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Exterior differential systems and Euler–Lagrange partial differential equations, by Robert Bryant, Phillip Griffiths, and Daniel Grossman, University of Chicago Press, 2003, 216 pp., $45.00 (cloth), ISBN 0-226-07793-4; $17.00 (paper), ISBN 0-226-07794-2

The book under review is concerned with certain geometric aspects of problems arising in the calculus of variations. It can be viewed as a sequel to the second author's earlier work, [7], which dealt with variational problems for curves, that is (in local coordinates) functions of a single independent variable. Here the focus shifts to first order variational principles for hypersurfaces, i.e. a single function of several variables and, in the final chapter, more general submanifolds, i.e. vector-valued functions. The geometric approach analyzes the behavior of variational problems (and differential equations) under changes of variables, including the equivalence and canonical form problems, that is, when can one variational problem be mapped to another (simpler) one by a change of variables. Symmetries of variational problems arise as self-equivalences and are connected with conservation laws via the general Noether correspondence. A wide variety of applications arises not just in geometry, but also in theoretical physics, engineering, image processing, and so on. Geometrical constructions also have a direct impact on the analysis of partial differential equations, variational problems, and, increasingly, numerical solution methods, [8].

To understand the book's scope, let's begin by defining the two terms in the title. An exterior differential system refers to a system of differential equations defined by a collection of differential forms. As an elementary example, the differential form equation

$$du \wedge dy - dv \wedge dx = 0 \tag{$*$}$$

requires the planar vector field $\mathbf{u}(x, y) = (u(x, y), v(x, y))$ to satisfy the incompressibility equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.$$

Many important physical systems, such as Monge-Ampère, Maxwell, and Yang–Mills, are naturally reformulated as exterior differential systems. To be more precise, an *exterior differential system* on a (smooth) manifold M is determined by a collection (either finite or infinite) of differential forms $E = \{\omega_\kappa\}$. Solutions are characterized as *integral submanifolds* $N \subset M$, of a prescribed dimension, that are required to annihilate all the differential forms in the system E. Thus, each $\omega_\kappa\,|_N = 0$, indicating the vanishing of the pull-back of the differential form to the submanifold N. In local coordinates, writing $N = \{\mathbf{u} = \mathbf{f}(\mathbf{x})\}$ as the graph of a function, one reduces the exterior differential system to a system of differential equations for the components of $\mathbf{u}$. Often, the integral submanifolds are also required to satisfy an independence condition, meaning that one or more differential forms remain non-zero on the integral submanifold. For instance, in formulating the incompressibility condition in differential form language $(*)$, one should require

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that $dx \wedge dy\,|_N \neq 0$ to ensure that the two-dimensional integral submanifold N is, at least locally, the graph of a single-valued smooth vector field.

Reformulating systems of differential equations as exterior differential systems offers significant advantages due to the natural behavior of differential forms under smooth maps, e.g., changes of variables. Moreover, integrability conditions for overdetermined systems are encoded by the vanishing of the square of the exterior derivative: $d^2 = 0$. For instance, the equation $\nabla \times \mathbf{u} = \mathbf{v}$ for a vector potential in $\mathbb{R}^3$ can be recast as an exterior differential system $\Omega = d\omega - \varphi = 0$ governed by a single two-form Ω, and the integrability condition $\nabla \cdot \mathbf{v} = 0$ stating incompressibility of the vector field becomes merely $d\Omega = 0$.

The modern theory of exterior differential systems was inaugurated almost exactly a century ago by Élie Cartan as the key tool for his penetrating studies of Lie pseudo-groups, [5]. Cartan's remarkable insight into Poincaré's newly established calculus of differential forms led to the fundamental existence theorem for analytic exterior differential systems, now known as the Cartan–Kähler Theorem, which is a far-reaching generalization of the classical Cauchy–Kovalevskaya Theorem, as well as the complete solution to the basic equivalence problem between exterior differential systems. It is worth noting that the first two authors of the book collaborated on the definitive modern text on exterior differential systems, [3]. As modern heirs to Cartan's vision, we all are, in many respects, still coming to grips with these phenomenal advances and their impact and potential in geometry, differential equations, the calculus of variations, and, increasingly, real-world applications such as computer vision, [15].

The second term in the book's title refers to the basic differential equations governing the (classical) solutions to variational problems. All sufficiently regular local minimizers (and maximizers) of a variational principle

$$\mathcal{I}[\mathbf{u}] = \int L(\mathbf{x}, \mathbf{u}^{(n)})\, dx^1 \wedge \cdots \wedge dx^p \tag{$**$}$$

must be solutions of the associated Euler–Lagrange equations. (But see [2] for simple examples of variational problems whose minimizers do not satisfy the Euler–Lagrange equations.) In practice, one constructs the Euler–Lagrange equation by taking the first variation $\delta\mathcal{I}[\mathbf{u}] = \mathcal{I}[\mathbf{u} + \delta\mathbf{u}]$ and then integrating by parts (omitting the boundary contributions, whose vanishing must be ensured by the boundary conditions) to rewrite the variation

$$\delta\mathcal{I}[\mathbf{u}] = \int \left(\mathbf{E}(L) \cdot \delta\mathbf{u}\right)\, dx^1 \wedge \cdots \wedge dx^p$$

as a pairing between the variation $\delta\mathbf{u}$ and the Euler–Lagrange expression $\mathbf{E}(L)$. Since $\delta\mathcal{I}[\mathbf{u}]$ must vanish at critical points $\mathbf{u}$, they must satisfy the Euler–Lagrange equation $\mathbf{E}(L) = 0$. Of course, being a solution to the Euler–Lagrange equation is only a necessary condition for a minimizer; sufficient conditions are based on the sign of the second variation $\delta^2\mathcal{I}[\mathbf{u}]$. In multivariable calculus, not every vector field is a gradient; for similar reasons, not every system of differential equations is a system of Euler–Lagrange equations, and the inverse problem of the calculus of variations, i.e. deciding when a system of differential equations is equivalent to an Euler–Lagrange system, while solved under certain restrictions, remains a challenging problem in general.

The most natural means of connecting the calculus of variations and differential forms rests on the modern *variational bicomplex*, whose roots can be found in the work of Hilbert and Cartan on invariant integrals, of Helmholtz and Jesse Douglas on the inverse problem, followed by the Belgian mathematicians Lepage, DeDonder and Dedecker, and culminating in the general constructions of Vinogradov, Tsujishita, Ian Anderson, and their collaborators in the 1970's and early 1980's. The bicomplex is constructed on the infinite jet bundle J^∞, which is defined as the space of equivalence classes of submanifolds $N \subset M$ of a prescribed dimension under the equivalence relation of infinite order contact — or, in more prosaic local coordinate language, as the space of Taylor series of functions. Specification of local coordinates on M as independent and dependent variables $(\mathbf{x}, \mathbf{u})$ naturally splits the differential one-forms on J^∞ into horizontal forms, $dx^1, \ldots, dx^p$, and contact forms, which are characterized by the fact that they vanish on all jets of submanifolds: $\theta\,|_{j_\infty N} = 0$. As a result, the space of differential forms $\Omega = \bigoplus_{i,j} \Omega^{i,j}$ on J^∞ decomposes, where each summand $\Omega^{i,j}$ is spanned by wedge products of i horizontal and j contact forms. (In a coordinate-free approach, [18], one works with filtrations rather than direct sums.) The exterior derivative also splits, $d = d_H + d_V$, into horizontal and vertical (contact) components, so $d_H \colon \Omega^{i,j} \to \Omega^{i+1,j}$, while $d_V \colon \Omega^{i,j} \to \Omega^{i,j+1}$. Thus, $d_H^2 = 0 = d_V^2$, while $d_H d_V + d_V d_H = 0$, thereby defining the *variational bicomplex*. Like the usual deRham complex, both the vertical differential d_V and the horizontal differential d_H are locally exact, meaning the kernel of its action on $\Omega^{i,j}$ is the image of the preceding differential. (For the vertical differential, this follows from the standard Poincaré lemma, but local exactness of d_H is deeper.) However, *unlike the deRham complex*, the vertical complexes never terminate, while in the horizontal direction there is no final map since not every differential form in $\Omega^{p,i}$ lies in the image of d_H. Indeed, if this were not true, every variational integral $(**)$ would only depend on the boundary values of the function $\mathbf{u}$, and so there would be no calculus of variations!

The reason for the term "variational bicomplex" is that geometric constructions in the calculus of variations have natural interpretations as objects in the bicomplex. Thus, the integrand or *Lagrangian* of a variational problem $(**)$ is a differential p-form

$$\lambda = L(\mathbf{x}, \mathbf{u}^{(n)})\, dx^1 \wedge \cdots \wedge dx^p \in \Omega^{p,0}.$$

Its vertical differential $d_V\lambda \in \Omega^{p,1}$ corresponds to the first variation, while integration by parts corresponds to quotienting by the image of the horizontal differential $d_H \colon \Omega^{p-1,1} \to \Omega^{p,1}$. The elements of the quotient space are known as *source forms* and can be identified with systems of differential equations. In particular, the image of $d_V\lambda$ is the Euler–Lagrange source form for the given Lagrangian. Local exactness of the vertical subcomplex (after quotienting) gives a solution to the direct inverse problem: a source form defines a system of Euler–Lagrange equations, so $\mu = E(\lambda)$ for some Lagrangian form $\lambda \in \Omega^{p,0}$ if and only if $d_V\mu = 0$ as an element of the quotient $\Omega^{p,2}/d_H\Omega^{p-1,2}$ — these are the classical Helmholtz conditions or, equivalently, the condition of formal self-adjointness of the Fréchet derivative of the system of differential equations, [13]. Furthermore, classical conservation laws are naturally interpreted as differential forms $\xi \in \Omega^{p-1,0}$ with the property that the horizontal differential $d_H\xi \in \Omega^{p,0}$ vanishes on solutions to the Euler–Lagrange equations. Noether's theorem, relating symmetries of the variational problem to conservation laws, reduces to a simple exterior differential algebra identity in the bicomplex framework. The global cohomology of the bicomplex plays a crucial role in the geometric study of differential equations and the calculus of variations: conservation laws, characteristic classes, etc., appear as cohomology representatives and can thus be analyzed by powerful methods from homological algebra, particularly spectral sequences. Further information and a wide range of interesting applications in mathematics and physics can be found in [1, 17, 18].

However, the book under review takes a rather different tack when connecting the calculus of variations with differential forms. Unlike the universally applicable and natural bicomplex constructions, their reformulation of variational problems as exterior differential systems tends to be ad hoc and not as well motivated. Moreover, it is not immediately apparent how to extend their approach to the more challenging situations of higher order variational problems. Still, the authors provide very valuable insight into how the amazing tools Cartan bequeathed us can be successfully applied to a wide variety of deep and interesting areas of differential geometry and analysis. The theory is illustrated by a number of examples arising in geometry and the applications of partial differential equations, including the wave and Monge-Ampère equations, minimal surfaces, and harmonic maps. A significant fraction of the text is devoted to the study of conformally invariant systems and includes results on the classification and applications of symmetries and conservation laws. Additional topics of interest include intrinsic characterization of the second variation, the Poincaré–Cartan form, and the connections between higher order symmetries and conservation laws, integrability, and Bäcklund transformations.

On the whole, the book is well written, although it tends to cover the basics too rapidly for the non-expert. It is not an introductory text: To fully appreciate the development, the reader needs to be reasonably familiar with exterior algebra, differential forms, basics of exterior differential systems, as well as the Cartan equivalence method. (The latter is reviewed, but far too quickly, within the context of Monge–Ampère systems.) Moreover, despite the sophisticated mathematical machinery, the overall aims are fairly modest, mostly concentrating on the "classical" situation of first order variational problems which, until the final chapter, only involve a single function with scalar Euler–Lagrange equation, thus avoiding significant and as yet unresolved issues in the higher order cases. The introduction includes a nice, albeit abbreviated, survey of some historical highlights, mentioning some of the contemporary contributors, but without citations. Indeed, a significant weakness is the lack of references to the literature, both classical and modern, coupled with a single-minded concentration on the authors' own approach to the subject that avoids any serious comparisons with alternative viewpoints such as those provided by the variational bicomplex. The reader will find it a challenge to place their results within a broader research context or to track down relevant articles or books to help supplement the sometimes cryptic presentations of basic ideas and techniques.

Let me conclude by briefly surveying the individual chapters:

Chapter 1 introduces the basic themes of the text: contact geometry, the equivalence and inverse problems for first order Lagrangians in one dependent variable, the Poincaré–Cartan form, which first arose as the integrand in Hilbert's invariant integral and plays a key role in the classical sufficiency conditions for minimizers, as well as the most basic version of Noether's Theorem that relates geometric symmetries of variational problems and conservation laws of their Euler–Lagrange equations. Simple applications to Euclidean-invariant variational problems occupy the final section.

Chapter 2 covers the geometry of the Poincaré–Cartan forms in the context of first order variational problems in a single dependent variable. Section 2.1 applies the Cartan equivalence method to hyperbolic Monge–Ampère systems, a key outcome being a solution to the inverse problem, establishing conditions that the system arise from a variational principle. The method is also used to establish conditions for such systems to be equivalent to the scalar wave equation, $u_{xy} = 0$, reproducing a classical result known to Darboux; more recent, deeper work on the fascinating topic of Darboux integrable equations, [9], is not mentioned.

Chapter 3 applies the techniques developed in the preceding chapters to study conformally invariant variational problems. One of the main applications is to determine the symmetries and conservation laws of the conformally invariant nonlinear Laplace equation $\Delta u = u^{(n+2)/(n-2)}$; however, the sophisticated machinery is not necessary as this result is readily established using Lie symmetry group methods and the classical Noether theorem, [13]. A particular case of an analytical uniqueness theorem due to Pohozaev is quoted, again without providing a wider context or how such results are connected with transformation groups via the Noether identity; a good source is the recent book by Reichel, [16]. Incidentally, on p. 100, the last name of the French mathematician Émile Cotton is misspelled.

Chapter 4 briefly surveys three additional topics. The first section develops the second variation from a geometric viewpoint. I was surprised that there is no mention of basic analytical constructions, such as strong ellipticity and the Legendre–Hadamard positivity condition. The second section discusses the generalized Poincaré–Cartan form proposed by Betounes for first order variational problems involving more than one dependent variable, but does not mention competing versions due to Weyl, [19], and Carathéodory, [4]. The choice of Poincaré–Cartan form has significant repercussions in the still not completely satisfactory establishment of a field theory for multiple integrals, and some discussion of the strengths and weaknesses of the different approaches would have been useful. For higher order variational problems, matters are even less well understood, and there is considerable controversy over whether an appropriate Poincaré–Cartan form even exists. Two excellent surveys are by Kastrup, [10], and Gotay, [6], and further results in this direction, based on the Cartan equivalence method, can be found in the reviewer's paper [14].

The final section deals with higher order symmetries and conservation laws. The authors fall into the unfortunately common trap of calling this a "generalization of Noether's Theorem" (and, even worse, leaving it as a conjecture), when, in point of fact, the full version already appears in Noether's original paper, [12]. Kosmann-Schwarzbach, [11], has recently published a wonderful overview of the convoluted and at times embarrassing history of this famous result. The discovery of the soliton has revealed the deep connections between higher order symmetries/conservation laws and the integrability of the underlying partial differential equations, and the book concludes with a discussion of integrability and Bäcklund transformations in the context of pseudo-spherical (constant negative curvature) surfaces.

In summary, despite its narrow focus and lack of references, the book is a welcome addition to the current literature in this active and applicable area of mathematical research, providing a summary of the authors' personal approach, but falling short of a comprehensive guide to the field.

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Peter J. Olver
University of Minnesota