BOOK REVIEWS

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The structure of spherical buildings, by Richard M. Weiss, Princeton University Press, Princeton, NJ, 2003, xiv+135 pp., $45.00, ISBN 0-691-11733-0

Buildings were introduced by Jacques Tits in order to provide a unified geometric framework for understanding semisimple complex Lie groups and, later, semisimple algebraic groups over an arbitrary field. "Geometric" here is to be understood in the sense of incidence geometry, as exemplified by the close relationship between projective geometry and the special linear group. Buildings are still an active area of research, and they have found many applications beyond those originally envisaged by Tits.

Tits's definition of a building evolved gradually during the 1950s and 1960s and reached a mature form in about 1965. Tits outlined the theory in a 1965 Bourbaki Seminar exposé [5] and gave a full account in [6]. At that time Tits thought of a building as a simplicial complex that can be expressed as the union of a family of subcomplexes called "apartments", subject to a few axioms that will be stated below. Each apartment is made up of "chambers", which are the top-dimensional simplices. This viewpoint might be called the "old-fashioned" approach to buildings.

In the more "modern" approach, introduced by Tits in a 1981 paper [7], one forgets about all simplices except the chambers, and one recasts the definition entirely in terms of "chamber systems". This definition also evolved over a period of years, and it reached a mature form in the late 1980s. An important catalyst was the theory of twin buildings, which was being developed by Ronan and Tits. The final version of the modern definition can be found in [8], where a building is viewed as a set $\mathcal{C}$ (the "chambers"), together with a "Weyl-group-valued distance function" subject to a few axioms.

For either approach to the subject one starts with Coxeter groups.

1. Coxeter groups and Coxeter complexes

A *Coxeter group* of rank n is a group generated by n elements of order 2, subject to relations that give the orders of the pairwise products of the generators. Thus

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The reviewer is grateful to Bill Casselman for drawing Figure 1. It first appeared in the *Notices Amer. Math. Soc.* **49** (2002), 1245.

The introduction to buildings in this review is adapted from the introductory chapter of the forthcoming second edition of [1], by permission of Springer.

the group of order 2 is a Coxeter group of rank 1, and the dihedral group D_{2m} of order $2m$ ($m \geq 2$) is a Coxeter group of rank 2, with presentation

$$D_{2m} = \langle s, t\,; s^2 = t^2 = (st)^m = 1 \rangle.$$

The infinite dihedral group

$$D_\infty = \langle s, t\,; s^2 = t^2 = 1 \rangle$$

is also a rank 2 Coxeter group; there is no relation for the product st because it has infinite order. Readers who have studied Lie theory have seen Weyl groups, which are the classical examples of (finite) Coxeter groups. For example, the symmetric group S_3 on 3 letters, which is the same as the dihedral group of order 6, is the Weyl group of type A_2. And the symmetric group S_4 on 4 letters is the Weyl group of type A_3, with presentation

$$S_4 = \langle s, t, u\,; s^2 = t^2 = u^2 = (st)^3 = (tu)^3 = (su)^2 = 1 \rangle.$$

Some infinite Coxeter groups also arise in Lie theory as "affine Weyl groups". For example, D_∞ is the affine Weyl group of type $\tilde{\mathrm{A}}_1$, and the Coxeter group W with presentation

$$W = \langle s, t, u\,; s^2 = t^2 = u^2 = (st)^3 = (tu)^3 = (su)^3 = 1 \rangle \tag{1}$$

is the affine Weyl group of type $\tilde{\mathrm{A}}_2$.

The given set S of generators of order 2 should be viewed as part of the structure, but one often suppresses it for simplicity. When we need to be precise, we will talk about the *Coxeter system* (W, S) rather than the Coxeter group W. The system (W, S) is said to be *reducible* if S admits a partition $S = S' \amalg S''$ such that all elements of S' commute with all elements of S''. In this case W splits as a direct product $W' \times W''$ of two Coxeter groups of lower rank.

Every finite Coxeter group can be realized in a canonical way as a group of orthogonal transformations of Euclidean space, with the generators of order 2 acting as reflections with respect to hyperplanes. Thus D_{2m} acts on the plane, with s and t acting as reflections through lines that meet at an angle of π/m. And S_4 admits a reflection representation on 3-dimensional space, obtained by starting with the obvious action of S_4 on $\mathbb{R}^4$ and restricting to the subspace $x_1 + x_2 + x_3 + x_4 = 0$. More geometrically, we get this action by viewing S_4 as the group of symmetries of a regular tetrahedron.

Given a finite Coxeter group W and its reflection representation on Euclidean space, consider the set of hyperplanes whose reflections belong to W. If we cut the unit sphere by these hyperplanes, we get a cell decomposition of the sphere. The cells turn out to be spherical simplices, and we obtain a simplicial complex $\Sigma = \Sigma(W)$ (or $\Sigma(W, S)$) triangulating the sphere. This is called the *Coxeter complex* associated with W.

For D_{2m} acting on the plane, Σ is a circle decomposed into $2m$ arcs. For the action of S_4 on $\mathbb{R}^3$ mentioned above, Σ is the triangulated 2-sphere shown in Figure 1. There are 6 reflecting hyperplanes, which cut the sphere into 24 triangular regions. Combinatorially, Σ is the barycentric subdivision of the boundary of a tetrahedron, as indicated in the picture. (One face of an inscribed tetrahedron is visible.) The vertex labels in the picture will be explained later.

A similar but more complicated construction yields a Coxeter complex associated with an arbitrary Coxeter group W. For example, $\Sigma(D_\infty)$ is a triangulated line, with the generators s and t acting as affine reflections with respect to the endpoints

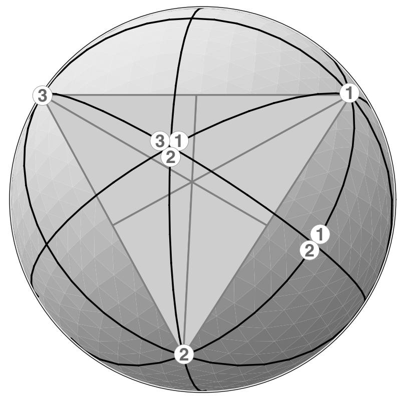

FIGURE 1. The Coxeter complex of type A_3.

of an edge. And the Coxeter complex for the group W defined in (1) is the Euclidean plane, tiled by equilateral triangles. (The generators s, t, u act as reflections with respect to the sides of one such triangle.)

We are now in a position to describe the old-fashioned approach to buildings; Coxeter complexes form the apartments.

2. BUILDINGS: THE OLD-FASHIONED APPROACH

We begin with the canonical example of a building: Let k be a field and let $\Delta = \Delta(k^n)$ be the abstract simplicial complex whose vertices are the nonzero proper subspaces of the vector space k^n and whose simplices are the chains

$$V_1 < V_2 < \cdots < V_r$$

of such subspaces. Every simplex σ is contained in a subcomplex, called an apartment, which is isomorphic to the Coxeter complex associated with the symmetric group on n letters. To find such an apartment, choose a basis $e_1, e_2, \ldots, e_n$ of k^n such that every subspace V_i that occurs in σ is spanned by some subset of the basis vectors. We then get an apartment containing σ by taking *all* simplices whose vertices are spanned by subsets of the basis vectors.

Figure 1 shows an apartment for the case $n = 4$. The labels on the vertices indicate which basis vectors span the corresponding subspace. Thus the vertex labeled 2 is the line spanned by e_2, the vertex labeled by both 1 and 2 is the plane spanned by e_1 and e_2, and the vertex labeled by 1, 2, and 3 is the 3-dimensional space spanned by e_1, e_2, e_3. These three subspaces form a chain, so they span a 2-simplex in Δ.

For a second example of a building, take any simplicial tree with no endpoints (i.e., every vertex is incident to at least two edges). Any copy of the real line in the tree is an apartment, isomorphic to the Coxeter complex associated with the infinite dihedral group.

As these examples suggest, a building is a simplicial complex that is the union of "apartments", each of which is a Coxeter complex. There are axioms that specify how the apartments are glued together. They are easy to write down, as we will do below, but not so easy to grasp intuitively until one works with them for a while.

The following definition is a slight variant of the one given by Tits [5] in 1965: A *building* is a simplicial complex that can be expressed as the union of subcomplexes Σ, called *apartments*, satisfying the following axioms:

(B0) Each apartment is a Coxeter complex.
(B1) Any two simplices are contained in an apartment.
(B2) Given two simplices σ, τ and two apartments Σ, Σ' containing both of them, there is an isomorphism $\Sigma \to \Sigma'$ fixing σ and τ pointwise.

The simplices of top dimension are called *chambers*, while those of codimension 1 are called *panels*. Note that a Coxeter complex is itself a building, with a single apartment. In fact, Coxeter complexes are precisely the *thin* buildings. (This means that every panel is a face of exactly two chambers.) The more interesting buildings are the *thick* buildings, i.e., those in which every panel is a face of at least three chambers. The building $\Delta(k^n)$ described above is thick, and a tree is a thick building if and only if every vertex is incident to at least three edges.

There is a well-defined Coxeter system (W, S) associated with a building Δ, such that the apartments are all isomorphic to $\Sigma(W, S)$. One says that Δ is a building of type (W, S), and one calls W the *Weyl group* of Δ. Much of the terminology (rank, reducibility,...) from the theory of Coxeter groups is carried over to buildings. In addition, one says that a building is *spherical* if W is finite (in which case the apartments are triangulated spheres). Thus $\Delta(k^n)$ is an irreducible spherical building of rank $n-1$, while a tree with no endpoints is an irreducible non-spherical building of rank 2.

3. Buildings: The modern approach

Let Δ be a building of type (W, S) as above, and let $\mathcal{C} = \mathcal{C}(\Delta)$ be its set of chambers. It turns out that there is a natural way to define a W-valued "distance function"

$$\delta \colon \mathcal{C} \times \mathcal{C} \to W$$

that describes the relative position of any two chambers. Intuitively, $\delta(C, D)$ for $C, D \in \mathcal{C}$ is something like a vector pointing from C to D. Giving the definition of δ would take us too far afield; all we can say briefly is that $\delta(C, D)$ contains information about the totality of minimal galleries from C to D. Here a *gallery* is a finite sequence of chambers such that any two consecutive ones have a common panel, and it is *minimal*, or a *geodesic*, if there is no shorter gallery with the same first and last chambers. It turns out that one can completely reconstruct the building Δ from the data consisting of the Coxeter system (W, S), the set $\mathcal{C}$ of chambers, and the function δ. Moreover, one can write down simple axioms that these data must satisfy in order that they come from a building.

For the modern approach to buildings, then, one simply starts with a Coxeter system (W, S), a set $\mathcal{C}$, and a function $\delta \colon \mathcal{C} \times \mathcal{C} \to W$ satisfying some axioms. We will not state the axioms here, but their most striking feature is that they contain nothing resembling the existence of apartments. Indeed, in proving that the modern definition is equivalent to the old-fashioned one, the key step is to use the axioms to prove the existence of apartments.

The modern approach is both more abstract and more elementary than the old-fashioned approach. It is more abstract because the geometric intuition is gone. Thus one no longer visualizes chambers as regions cut out by hyperplanes, and one no longer visualizes apartments as simplicial complexes associated with reflection groups. But it is more elementary because the underlying mathematical object can be boiled down to nothing more than a graph with colored edges. (The vertices of the graph are the elements of $\mathcal{C}$, and two such vertices C, D are connected by an edge with "color" $s \in S$ if and only if $\delta(C, D) = s$.)

4. The classification theorem

One of Tits's greatest achievements is the classification of thick, irreducible, spherical buildings of rank at least 3, proved in his Springer Lecture Notes [6]. Roughly speaking, the result is that such buildings correspond to simple algebraic groups (of relative rank at least 3) defined over an arbitrary field. The rank restriction cannot be avoided. The buildings of rank 2, for example, include those of type A_2, which are essentially the same as projective planes. And the problem of classifying projective planes, even the finite ones, seems to be hopeless.

If, however, one imposes a certain symmetry condition on the buildings (called the *Moufang property*), then the classification extends to rank 2. This result is due to Tits and Weiss and can be found in [9], which also includes a simplified proof of the original classification theorem. The simplification is possible because the case of rank ≥ 3 can be treated much more systematically once the rank 2 Moufang buildings have been classified.

At the center of the proof of the classification theorem is Tits's extension theorem for "local isomorphisms" of buildings ([6], Theorem 4.1.2). The extension theorem implies, incidentally, that all thick, irreducible, spherical buildings of rank ≥ 3 have the Moufang property; this explains why Tits did not need to add the latter as a hypothesis in his classification theorem. Tits's proof of the extension theorem is quite intricate. In spite of the fact that there have been three textbooks on buildings [1, 2, 3] and one textbook-length survey [4] prior to the book under review, none of these tries to prove the extension theorem or even to sketch a proof.

5. Weiss's book

Weiss's book remedies this deficiency in the literature. There is good reason to want to do this because Tits's proof of the extension theorem has been simplified, primarily by Mark Ronan. The simplified proof, which retains the basic shape of Tits's orginal proof, makes systematic use of the W-valued function δ, which was not available when Tits wrote [6]. Weiss also includes an account of the Moufang property and some of its most important consequences. And the final chapter contains an overview of the simplified proof of the classification theorem ([9], Chapter 40) mentioned above.

The book is clearly and carefully written and is completely self-contained except for the references to [9] in the final chapter. A reader can start with no prior knowledge of Coxeter groups or buildings and end up with all the basic facts about spherical buildings needed for the proof of the classification theorem. It is remarkable that the author is able to accomplish his goal in a mere 130 pages. Part of what makes this possible is his deliberate decision to treat buildings entirely from the modern viewpoint, with no excursions to mention alternate definitions or different ways of proving things. For example, the fact that a finite Coxeter group can be realized as a reflection group is mentioned only briefly on p. 14 and is never referred to again. And this discussion is the only place in the book where there is any hint that a chamber can be thought of as anything other than a vertex of a graph.

Weiss's book is an extremely valuable addition to the literature. Most readers will want to supplement it by learning something about the old-fashioned viewpoint from another source. The old-fashioned and modern approaches complement each other nicely, and both are useful for a full appreciation of the theory and for many of the applications.

References

[1] K. S. Brown, *Buildings*, Springer-Verlag, New York, 1989. MR969123 (90e:20001)
[2] P. Garrett, *Buildings and classical groups*, Chapman & Hall, London, 1977. MR1449872 (98k:20081)
[3] M. Ronan, *Lectures on buildings*, Perspectives in Mathematics, vol. 7, Academic Press Inc., Boston, MA, 1989. MR1005533 (90j:20001)
[4] R. Scharlau, *Buildings*, Handbook of Incidence Geometry (F. Buekenhout, ed.), North-Holland, Amsterdam, 1995, pp. 477–645. MR1360726 (97h:51018)
[5] J. Tits, *Structures et groupes de Weyl*, Séminaire Bourbaki: Volume 1964/1965, New York-Amsterdam, 1966. Exp. no. 288. MR1608796
[6] J. Tits, *Buildings of spherical type and finite BN-pairs*, Lecture Notes in Mathematics, vol. 386, Springer-Verlag, Berlin, 1974. MR0470099 (57:9866)
[7] J. Tits, *A local approach to buildings*, The Geometric Vein (C. Davis, B. Grünbaum, and F. A. Sherk, eds.), Springer-Verlag, New York, 1981, pp. 519–547. MR661801 (83k:51014)
[8] J. Tits, *Twin buildings and groups of Kac-Moody type*, Groups, Combinatorics & Geometry (M. Liebeck and J. Saxl, eds.), London Mathematical Society Lecture Note Series, vol. 165, Cambridge University Press, Cambridge, 1992, pp. 249–286. MR1200265 (94d:20030)
[9] J. Tits and R. M. Weiss, *Moufang polygons*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2002. MR1938841 (2003m:51008)

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