

Meromorphic functions and linear algebra, by Olavi Nevanlinna, Fields Institute
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Usually a book review will concentrate on the mathematics in the text, but part of the appeal of this work is how the author weaves his own personal odyssey into it. Olavi Nevanlinna is a member of the family that produced the (Nevanlinna) theory of meromorphic functions in the 1920s. As a young man, he knew the brothers Rolf (1895-1980), the founder of this theory, and Frithiof Nevanlinna (1894-1977), the author's grandfather, but this was long after their active participation in meromorphic function theory. In the 1980s he "dreamed" that it would be "romantically pleasant to allude to [this theory] in some [of my own] work," on the principle that "[w]hat is beautiful is useful, functional" (he contrasts this unity principle with the famous Alvar Aalto chair [p. 6]). By the mid '90s, conscious of the coming centenary of Rolf Nevanlinna, he obtained a result in the classical theory (concerning the approximate degree of an entire or meromorphic function), thereby opening another area of mathematics to the "Nevanlinna formalism". This book confirms the success of this linkage. Among more established applications of the Nevanlinna theory to other parts of mathematics we note as a fragmentary list those to several complex variables [4], complex geometry [7], algebraic geometry [8], potential theory [6], quasiregular mappings [10] and number theory [11]. The standard references for classical Nevanlinna theory remain [3], [5], [9].

Since this book highlights research of the past decade, the number of relevant references is modest, and the author is able to present a leisurely, very charming, largely self-contained development that will be absorbed comfortably by most interested readers: mathematicians specializing in numerical linear algebra or numerical analysis, as well as those interested in functional analysis or classical complex analysis.

The influence of the name Nevanlinna in Finland has no counterpart in the world of American mathematics: not only is it attached to a high point in mathematical research, but the family has been a significant force in the country's scientific history for well over a century (for pre-1918, see the monograph [1]). The focus of [1] concludes as the Nevanlinna brothers began their research career. In later years, Frithiof left his academic career for one in the insurance industry (where there have been long-standing connections with the Scandinavian academic world), but his role during the early development of the theory has had significant impact even in recent times [2].

Rolf Nevanlinna's research career began as the theory of entire functions of one complex variable was near its zenith, led by figures such as Hadamard, Borel, and Valiron. From the viewpoint of our review, the most important result was Picard's theorem: an entire function which omits two finite complex values must be constant. In searching for 'elementary' proofs of the Picard theorem, important use had been

made of the increasing real function

$$(1) \quad M(r) = \max_{|z| \leq r} |f(z)|,$$

clearly inappropriate for studying general meromorphic functions. Nevanlinna's achievement was to find a replacement for (1): if f is (nonconstant) meromorphic in $\{|z| < R\}$ (with $f(0) \neq \infty$) let $n(t, \infty)$ be the number of poles in $\{|z| < t\}$ and for $0 < r < R$ set $N(r, f) = \int^r n(t, \infty) t^{-1} dt$. Nevanlinna then defines his *characteristic* of f as

$$(2) \quad T(r, f) = N(r, \infty) + \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta \quad (0 < r < R).$$

If f is rational of degree d , then $T(r, f) = d \log r + O(1)$, and $T(r) \uparrow \infty$ as $r \rightarrow \infty$. It is difficult for an outsider to appreciate how significant was Nevanlinna's organizing the expression in (2); indeed his so-called first fundamental theorem, that

$$(3) \quad T(r, f) - T(r, 1/(f - a)) = O(1) \quad (a \in \mathbb{C}),$$

becomes little more than a reinterpretation of the well-established Jensen theorem. (The second fundamental theorem is far deeper, but other than in the final chapter, this book uses primarily the mathematics of the level needed to give (3).) Even in the case of entire functions (where $N(r, \infty) \equiv 0$), (2) is more fundamental than (1).

Olavi Nevanlinna is a classical applied mathematician, and his perspective has as starting point the elementary fact that if A is an operator on a Hilbert space, then the operator-valued function

$$(4) \quad \lambda \implies (\lambda I - A)^{-1}$$

(the resolvent) is analytic outside the closed set $\sigma(A)$, the *spectrum* of A . This is one of the many applications of versions of the Cauchy theorem and the Cauchy integral formula applied to the function $f(A)$, where f is analytic. The principle motivating this book thus becomes: if one Nevanlinna might study meromorphic non-entire functions with such profit, then expressions such as (4) could be considered by another Nevanlinna as meromorphic operator-valued functions, and so allow λ to penetrate the spectrum. This philosophy leads to another application of the Nevanlinna formalism, one which also makes many interesting intersections with earlier work in operator theory. The exposition is illuminated with many clear examples, in which A is either a $d \times d$ matrix ($A \in \mathbb{M}_d$) or an operator on a Hilbert or Banach space.

The book has ten (usually short) chapters, laced throughout with personal observations and comments. There is also a detailed prologue which unveils the orientation of the text. After two introductory chapters (the second is a self-contained sketch of classical Nevanlinna theory) we are introduced to analytic vector-valued functions. Remarkable is that the functions $\|f\|$ and $\log \|f\|$ are subharmonic when f is a Banach-space-valued analytic function. This gives the maximum principle and applications to the behavior of the spectral radius.

In place of (2) the author introduces two characteristics, $T_\infty(r, f)$ and $T_1(r, f)$, where now f is a *meromorphic mapping* of $\{|z| < R\}$ to a Banach space X . The definition of a meromorphic map is clear when z is away from poles (power series!), and z_0 is a *pole* of f if near z_0 there is a positive integer m so that $(z - z_0)^m f(z)$ is analytic at z_0 . If f is a meromorphic mapping, $T_\infty(r, f)$ may be defined in a

manner imitative of (2), and many classical elementary properties transfer at once to this context. For example, in order to guarantee that

$$T_\infty(r, fg) \leq T_\infty(r, f) + T_\infty(r, g) + O(1),$$

X need only be assumed to be a Banach algebra. However, the analogue of the basic estimate (3) is far more subtle. Simple examples (p. 43) show that $1/f$ need not be meromorphic in the plane when f is, and that even when it is, nothing as simple as (3) can hold: even if the range of f is \mathbb{M}_d , the only general relation is that $T_\infty(r, f^{-1}) = O(dT_\infty(r, f))$.

This leads to two chapters which introduce T_1 , a less intuitive expression, but one whose properties mimic more faithfully those of [9]. This requires about twenty pages (through p. 68), but is developed in small, well-motivated steps so that the reader can vicariously experience the author's thought processes. Two key steps are required, and these are carefully developed in the test case that f is an \mathbb{M}_d -function. The first principle is that if $A \in \mathbb{M}_d$, then the entire function

$$z \implies I - zA$$

should be viewed as a polynomial of degree d rather than of degree one, and this is realized by a nice application of the classical factorization

$$A = Q\Lambda Q^*,$$

where Λ is upper triangular and Q is unitary.

The other ingredient is that all *singular values* of a matrix A should be used, where $\sigma_j \geq 0$ is a singular value of A if σ_j^2 is an eigenvalue of A^*A . The author defines the *total logarithmic size* of A by

$$s(A) = \sum_1^d \log^+ \sigma_j(A),$$

and the term corresponding to the integral in (2) becomes

$$m_1(r, f) = \frac{1}{2\pi} \int_0^{2\pi} s(f(re^{i\phi})) d\phi;$$

the analogue of $N(r, f)$ is more straightforward (in contrast, $\|A^{-1}\|$ will depend only on the smallest eigenvalue of A). It follows from earlier work of B. Aupetit that m_1 is subharmonic, and as a result $T_1 = m_1 + N_1$ preserves many properties of (2), especially (3). (The definition of s , and so of T_1 , can also be extended to certain operators.)

The remaining chapters develop these tools and introduce some applications. Much of this works out the Nevanlinna calculus in the operator-theoretic context, since at present the number of applications outside this internal framework is limited. The author introduces in Chapter 7 the idea of an operator A being almost algebraic: there is a fixed sequence $\{a_j\}$ so that if $p_j(\lambda) = \lambda^j + a_1\lambda^{j-1} + \dots + a_j$, then $\|p_j(A)\|^{1/j} \rightarrow 0$, and observes that this is equivalent to the function in (4) being meromorphic in the plane (the $\{p_j\}$ are called spectral polynomials). This yields as a consequence information on how the expression in (4) can be approximated by polynomials in A .

One interesting topic is in Chapter 8, where the author discusses conditions that ensure that $\{\|A^n\|\} = O(1), n \geq 1$: A is *power-bounded*, a condition considered crucial for stability in numerical analysis. These involve his characteristic T_∞ and

the classical Kreiss resolvent condition and are applications of his earlier work on approximate degree. The notion of approximate degree is somewhat different from the well-developed Wiman-Valiron theory, which is not mentioned here. The final chapter relates the classical notion of the (Nevanlinna) defect of the resolvent function (for $A \in \mathbb{M}_d$) to the eigenvalues of A ; the resulting defect relation is sharp only for matrices A nilpotent of degree two.

The presentation is clear and very readable, with only a few small errors in English. As with most books printed from a LaTeX file, there are some minor typographical slips. The book has no index nor index of notation, but the detailed table of contents is useful.

The author closes with a dedication to his father and some thoughts which indicate his serenity in seeing his work merge with his family's earlier contributions: "I grew up in the independent Finland. When [my father and I] go fishing we use a rowboat, and the lake remains quiet."

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