congruent.

In *Cartesian geometry* (*analytic geometry*) we add the measure of length to the geometry, and the transformations that preserve lengths are the *isometries* (reflections, rotations, translations, and glides).

The transformations of the *spherical geometry* (*double-elliptic* or *elliptic*) are the rotations, reflections (through great circles), and their compositions. [Note that dilations do not take the sphere to itself.] The notions that are preserved are triangles, segments, angles, and their congruencies, but also length of segments have a natural definition in terms of the angle subtended by the segment (great circle arc) from the center of the sphere.

The transformations of *hyperbolic geometry* are all the transformations that can be obtained by compositions of reflections over geodesics. In the Poincaré models these are the transformations that take the boundary to itself and take all (semi)circles to (semi)circles and preserve betweenness and angles. Hartshorne describes these transformations (Sections 37-39) as compositions of inversions in circles (which is a topic in Euclidean geometry that is interesting in its own right).

For a recent discussion of various geometries from the transformation point-of-view see [PrTi].

## 8. CONCLUSIONS

This book as a *Companion to Euclid* (the title of an earlier version) produces what it claims. It is a reasonably complete description of the Building Structures strand since Euclid. It has many detailed historical comments (but leaves out almost all mention of contributions from non-Greek and non-European sources). There are

large numbers of challenging exercises at the end of each of the 47 sections. I learned many things reading this book and my knowledge of the Building Structures strand has been enhanced. I judge that the book is suitable in level as a text for well-prepared mathematics majors. However, I think that any such course should be in a context where the students are aware of geometry from the other two strands. This book could contribute as a part of a balanced undergraduate geometry curriculum.

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DAVID W. HENDERSON

CORNELL UNIVERSITY

E-mail address: [dwh2@cornell.edu](mailto:dwh2@cornell.edu)