BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 42, Number 2, Pages 251–256
S 0273-0979(05)01046-3
Article electronically published on January 12, 2005

Bergman spaces, by Peter Duren and Alex Schuster, Mathematical Surveys and Monographs, vol. 100, American Mathematical Society, Providence, RI, 2004, x+318 pp., \$79.00, ISBN 0-8218-0810-9

1. Hilbert spaces of holomorphic functions

A holomorphic function f defined in the $\mathbb{D}$, the unit disk of the complex plane, has a power series representation $f(z) = \sum_{n=0}^{\infty} a_n z^n$. The back-and-forth between the analytic properties of f and the properties of the sequence $\{a_n\}$ is one of the basic sources of the very productive interaction between the theory of holomorphic functions and functional analysis. In particular, some natural operators on Hilbert space can be analyzed with great precision using tools from function theory. Consider for instance the three Hilbert spaces of (one-sided) sequences of complex numbers $\{a_n\}_0^{\infty}$ defined by the three conditions

$$\sum \frac{1}{n+1} |a_n|^2 < \infty,$$
$$\sum |a_n|^2 < \infty,$$
$$\sum (n+1) |a_n|^2 < \infty.$$

On each of these spaces one can define a fundamental operator, the shift operator, S, which shifts sequence entries one place to the right; $S((a_0, a_1, a_2, ...)) = (0, a_0, a_1, a_2, ...)$. After checking that each operator is continuous a basic goal is to describe its invariant subspaces (the closed subspaces mapped into themselves). By regarding the sequences as power series coefficients of holomorphic functions we are led to consider Hilbert spaces of holomorphic functions on $\mathbb{D}$ for which

$$\text{(BERG)} \qquad \begin{aligned} \|f\|_{A^2}^2 &:= \frac{1}{\pi} \int_{\mathbb{D}} |f(z)|^2 \, dxdy < \infty, \\ \|f\|_{H^2}^2 &:= \frac{1}{2\pi} \int_{\partial\mathbb{D}} |f(z)|^2 \, d\theta < \infty, \\ \|f\|_{D}^2 &:= \frac{1}{\pi} \int_{\mathbb{D}} |f'(z)|^2 \, dxdy < \infty, \end{aligned}$$

respectively (with a technical caveat on the middle formula). These spaces are the Bergman space, A^2, the Hardy space, H^2, and the Dirichlet space D. This changed viewpoint casts questions about S as questions about M_z, the multiplication operator defined by $(M_z f)(z) = zf(z)$. If we were working with the ring of all functions holomorphic in the disk, without topology, then the invariant subspaces of M_z would be exactly the ideals of the ring. The invariant subspaces play a comparably important role in the study of these Hilbert spaces. In each case some facts about the invariant subspaces of M_z are quite easy to obtain. Given any finite subset Z of $\mathbb{D}$, let p be a polynomial whose zero set is Z and, in each of the three spaces, let $[p]$ be the smallest invariant subspace containing p. It is easy to see that $[p]$ consists exactly of functions of the form pf with f in the Hilbert space and also that $[p]$

2000 *Mathematics Subject Classification.* Primary 30H05; Secondary 30-02, 46E15, 46E20, 47B38.

is the subspace of functions which vanish on Z. These invariant subspaces have finite codimension and with a small bit of work one also finds that, after adjusting to allow p to have repeated zeros, these are the only invariant subspaces of finite codimension. It is now also really easy to ask very hard questions. For instance, what are the related results for infinite sets Z? Which sets Z should be considered? What will replace the polynomial p? Will the invariant subspace consist "exactly of functions of the form..."? Do such constructions give all the invariant subspaces?

The book by Duren and Schuster [DS] can bring a student or researcher from a basic background in analysis to the research frontier on these and related questions for the Bergman space. Before discussing the Bergman space, however, we recall the elegant classical results for the Hardy space which defined the landscape in which Bergman space theory evolved. (The Dirichlet space has yet to earn a book of its own, but recent research progress could justify one.)

2. The Hardy space

$Z = \{z_i\} \subset \mathbb{D}$ is a zero set for the Hardy space; that is, there is an $f \in H^2$ which vanishes exactly on Z iff the Blaschke condition is satisfied: $\sum(1-|z_i|) < \infty$. (Note for later an easy consequence: the union of two zero sets is a zero set.) Associated with such a Z is a Blaschke product $B_Z \in H^2$ with exactly those zeros. The invariant subspace of functions which vanish on Z is exactly $[B_Z]$, which in turn equals $\left\{B_Z f : f \in H^2\right\}$. There are also invariant subspaces of the form $[S]$ where $S(z) \in H^2$ is a singular inner function, a nonvanishing function which approaches zero as z tends toward various points of $\partial\mathbb{D}$. The basic factorization result for the Hardy space is that any $f \in H^2$ can be factored uniquely as $f = BSO$ where B is a Blaschke product, S is a singular inner function, and O is a zero-free function in H^2 which is not divisible (in H^2) by any singular inner function. The product BS is called the inner factor of f; O is the outer factor. Each of the three factors B, S, and O can be characterized intrinsically and each is given by an explicit formula. The basic result about invariant subspaces is due to Beurling [B].

Theorem 1. *Given $f \in H^2$ with factorization BSO, then $[f]$, the smallest invariant subspace containing f, is $BSH^2 = \{BSh : h \in H^2\}$. These are the only invariant subspaces.*

This is a complete and satisfying description of the invariant subspaces. The possible B's are parametrized by sequences satisfying the Blaschke condition, the possible S's are parametrized by positive singular measures on the circle, and the pairs (B,S) parametrize the full collection of invariant subspaces. The cyclic vectors for M_z, those $f \in H^2$ for which $[f] = H^2$ (the analogs of invertible elements in the ring) are precisely the outer functions, those with trivial inner factor, $B(z)S(z) \equiv 1$.

Another fundamental result about H^2 is a description of the interpolating sequences: those sequences $Z = \{z_i\} \subset \mathbb{D}$ which are so scattered that one can freely interpolate the values of an H^2 function, subject only to the natural size restriction. That restriction can be described using the point evaluation functionals. Evaluation of functions in H^2 at a point $z \in \mathbb{D}$ is a continuous linear functional and hence, by Hilbert space basics, can be realized by an inner product, $f(z) = \langle f, k_z \rangle$, for a unique k_z in H^2. General theory then insures that if $f \in H^2$, then the function $z_i \to f(z_i)/\|k_{z_i}\| = (1-|z_i|)^{1/2} f(z_i)$ is a bounded function on Z. Z is called an interpolating sequence for H^2 if all the functions on Z which are obtained this way are in $l^2(Z)$, and if, furthermore, every function in $l^2(Z)$ can be obtained this way. Analogous definitions are used for both A^2 and D. A complete description of such sequences was given by Shapiro and Shields [SS] building on the earlier work of Carleson [C]. For any z in $\mathbb{D}$ set $\tilde{k}_z = k_z / \|k_z\|$ and note that, by the Cauchy-Schwarz inequality, for any z, w, $\left|\left\langle \tilde{k}_z, \tilde{k}_w \right\rangle\right| \leq 1$. If Z is an interpolating sequence, then it must be possible, given any i, j, to find $f \in H^2$ so that $(1-|z_i|)^{1/2} f(z_i) = 0$, $(1-|z_j|)^{1/2} f(z_j) = 1$, and to do this with control of the size of $\|f\|$. This implies a weak separation condition on the points of Z which is necessary for Z to be an interpolating sequence: there is $\varepsilon > 0$ so that for all i, j,

$$\text{(SEP)} \qquad \left|\left\langle \tilde{k}_{z_i}, \tilde{k}_{z_j} \right\rangle\right| \leq 1 - \varepsilon.$$

An equivalent geometric statement is that there is a uniform lower bound on the hyperbolic distances $d(z_i, z_j)$. Shapiro and Shields showed that the necessary and sufficient condition for Z to be an interpolating sequence is that (SEP) hold and that the sequence satisfy a very explicit, although somewhat technical, density condition (that $\sum \|k_{z_i}\|^{-2} \delta_{z_i}$ be a Carleson measure).

3. The Bergman space

Much less is known about the Bergman space and most (but not all) questions seem fundamentally more difficult than their analogs on the Hardy space. For instance it is a result of Horowitz [Ho] that, in contrast to H^2, there are pairs of zero sets for A^2 whose union is not a zero set. A complete description of zero sets for A^2 is not known. What is known, and there is a great deal, is in Chapter 4 of [DS]. The description of the invariant subspaces for M_z acting on A^2 must be even more complicated than the description of the zero sets. If V is an invariant subspace for M_z in H^2, then $\dim(V/M_z(V)) = 1$, but for the Bergman space this index can take any value. This was first shown using abstract techniques by Apostol, Bercovici, Foias, and Pearcy [ABFP]. [DS] presents the more recent constructive examples that exhibit this behavior.

One of the reasons for the increase in complication as we pass from H^2 to A^2 is that functions in H^2 have nontangential boundary limits *a.e.* and their behavior near the boundary is (relatively) well understood. However, even after the deep investigations beginning with Korenblum's work, [K1], [K2], the behavior of A^2 functions near the boundary is not well understood. Some of what is known is in [DS] and more is in [HKZ].

Around 1990 there were two fundamental advances in the theory of the Bergman space. First, Håkan Hedenmalm found a class of functions that could be viewed as analogs of Blaschke products and inner functions in H^2 [He]. Building on that and the work of Aleman, Richter, and Sundberg [ARS] we now have the framework of a theory of invariant subspaces for M_z on A^2 which has substantial similarities to the theory in the Hardy space. We can give some of the flavor here. For the most straightforward case, consider the invariant subspace V_Z of functions which vanish on a set Z, and assume for convenience $0 \notin Z$. Consider the function f_Z in V_Z which maximizes $\operatorname{Re} f(0)$ subject to $\|f\|_{A^2} = 1$. In the Hardy space such a construction will single out the Blaschke product for the zero set. Here it produces a function with some similar fundamental properties: f_Z will vanish on Z and nowhere else, f_Z generates the invariant subspace $[f_Z] = V_Z$, f_z is a contractive divisor; that is, if $h \in V_Z$, then $h/f_Z \in A^2$ and $\|h/f_Z\| \leq \|h\|$ (for the Hardy space there would be equality here), and hence $[f_Z] = \left\{f_Z h : h \in A^2\right\}$. As in H^2, f_Z is rational if Z is finite. Furthermore, any $f \in A^2$ can be written as a (Bergman) inner function of roughly this type times a (Bergman) outer function O which is cyclic for M_z, $[O] = A^2$. Again, both the inner and outer functions have explicit intrinsic descriptions. However, the situation is not as clean as in the Hardy space: the factorization is not unique, the factors are not given by explicit formulas and their analytic properties remain quite mysterious, there are invariant subspaces which are not of the form $[f]$ for a single f (but such subspaces are generated by the inner functions they contain), etc.

What about interpolating sequences? In the 1980's there were some weak results by the reviewer [R], who concluded there was no clean necessary and sufficient condition for a sequence to be an interpolating sequence for the Bergman space. In a striking 1993 paper Kristian Seip [S1] gave a clean necessary and sufficient condition. In addition to the Bergman space version of (SEP) which is again necessary, and for the same reason, the sequence must have an appropriate density of at most 1/2. (For the Bergman space modeled on L^p the crucial density is $1/p$.) The relevant density is an asymptotic average density with a slightly involved definition. However, as an example, for $a > 0, b \in \mathbb{R}$ the image of the set $\{a^n(mb+i) : n, m \in \mathbb{Z}\}$ under a conformal map of the upper half plane to the unit disk has density $2\pi/(b \log a)$. In the same paper, using similar densities, Seip also described the sampling sequences for the Bergman space, a related notion which has only a trivial analog in the Hardy space. The sampling sequences for A^2 must have density at least 1/2.

4. This book

In this book the authors consider the classical Bergman space, defined by (BERG), and also the Banach and quasi-Banach spaces of functions obtained when the exponent 2 in (BERG) is replaced by p, $0 < p < \infty$. The first four chapters collect function-theoretic background material and develop basic results such as growth estimates for functions, coefficient estimates (non-trivial as soon as $p \neq 2$) as well as what is known about zero sets. The rest of the book develops the two newer themes: two chapters are on sampling and interpolation and three on Bergman inner and outer functions and aspects of the invariant subspace theory.

The authors have included prerequisite material as well as lots of detail and explicit examples. As a result, even though there are no problems at the end of chapters, the book would be a very good choice for a graduate course, a reading course, or seminar presentations. Of course some topics are omitted. For instance, all the topics in the book are also studied when $dxdy$ in (BERG) is replaced by $(1-|z|^2)^\alpha dxdy$ with $\alpha > -1$, that is, not included. But in general the notes and references provide a good road map to topics not included.

There is substantial overlap in topics between this book and the recent book of Hedenmalm, Korenblum and Zhu [HKZ]. That book, with its substantially broader scope, will be invaluable to workers in the field, but it is more demanding and perhaps less well suited to a student new to the area.

The material on sampling and interpolation is also covered in Seip's new monograph [S2] where it is the author's intent "to view the sizable literature on interpolating sequences for spaces of analytic functions as one subject." In particular it has the first presentation in a book of recent work characterizing interpolating sequences for D [Bo]. Seip's book is a wonderful complement to the one being reviewed.

5. A broader view

Of course there are many generalizations of these themes. Hardy and Bergman spaces are studied for general domains in $\mathbb{C}$ and $\mathbb{C}^n$. In fact Bergman spaces can be used to study the geometry of those domains, a viewpoint that was championed by Stefan Bergman and is still important. Also, beyond M_z, a wide variety of operators and operator algebras acting on Bergman and Hardy spaces are studied systematically.

More surprisingly, and perhaps more fundamentally, both the theory of the Hardy space and of the Bergman space have strong resonances outside the study of spaces of holomorphic functions. There is a mature and powerful real variable Hardy space theory which has distilled some of the essential oscillation and cancellation behavior of holomorphic functions and then found that behavior ubiquitous. A good introduction to that is [CW]; a more recent and fuller account is in [St]. The theory of the Bergman space is less well developed, but already there are deep and unexpected relations with other areas. For example, the study of the Bergman inner functions turns out to be entwined with the theory of the bi-Laplace equation: $\Delta^2 u = \Delta(\Delta u) = 0$ and its Green function. In a different direction, work on interpolating and sampling sequences for the Bergman space is a function-theoretic analog of recent work on wavelet frames and bases in Euclidean space. The Möbius group acting on the disk plays a role similar to the $Ax + B$ group acting on the Euclidean phase space and similarities between basic results in the two areas are clarified by casting both in the language of the associated hyperbolic geometries. That view of things is set out in [AAG].

What comes next for the Bergman space? I recall the words of a sportscaster about the Boston Red Sox after they lost the 1967 World Series: "They're a young team and their future is still ahead of them."

References

[ARS] Aleman, A., Richter, S., Sundberg, C., Beurling's theorem for the Bergman space. Acta Math. 177 (1996), no. 2, 275–310. MR1440934 (98a:46034)

[AAG] Ali, S., Antoine, J-P., Gazeau, J-P., Coherent states, wavelets and their generalizations. Graduate Texts in Contemporary Physics. Springer-Verlag, New York, 2000. MR1735075 (2002m:81092)

[ABFP] Apostol, C., Bercovici, H., Foias, C., Pearcy, C., On the theory of the class $A_{\aleph_0}$ with applications to invariant subspaces and the Bergman shift operator. Advances in invariant subspaces and other results of operator theory (Timişoara and Herculane, 1984), 43–49, Oper. Theory Adv. Appl., 17, Birkhäuser, Basel, 1986. MR0901058 (88k:47005)

[B] Beurling, A., On two problems concerning linear transformations in Hilbert space, Acta Math. 81 (1948). MR0027954 (10:381e)

[Bo] Böe, B., Interpolating sequences for Besov spaces. J. Funct. Anal. 192 (2002), no. 2, 319–341. MR1923404 (2003h:46044)

[C] Carleson, L., An interpolation problem for bounded analytic functions. Amer. J. Math. 80 (1958), 921–930. MR0117349 (22:8129)

[CW] Coifman, R., Weiss, G. Extensions of Hardy spaces and their use in analysis. Bull. Amer. Math. Soc. 83 (1977), no. 4, 569–645. MR0447954 (56:6264)

[DS] Duren, P., Schuster, A., Bergman spaces. Mathematical Surveys and Monographs, 100. American Mathematical Society, Providence, RI, 2004. MR2033762

[He] Hedenmalm, H., A factorization theorem for square area-integrable analytic functions. J. Reine Angew. Math. 422 (1991), 45–68. MR1133317 (93c:30053)

[HKZ] Hedenmalm, H., Korenblum, B., Zhu, K., Theory of Bergman spaces. Graduate Texts in Mathematics, 199. Springer-Verlag, New York, 2000. MR1758653 (2001c:46043)

[Ho] Horowitz, C., Zeros of functions in the Bergman spaces. Duke Math. J. 41 (1974), 693–710. MR0357747 (50:10215)

[K1] Korenblum, B., An extension of the Nevanlinna theory. Acta Math. 135 (1975), no. 3-4, 187–219. MR0425124 (54:13081)

[K2] ——, A Beurling-type theorem. Acta Math. 138 (1976), no. 3-4, 265–293. MR0447584 (56:5894)

[R] Rochberg, R., Interpolation by functions in Bergman spaces. Michigan Math. J. 29 (1982), no. 2, 229–236. MR0654483 (83e:30039)

[S1] Seip, K., Beurling type density theorems in the unit disk. Invent. Math. 113 (1993), no. 1, 21–39. MR1223222 (94g:30033)

[S2] Seip, K., Interpolation and sampling in spaces of analytic functions. University Lecture Series, 33. American Mathematical Society, Providence, RI, 2004. MR2040080

[SS] Shapiro, H. S., Shields, A. L., On some interpolation problems for analytic functions. Amer. J. Math. 83 (1961), 513–532. MR0133446 (24:A3280)

[St] Stein, E., Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals. With the assistance of Timothy S. Murphy. Princeton Mathematical Series, 43. Monographs in Harmonic Analysis, III. Princeton University Press, Princeton, NJ, 1993. MR1232192 (95c:42002)

Richard Rochberg
Washington University in St. Louis
E-mail address: rr@math.wustl.edu