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*Knots*, by G. Burde and H. Zieschang, de Gruyter Stud. Math., vol. 5, Walter de Gruyter, New York, 2003, xii+559 pp., $69.95, ISBN 3-11-017005-1

*Algebraic invariants of links*, by Jonathan Hillman, Series on Knots and Everything, Volume 32, World Scientific Co. Pte. Ltd., Singapore, 2002, xii+305 pp., $42.00, ISBN 789812-381545

The discovery 18 years ago by Vaughan Jones of a powerful new polynomial invariant for knots and links began a revolution in knot theory. Of the nearly 8,000 *Mathematical Reviews* about knots, more than two thirds have appeared after 1985. Connections between knot theory and other areas of mathematics as well as physics are no longer surprising. Into this heady environment come two new publications, both revisions of important books on knots and links, both surveying knot theory from an earlier, mostly algebraic point of view.

*Knots*, 2nd edition, by Gerhard Burde and Heiner Zieschang, revises and expands the first edition, which was published in 1985. In this book, which has a strongly group-theoretic flavor, links appear, but high-dimensional knots do not. *Knots* is intended as a textbook, and according to its authors, inspiration was derived from Dale Rolfson's classic *Knots and Links*, Publish or Perish Press, published in 1976 and reprinted in 1990 (with corrections).

*Algebraic Invariants of Links*, by Jonathan Hillman, is a greatly revised and expanded version of Hillman's *Alexander Ideals of Links*, Springer-Verlag Lecture Notes in Mathematics, Volume 895, published in 1981. Intended more as a reference rather than a textbook, *Algebraic Invariants of Links* emphasizes links and high-dimensional knots. It is more technical than *Knots* and requires a deeper algebraic background from the reader.

Together these two books offer an algebraic view of knot theory that is both panoramic and penetrating.

There are several superb surveys of knot theory ([G79], [KW79], [E99], [LO03], [L03], [T85] for example), and it is not our purpose to compete with them. Rather, we intend to describe some of the algebraic ideas that are central to the two books under review. Details about the books themselves will be found in the final section.

## I. Origins of the subject

Knots and links are an ancient part of human culture. For centuries they have served to ornament and secure. Tying a knot in a handkerchief or some other material remains a way of recalling to mind something that must be done. Tying several knots can record a number. An early example of the practice is found in *The Persian Wars*, written by Herodotus in 440 B.C.:

> *The king took a leather thong and tied sixty knots in it. He called together the Ionian rulers, and addressed them as follows: "...From the time that I leave you to march into Scythia, untie one of the knots each day. If I do not return before the knots are untied, then leave your station, and sail back to your homes."*

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It is puzzling that mathematical questions about knots and links do not appear until the 19th century. One possible explanation is that mathematicians had not yet formulated the topological concepts needed to ask such questions. Another reason is that a mathematician's knot differs from that of a workman or a sailor by having no loose ends: a *knot* is an embedded circle in 3-space $S^3$. (For simplicity we will work only in the smooth setting, where functions have partial derivatives of all orders.) A *link* is a finite collection of mutually disjoint knots, each knot referred to as a *component* of the link. Two knots or links are regarded as the same if they are ambiently isotopic, that is, if one can be smoothly deformed into the other. Other, weaker equivalence relations such as concordance and link homotopy are considered as well.

The earliest significant theorem of the subject is one about links. In 1833 Gauss showed that "number of intertwinings", what we now call the linking number of two knots, can be computed by an integral [G33]. (Gauss's integral finds the degree of a map from a torus, which parametrizes the link, to the 2-sphere.) It is likely that Gauss was motivated by the problem of determining the smallest region of the celestial sphere, the *zodiacus*, onto which the orbits of two heavenly bodies can be projected [E99].

The fundamental problem of knot theory is that of classification. Gauss's student and protegé, J.B. Listing, came very close to posing it. In his 1848 paper *Vostudien zur Topologie* [L48], which introduced the term topology, Listing used polynomials to encode knot diagrams combinatorially, hoping to create a practical calculus. He could prove very little, but he did show that the figure eight knot (now sometimes called Listing's knot) is isotopic to its mirror image.

The most important figure in 19th century knot theory was the Scottish physicist P.G. Tait. Motivated by Von Helmholtz's investigation of vortex motion and by the theory of the vortex atom proposed by William Thompson (later Lord Kelvin), Tait began compiling tables of knots [T77], [T84], [T84′]. After Helmholtz had shown that a closed orbit in a frictionless and incompressible fluid can be neither created nor destroyed, Kelvin proposed that atoms are eternal knotted vortices rotating in an aether. Kelvin's theory appealed to Tait on both scientific and philosophical levels. In [T76] Tait wrote: "Thus this property of rotation may be the basis of all that to our senses appeals as matter." He lamented, "Unfortunately, it appears impossible for us to form, even with an imperfect fluid like air or water, a vortex-filament of any more complex character than that simple circle."

James Clerk Maxwell had been a school friend of Tait. They attended Edinburgh Academy together and remained lifelong friends. Maxwell's monumental *Treatise on Magnitism and Electricity* [M91] employed many of the terms and ideas of Listing's *Topologie.* It also explained a deep physical application of Gauss's linking integral: If one knot is regarded as a wire through which electric current passes, then the integral expresses the work done against the induced magnetic field by a charged particle traveling along a path described by the other knot. Maxwell was struck by the idea that two knots can have zero linking number and yet be inseparably "interlocked". An illustration of such a link appears in the *Treatise* (Volume II, p. 43). Perhaps it was this image that caused Maxwell to pen the following verse [K11]:

*It's monstrous, horrid, shocking,*
*Beyond the power of thinking,*
*Not to know, interlocking*
*Is no mere form of linking.*

Tait employed knot *diagrams*, drawings that represent knots and links as curves with transversal intersection points, called crossings. The simple *trompe l'oeil* effect of breaking an arc indicates that an arc dips below the plane of projection. Tait understood that certain simple local changes in the diagram would not alter the isotopy class of the knot. What he apparently did not know was that successive applications of just three special changes are also sufficient to convert any diagram representing one knot to another. The latter fact was proved by K. Reidemeister in 1932 [R32], and the three changes are now called *Reidemeister moves*. As a consequence, knots and links can be studied formally as equivalence classes of diagrams. Any quantity defined for a diagram that is unchanged by each of the three Reidemeister moves is necessarily an invariant.

Tait's approach was intuitive. He had no effective tools to tell that two knots were different. Rigorous methods for distinguishing knots would have to wait until the next century. Indeed, several of Tait's conjectures were settled only in recent years using the Jones polynomial [K87], [M87], [MT93], [T87].

It is ironic that while knot theory is the offspring of a fanciful physical theory, its more serious connections with physics have only recently been glimpsed. In 1987 M. Atiyah speculated that a relationship might exist between the Jones polynomial and Floer homology [A88]. The latter is homology theory of 3-manifolds based on consideratons of gauge theory and instantons. One year later E. Witten presented a formulation of the Jones polynomial in terms of topological quantum field theory [W89], [W89′]. It is likely that any discovery of a direct connection between the Jones polynomial and the algebraic themes of *Knots* and *Algebraic Invariants of Links* would begin yet another revolution in knot theory.

## II. Knot theory in the twentieth century

High-dimensional knots first appeared in a paper by E. Artin in 1925 [A25]. An *$n$-knot*, for $n \geq 1$, is an embedded $n$-sphere in $S^{n+2}$. Similarly, an *$n$-link* is a collection of $n$-knots. (Since an $n$-knot is a special case of an $n$-link, we will occasionally use just the latter term for the sake of simplicity. We will refer to 1-links simply as links, reserving the more technical term for the higher dimensional objects.) As in the classical case of $n = 1$, two $n$-links are regarded as the same if they are ambiently isotopic.

Knot theory is intrinsically connected with the study of manifolds, a fact that itself bestows a raison d'etre on the subject. Every closed orientable 3-manifold arises as a covering space of $S^3$ branched over a link [A20].

On a simpler level, the complementary space $S^{n+2} \setminus \ell$ of an $n$-link is determined up to homeomorphism by the isotopy class of $\ell$, and hence any topological invariant of the complement is an invariant of $\ell$. A particularly important invariant is the Poincaré or fundamental group $\pi_1(S^{n+2} \setminus \ell)$, called simply the group of $\ell$ and often denoted by $\pi(\ell)$ (abbreviated further by $\pi$, if the link is understood). Efforts of Dehn, Reidemeister, Seifert and Wirtinger in the early 1900's resulted in an algorithm to determine a finite presentation of $\pi(\ell)$ for any link $\ell \subset S^3$ from a diagram.

High-dimensional knots are more difficult to visualize than classical knots. In general one cannot simply draw a picture. (Sequences of cross-sections, however, can help us see 2-knots [R93], [CS98].) Artin's paper provided a simple and direct construction called "spinning". Given any $n$-link $\ell$, one obtains an $(n+1)$-link with the same group as $\ell$. Consequently, the sets $\mathcal{G}_n$ of $n$-knot groups are nested. In fact,

$$\mathcal{G}_1 \subsetneq \mathcal{G}_2 \subsetneq \mathcal{G}_3 = \cdots = \mathcal{G}_n = \cdots .$$

(The first proper inclusion was shown by Kervaire [K64]; for the second, see Farber [F75], Gutierrez [G72], Levine [L77], for example.)

If $\pi$ is the group of a $\mu$-component $n$-link, then:

(1) $\pi$ is finitely presentable;
(2) $\pi$ is generated by the conjugates of $\mu$ elements;
(3) the abelianization $\pi/\pi'$ is isomorphic to $\mathbb{Z}^\mu$;
(4) if $n > 1$, then $H_2(\pi, \mathbb{Z}) = 0$.

In 1964, M. Kervaire used surgery techniques to prove the converse statement when $n \geq 3$ [K64]: any group satisfying conditions (1) – (4) is the group of 3-link. In the hands of Haefliger, Levine, Cappell and Shaneson, and others, surgery theory became a powerful tool for high-dimensional knot theory. See [LO03] for an overview.

While high-dimensional knot groups are characterized, open problems and questions remain in lower dimensions. Is there an intrinsic characterization of $n$-knot groups, for $n = 1, 2$? Is every finitely generated subgroup of a 2-knot group necessarily finitely presentable? Is every epimorphism from the commutator subgroup of a 1-knot group to itself necessarily an isomorphism?

For classical links, the group is a complete invariant in the following sense. For convenience, we orient the components of a link and require that isotopy respect them. We can select tubular neighborhoods $V_i$ for the $\mu$ components of a link $\ell$. Each $V_i$ is a solid torus, and we can easily arrange that $V_i \cap V_j \neq \emptyset$ whenever $i \neq j$. A loop $m_i \subset \partial V_i$ that is noncontractible in the boundary but bounds a disk in $V_i$ is called an *ith meridian* of $\ell$; a loop $l_i \subset \partial V_i$ that meets $m_i$ transversally in a single point and is null-homologous in $S^3 \setminus \ell_i$ is called an *ith longitude*. Well-defined orientations for $m_i$ and $l_i$ are induced by the orientation of the corresponding component of $\ell$. After choosing a path from a base point to $m_i \cap l_i$, we can regard $m_i$ and $l_i$ as elements of $\pi(\ell)$. The conjugacy class $\langle m_i, l_i \rangle^\pi$ of subgroups in $\pi(\ell)$ generated by $m_i$ and $l_i$ is well-defined, independent of the choice of base path. The *group system* of $\ell$ is the tuple

$$(\pi; \langle m_1, l_1 \rangle^\pi, \ldots, \langle m_\mu, l_\mu \rangle^\pi).$$

A theorem of F. Waldhausen [W68] implies that two links are identical if and only if their group systems are the same in the sense that there exists an isomorphism of their groups that maps corresponding pairs of meridian and longitude elements to each other. It follows that a link is trivial (that is, the boundary of mutually disjoint disks) if and only if its group is free.

While the group system of a link is a complete invariant, extracting information from it is a formidable task. An algebraic approach to knot theory seeks invariants of the group system that are computable and yet nontrivial. The most basic ones are derived from the homology of covering spaces of the link complement.

The homology group $H_1(S^{n+2}\backslash\ell) \cong \pi/\pi'$ (integer coefficients understood) is free abelian with generators $t_1, \ldots, t_\mu$ corresponding to meridians of $\ell$. It is useful to regard it as a multiplicative group $\Pi = \langle t_1, \ldots, t_\mu\rangle$. The *universal abelian covering* $\tilde{X}$ of $S^{n+2}\backslash\ell$, the cover corresponding to $\pi'$, is a storehouse of all *abelian invariants* of $\ell$.

The group $\Pi$ acts on $\tilde{X}$ as the covering transformation group, and each homology group $H_i\tilde{X}$, regarded as a module over the integral group ring $\mathbb{Z}\Pi$, is finitely generated. Identifying $\mathbb{Z}\Pi$ with the Laurent polynomial ring $\Lambda$ in the variables $t_1, \ldots, t_\mu$ allows us to regard $H_i\tilde{X}$ as a $\Lambda$-module.

Although $\Lambda$ is Noetherian and a unique factorization domain, there is no really satisfactory theory of finitely generated $\Lambda$-modules. Fortunately, computable invariants exist. For each nonnegative integer $d$, the *dth elementary ideal* $E_d(\ell) = E_d(H_1\tilde{X})$ is the ideal of $\Lambda$ generated by the $(n-d)$-minors of an $m$-by-$n$ presentation matrix for $H_1\tilde{X}$. The greatest common divisor of its elements is the *dth characteristic polynomial* $\Delta_d(\ell) = \Delta_d(H_1\tilde{X})$. The polynomial $\Delta_0(\ell)$ is usually called the *Alexander polynomial* of $\ell$ and it is often abbreviated by $\Delta(\ell)$.

When $n > 1$, the link group carries much less information than it does in the classical dimension. (One considers only the group and meridian elements, since longitudinal elements are meaningless.) Given any $n$-knot, there are infinitely many with the same group. In this case invariants must be sought elsewhere, such as in higher homology groups and higher homotopy groups of $\tilde{X}$.

In the 1950's and 1960's knot theory was rejuvenated by the work of Ralph Fox and John Milnor. In his senior thesis, written under Fox's direction, Milnor introduced *link homotopy*, a deformation by which each component of a link is allowed to pass through itself (but not through any other) [M54]. In particular, small local knots, which have no effect on linking number, can be eliminated. The relation of link homotopy, clearly weaker than isotopy, gets closer to the heart of linking phenomena.

Milnor defined a group $\mathcal{G}$, the largest quotient of $\pi$ in which each meridian generator $m_i$ commutes with all of its conjugates; it is the largest common quotient of the groups of those links that are link homotopic to $\ell$. In the case of a knot, $\mathcal{G}$ is simply the infinite cyclic abelianization of $\pi$, as one would expect since all knots are link homotopic. However, for links of more than one component, $\mathcal{G}$ is a nontrivial invariant of link homotopy.

Milnor's group remains an object of study today. Let $N_i$ be the subgroup of $\mathcal{G}$ generated by the commutators $[g,[g,m_i]]$, where $g \in \mathcal{G}$. Following J. Levine [L88] we consider peripheral pairs $(m_i, l_iN_i,)$, where by abuse of notation the meridian $m_i$ is identified with its image in $\mathcal{G}$ while $l_iN_i$ is a coset of $N_i$ in $\mathcal{G}$. In a sense, $(\mathcal{G}(\ell); \langle m_1, l_1N_1\rangle^{\mathcal{G}}, \ldots, \langle m_\mu, l_\mu N_\mu\rangle^{\mathcal{G}})$ is a link homotopy analog of the group system of $\ell$ defined earlier. Levine conjectured that two links are link homotopic if and only if there is an isomorphism between their Milnor groups inducing maps that match corresponding peripheral pairs, and he verified this in [L88] for links of up to four components. James Hughes, a former student of Levine, showed that the above peripheral structure is finer than that considered by Milnor. However, his arguments [H93], [H98] suggest that Levine's peripheral structure might need to be made finer yet for links of more than four components.

Milnor introduced a sequence of numerical invariants that generalize Gauss's linking numbers [M54], [M57]. He defined these "higher linking numbers", called

$\bar{\mu}$-invariants, in terms of the coefficients of Magnus expansions of words representing the longitudes. Fox's "free differential calculus" provided the computational machine. However, they are usually difficult to compute. General $\bar{\mu}$-invariants are related to certain Massey cohomology products. Often they can be computed in a more geometric fashion via intersection theory, as T. Cochran has shown [C90]. Milnor showed that the invariants vanish if and only if the link is homotopically trivial. A complete classification of links up to link homotopy was achieved by N. Habegger and X.S. Lin [HL90]. They also provided an effective algorithm to decide if two links are link homotopic.

The study of singularities of surfaces in 4-space led Fox and Milnor to introduce yet another equivalence relation [FM66]. Knots $k_0 \subset \mathbb{R} \times \{0\}$ and $k_1 \subset \mathbb{R} \times \{1\}$ are *concordant* if together they form the boundary of an annulus embedded in $\mathbb{R} \times [0,1]$. Recall that all maps here are infinitely differentiable. The same relation with the weaker requirement that the annulus be topologically embedded is called *I-equivalence*. Clearly isotopy implies concordance implies I-equivalence. Isotopy is a stronger requirement than concordance, as we shall see. That concordance is stronger than I-equivalence is less obvious. Although the last result, due to C. Giffen [G76], is unpublished, a short exposition is presented in the first chapter of *Algebraic Invariants of Links*.

A knot that is concordant to the trivial knot is called a *slice knot*. The simplest nontrivial example is the *square knot*, the connected-sum of a left-hand trefoil and a right-hand trefoil. (The *connected sum* of two knots is obtained by splicing them together.) The set of concordance classes of knots under the connected sum operation forms an abelian group $\mathcal{C}$ in which the identity element is the class of slice knots. No complete set of invariants for knot concordance is known. However, in [COT03] Cochran, K. Orr and P. Teichner exhibited a new, geometrically defined filtration of $\mathcal{C}$:

$$\cdots \subset \mathcal{F}_{(n.5)} \subset \mathcal{F}_{(n)} \subset \cdots \subset \mathcal{F}_{(0.5)} \subset \mathcal{F}_{(0)} \subset \mathcal{C},$$

indexed by half-integers. The first few terms correspond closely to previously known concordance invariants. If the class of a knot is in $\mathcal{F}_{(1.5)}$, then all previously known concordance invariants of the knot vanish. Such invariants include those of A. Casson and C. McA. Gordon [CG75] as well as P. Gilmer [G83], [G93], P. Kirk and C. Livingston [KL99] and C. Letsche [L00].

Most known examples of slice knots, such as the square knot, bound disks in 4-space that project with only ribbon singularities, singularities that resemble a piece of a ribbon passing through the interior of another. Knots that bound such disks are called *ribbon*. One of the most stubborn open questions in knot theory asks whether every slice knot is ribbon. There are other important questions as well. One consequence of Freedman's surgery theory in the topological category is that every knot with Alexander polynomial one bounds a locally flat disk in the 4-ball. Must it bound a smooth disk (that is, must it be slice) [K97, Problem 1.36]? An algorithm exists for deciding whether a knot is trivial [H61], but the problem remains of finding an algorthithm for determining whether a knot is slice or ribbon [K97, Problem 1.34].

Concordance can be defined also for higher-dimensional knots and links, and much is known. Kervaire [K65] proved that all even-dimensional knots are slice, while Levine [L69] and Stoltzfus [S77] determined the concordance groups of odd-dimensional knots $k \subset S^{2n+1}, n > 1$. Still, questions remain. For example, it is not known if all even-dimensional *links* are slice.

## III. Details of the books

*Knots* consists of 15 chapters and four appendices that guide the reader through much of classical knot theory. Each chapter concludes with a brief history and exercises. The tables of knot invariants in the first edition of *Knots* contained errors that have been removed. An updated 40-page bibliography will be valued by anyone attempting to work in this swiftly moving field.

Chapters 1 and 2 are concerned with basic definitions and geometric concepts. Knot diagrams, Reidemeister moves, and the connected sum operation are explained. *Prime knots*, those knots that cannot be nontrivially expressed as connected sums, have an atomic significance in the subject just as primes do in number theory. Satellite knots, which generalize connected sums, are also described. Roughly, $k$ is a satellite of another knot $\hat{k}$ if it is contained nontrivially in a thickened copy of $\hat{k}$. That satellite knots can be prime gives an idea of the concept's subtlety.

Chapters 3, 4 and 5 cover material about knot groups $\pi$ and their commutator subgroups $\pi'$. Fibered knots can be characterized in such terms. They are knots for which $\pi'$ is finitely generated. (A more topological definition is given below.) The authors have taken the opportunity to correct an erroneous result about fibered satellite knots that appeared in the first edition.

Torus knots are so named because they can be isotoped to lie on a standardly embedded torus. In many ways they are the simplest nontrivial knots. Chapter 6 is devoted to a proof of an important result due to the authors: A nontrivial knot is a torus knot if and only if its group has nontrivial center.

Chapter 5 is devoted to the study of fibered knots. A knot $k$ is *fibered* if its complement admits a locally trivial fibration over the circle. In this case, $S^3$ minus an open tubular neighborhood of $k$ is diffeomorphic to $S \times [0,1]/\{(x,0) \sim (h(x),1)$, for some compact orientable surface $S$ with boundary, and some homeomorphism $h : S \to S$ called a *monodromy*. The trefoil and figure eight knot, the two nontrivial knots that can be drawn with fewer than 5 crossings, are both fibered. Fibered knots can be studied through the properties of their monodromy $h$. This point of view opens the door to geometry and dynamics. Regrettably, such topics are not addressed in *Knots*.

Chapters 8, 9 and 13 are substantial introductions to cyclic covering spaces and their homology groups. The reader will find a careful exposition of Fox's theory of derivations (the so-called Fox differential calculus) and several well-chosen examples. Although the authors minimize space devoted to links, they wisely included a section about the multivariable Alexander polynomials that are defined by them.

Classical knots and links can be studied from the point of view of braids, invented by Artin in 1925. Briefly, a *braid* is an isotopy class of any number of mutually disjoint strands that run strictly downwards from one level $\mathbb{R}^2 \times \{1\}$ to another $\mathbb{R}^2 \times \{0\}$. Braids with the same number of strands form a group under concatenation. Connecting the ends of a braid in a fixed manner, without introducing any new crossings, produces a knot or link. Such *closed braids* had already appeared: in 1923, J. Alexander had shown that any knot or link arises as a closed braid. The braid point of view has continued to be useful, suggesting connections with areas such as dynamical systems, robotics and statistical mechanics, just to name a few. Braids and representations of their groups motivated Jones's discoveries in the mid 1980's. Chapter 10 is devoted to this vital topic. A new feature in this edition of *Knots* is a proof of Markov's theorem, which states that two (oriented) links represented by braid closures are isotopic if and only if the corresponding braids differ by a finite sequence of elementary moves ("Markov moves").

In 1920 J. Alexander proved that every closed 3-manifold arises as a covering of $S^3$ branched over a link. Alexander's argument was completed independently by J. Birman and H. Hilden in 1975, and the result was then improved by Hilden and J. Montisinos in 1976. Chapter 11 contains proofs of the theorems of Alexander and Hilden-Montisinos. There is also a discussion of Heegaard diagrams for 3-manifolds.

Chapter 12 is devoted to 2-*bridge knots*, knots that have diagrams with only two local maxima. Montisinos links, which in a sense generalize 2-bridge knots, are also treated.

Chapters 14 and 15 bring the reader back to knot groups. Representations onto simpler groups have been a source of usable knot invariants. Metabelian representations are discussed in detail. Many knots possess diagrams that display rotational symmetry of some period. The authors use metabelian representations to investigate the form of the Alexander polynomial of periodic knots. The powerful periodicity results of Murasugi can be found here.

After reminding the reader that the group and its peripheral structure determine any knot, Chapter 15 investigates the subtle role that the meridian and longitude play. We are informed that the long-standing conjecture that knots themselves are determined by their complements was finally proved by Gordon and J. Luecke in 1989. Unfortunately, a proof of the Gordon-Luecke theorem is outside the scope of the book. Combining this with a theorem of W. Whitten [W87], we arrive at the appealing conclusion that there are at most two distinct prime knots with isomorphic groups.

The final chapter was added to the new edition, a very brief treatment of the HOMFLY polynomial (named after its main contributors: J. Hoste, A. Ocneano, K. Millet, W. Floyd, R. Lickorish and D. Yetter, but missing others: J. Conway, L. Kauffman, J. Przytycki and P. Traczyk). A generalization of the Jones polynomial, it can be computed in a simple combinatorial manner beginning with a knot or link diagram. However, deeper understanding comes from its derivation via a beautiful representation of the braid group into a Hecke algebra, the details of which can be found here.

The exposition of *Knots* is both careful and concise, and every topic chosen is essential to the subject. It is understandable that links are mentioned only briefly and a discussion of high-dimensional knots is completely omitted. However, it is regrettable that the word "geometric" is used here only in the most restricted way. Thurston's discovery that the complement of any knot that is neither a satellite nor a torus knot admits a hyperbolic structure afforded a powerful new approach to the subject.

Alas, one must make choices when writing such a book. Nevertheless, the topics covered here form a beautiful whole. *Knots* will be appreciated by anyone interested in the subject, especially from a group-theoretic point of view.

*Algebraic Invariants of Links* concentrates on links of more than one component as well as high-dimensional knots. The book has three parts: Abelian covers; Applications: special cases and symmetries; and Free covers, nilpotent quotients and completion.

Chapter 1 introduces the main definitions and equivalence relations of link theory. Boundary links, defined here, are in some ways more similar to knots than general links, and they appear throughout the book. An $n$-link is a *boundary link* if its components bound pairwise disjoint orientable manifolds (called *Seifert hypersurfaces*). An $n$-knot is trivially a boundary $n$-link, since a Seifert hypersurface bounding it can always be found. The algebraic attraction of boundary links becomes apparent in the classical case: a result of N. Smythe [S66] says that a $\mu$-component link $\ell$ is a boundary link if and only if its group maps onto the free group $F(\mu)$ of rank $\mu$, sending some set of meridians to a basis of $F(\mu)$. If the requirement on the meridians is dropped, then $\ell$ is said to be a *homology boundary link*.

Chapter 2 begins with a review of homology and cohomology with local coefficients. The Blanchfield pairing is introduced. A version of Poincaré duality for covering spaces, the pairing has proven to be essential in high-dimensional knot theory. Any $(2q-1)$-knot, for $q \geq 2$, is determined up to concordance by the equivalence class of its Blanchfield pairing in a certain Witt group.

While not a complete concordance invariant for 1-knots, the pairing is still of great interest. The chapter concludes with a discussion of signature invariants for odd-dimensional knots.

A review of elementary ideals and other determinantal module invariants is found in Chapter 3. It is followed by a discussion of maximal abelian covers in the next chapter. Chen groups, associated to any finitely generated group, are defined. For an $n$-link group $\pi$, they are the finitely generated abelian quotients $Ch(\pi, q) = \pi_q\pi''/\pi_{q+1}\pi''$, $q \geq 2$. (Here $\pi_q$ is the $q$th term of the lower central series of $\pi$.) It is shown that for any $\mu$-component link, the Alexander ideal $E_{\mu-2}(\ell)$ vanishes if and only if the Chen groups $Ch(\pi, q)$ agree with those of a trivial $\mu$-component link if and only if the longitudes of $\ell$ are contained in every $\pi'_q\pi''$. (Hillman proved this result in 1978, generalizing the case for $\mu = 2$ shown by Murasugi eight years earlier.)

Chapter 5 addresses the relationships between Alexander invariants of a link and those of its various sublinks. Intermediate abelian covering spaces of links such as the infinite cyclic "total linking number cover" are discussed. The chapter begins with the important Torres conditions, necessary conditions on the Alexander polynomial of a link, and generalizions for Alexander ideals and $n$-links due to Blanchfield, N. Sato and L. Traldi. A discussion of twisted Alexander polynomials, a topic that reappears constantly in knot theory, completes the first part of the book.

Chapter 6 is devoted to the study of knot modules. A *knot module* is a finitely generated $\mathbb{Z}[t, t^{-1}]$-module on which $t-1$ acts invertibly. The homology modules of the universal abelian cover of the complement of an $n$-knot are motivating examples. Levine and M. Farber have classified large classes of higher-dimensional knots in terms of their knot modules and associated pairings, while Levine and C. Kearton have shown that the concordance class of a $(2n+1)$-knot, for $n \geq 1$, is determined by the Blanchfield pairing on the middle-dimensional knot module. The classification problem for knot modules remains open. While 3-knot groups are classified, 3-knot modules are not.

In view of the wealth of sophisticated results in knot theory, it is amazing that many simple questions persist. One such question is: what polynomials arise as Alexander polynomials of 2-component links? Chapter 7 examines 2-component links. It begins with a review of important results in J. Bailey's unpublished Ph.D. thesis. It concludes with several examples, including a nontrivial 2-component ribbon link with group $\pi$ such that $\pi/\pi'' \cong F(2)/F(2)''$. In particular, $\ell$ has the same abelian invariants as a trivial link. The 3-component link that decorates the cover of the book is closely related: this example, due to Hillman, is a nonboundary link with the algebraic properties of a boundary link (in particular, its group maps onto $F(3)$).

Chapter 8 reviews results about symmetries of links and properties that are reflected in the Alexander invariants and concludes the second part of the book.

Exteriors of homology boundary links have covers with nontrivial free covering group. The third part of *Algebraic Invariants of Links* describes invariants of such covers. Chapter 9 surveys results of Sato, Du Val and Farber about modules over free group rings. Included is a theorem of Gutiérrez stating that a $\mu$-component $n$-link $\ell$ with $n \geq 3$ is trivial if and only if $\pi(\ell) \cong F(\mu)$, with basis a set of meridians, and higher homotopy groups $\pi_j(S^n \setminus \ell) = 0$ for $1 < j \leq [(n+1)/2]$. This result is a higher-dimensional version of the fact that a classical link is trivial if and only if its group is free.

The quotients of a link group by terms of its lower central series are I-equivalence invariants of the link. Chapter 10 explores properties of invariants associated to link exterior covers corresponding to canonical subgroups of the link group.

The last two chapters are introductions to Levine's work on algebraic closure and completions of groups, Le Dimet's high-dimensional disk links, and the link homotopy classification result of Habegger and Lin mentioned earlier. As Hillman explains in the introduction of his book, work in these areas is evolving. Many open questions and provocative conjectures are offered. This section will be most interesting to those looking for challenging research problems in link theory.

*Algebraic Invariants of Links* is masterful, offering a survey of work, much of which has not been summarized elsewhere. It is an essential reference for those interested in link theory. My only complaint is that the book is terse. The material presented could easily accommodate a book more than twice as long. Additional examples would help the reader. However, the criticism seems minor in view of the fact that there is no other book that covers such topics. *Algebraic Invariants of Links* is unique and valuable.

## IV. Concluding remarks

Knot theory is changing faster than anyone might have predicted. The combinatorial approach, which marked the emergence of the subject but was later overtaken when algebraic methods proved more effective, has returned with surprising vigor. Geometric perspectives have reshaped the methods and problems of the subject. Yet we continue to find that the development of our new structures requires a familiarity with earlier, algebraic concepts. (Consider, for example, our understanding of hyperbolic structures on knot complements, which relies so heavily on knot group representations.) *Knots* and *Algebraic Invariants of Links* support a foundation that will enable us to reach even higher levels of understanding of knotting and linking phenomena in this new century.

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