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Hodge theory and complex algebraic geometry I, II, by Claire Voisin, Cambridge University Press, New York, 2002, ix+322 pp., £65, ISBN 0-521-80260-1 (Vol. I); 2003, ix+351 pp., £65, ISBN 0-521-80283-0 (Vol. II)

1. Introduction

The task of reviewing Claire Voisin's two-volume work *Hodge Theory and Complex Algebraic Geometry* [V] is a daunting one, given the scope of the subject matter treated, namely, a rather complete tour of the subject from the beginning to the present, and given the break-neck pace of Voisin's clear, complete, but "take no prisoners" exposition. As is the case with most substantial mathematical treatises, digesting the content of these volumes can only occur by reconstructing and reorganizing their material for oneself, led forward, of course, by the clear beacon carried by the author, one of the foremost leaders in the field. Rather than talk descriptively about the mathematics, I have chosen to exemplify below a strand of my own adventure in understanding and reconstruction as I worked my way through the two volumes with a group of graduate students. I hope this will serve as an invitation to other readers to do likewise. Their efforts will be amply rewarded.

I beg my readers' indulgence for the longer-than-usual review that resulted. As Voisin does, I have tried to write in such a way that readers of varying levels of background or interest can enter or leave the exposition as suits their needs and still come away with a mathematical experience of some integrity and completeness. And, as Voisin does, we start at the beginning.

2. deRham cohomology

The most natural and fundamental objects in geometry are the differentiable manifolds, that is, topological spaces which are everywhere 'in microcosm' like Euclidean space in a consistent way as one moves from point to point. The differentiable structure on M is uniquely characterized by specifying which locally defined functions on M are the *differentiable* ones. The strongest form that a differential structure can take is to specify in a consistent way the functions which have a convergent Taylor series, the *real analytic* functions. Pioneering work of Whitney and Nash showed that every differentiable structure could be, in an appropriate sense, approximated by a real analytic one and that, again in an appropriate sense, real analytic structures had only trivial (infinitesimal) deformations.

Early on, the study of the differential forms on a manifold M, that is, of the indefinite integrals the manifold supports, became a fundamental tool for understanding its global geometry. Many topological properties of a differentiable manifold M are encoded in its algebra of differential forms. An easily defined degree-1 derivation d on forms, generalizing implicit differentiation

$$f \mapsto df$$

on functions, turned the algebra of differential forms into a complex, called the deRham complex after the theory's pioneer, Georges deRham. For a differential

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r-form ω and an r-dimensional polyhedron P on M, the several-variable generalization

$$\int_P d\omega = \int_{\partial P} \omega$$

of the Fundamental Theorem of Calculus (Stokes theorem) realized the homology of the complex of differential forms as 'cohomology', that is, as the dual to the homology of the complex of singular chains on M (or to homology of M as a CW-complex).

Work of Poincaré, E. Cartan, and deRham established a remarkable internal symmetry,

$$H^r(M) \leftrightarrow H^{m-r}(M), \tag{1}$$

(*Poincaré duality*) in the deRham cohomology groups of a compact and orientable manifold M of dimension m. Arguably the most prescient and productive vision of this symmetry was proposed by Hodge, who associated to each (Riemannian) metric μ on tangent vectors on M a signed involution

$$\omega \mapsto *\omega$$

on forms which 'lifted' the symmetry (1) to the level of differential forms. The 'adjoint'

$$d^* = \pm * \circ d \circ *$$

of d permitted Hodge to show that every cohomology class in $H^*(M)$ was represented by a unique form ω characterized by the equations

$$\begin{aligned} d\omega &= 0 \\ d^*\omega &= 0 \end{aligned}$$

or, equivalently, by the fact that it is the (closed) forms of minimal size in its deRham cohomology class. Such a form was called *harmonic* since it generalized the notion of harmonic function. These computations were facilitated by a fundamental property of any metric μ, the property that for appropriately chosen local coordinates around any given point, μ and its first derivatives are those of the standard Euclidean metric on Euclidean space.

(See S.-S. Chern's introduction to the English edition of [deR] for a very brief yet authorative history. In particular, Chern makes the central point that the passage to deRham cohomology permitted the 'localization' of the (co)boundary operator, which in turn led the way to the powerful notions of sheaf cohomology.)

3. Hodge theory

The subject matter of [V] lies within the realm of complex differential geometry, the case in which M is a complex manifold, that is, is everywhere 'in microcosm' like *complex* Euclidean space, in a consistent way as one moves from point to point. The complex structure on M is uniquely characterized by specifying which locally defined functions on M are the *holomorphic* ones. Since holomorphic functions are real analytic, an n-dimensional complex structure on M overlays a real analytic structure on the underlying real differential manifold M of real dimension $m = 2n$.

Analogously the introduction of a *hermitian* metric μ plays a central role in the theory. Our understanding of a compact complex manifold M is much enhanced if, for appropriately chosen local holomorphic coordinates $\{z_i\}$ around any point, the values of μ and its first derivatives at the point are those of the standard hermitian metric on complex Euclidean space. This is not automatic for complex manifolds. In fact, some complex manifolds admit no μ satisfying this property at each point, although complex projective space and all its complex submanifolds do. A metric with this property is called a *Kähler* metric, and a complex manifold which admits such a metric is called a *Kähler manifold*. The theory of Kähler manifolds, called *Hodge theory*, is driven by a second symmetry on $H^*(M) \otimes \mathbb{C}$ in the case in which M is a Kähler manifold. To describe it, complex-valued differential r-forms are said to be of type (p,q) if they can be written everywhere locally as sums of terms of the form

$$f dz_{i_1} \wedge \ldots \wedge dz_{i_p} \wedge d\overline{z_{j_1}} \wedge \ldots \wedge d\overline{z_{j_q}}$$

and every differential r-form has a unique decomposition into a direct sum of (p,q)-forms for $p+q=r$. Remarkably this decomposition passes to cohomology, in fact, to harmonic forms. The second symmetry on

$$H^r(M) \otimes \mathbb{C} = \sum\nolimits_{p+q=r} H^{p,q}(M)$$

is therefore induced by complex conjugation (at 90° to the first one induced above by the operator $*$).

In Parts I and II of Volume I, Voisin takes us through a lean, rapid, yet almost completely self-contained development of classical Hodge theory. As mentioned above, the prototypical example of a Kähler metric is the one induced via the standard (Fubini-Study) metric on complex projective space $\mathbb{P}^N$ via an embedding

$$M \hookrightarrow \mathbb{P}^N.$$

Powerful interactions between topology and linear algebra find their expression in this setting. Perhaps none is more powerful than the cohomological consequences of 'slicing' the n-dimensional complex manifold $M \subseteq \mathbb{P}^N$ into a one-parameter family of $(n-1)$-dimensional complex submanifolds by intersecting it with a linear family of hyperplanes of $\mathbb{P}^N$. Two fundamental facts are responsible for the power of this so-called *Lefschetz pencil* technique. The first is that the action of cup product with the $(1,1)$-cohomology class representing the hyperplane and the adjoint of that action generate a representation of $\mathfrak{sl}(2,\mathbb{C})$ on $H^*(M)$, the weight-spaces of which give a powerful semi-simple decomposition of $H^*(M)$. The second is that slicing by a linear 'pencil' of hyperplane sections opens the way to argue by induction on the dimension of the manifold M. The (finite set of) singular slices have only the simplest of singularities, called ordinary nodes, modeled topologically by contracting an imbedded real $(n-1)$-sphere (vanishing cycle) in a nearby slice to a point. Lefschetz used this second fact to show that a general slice M_0 of M has the same cohomology as M itself in dimensions $< (n-1)$ and that the $(n-1)$-st cohomology of M injects into that of M_t under the restriction map. Said otherwise, the relative cohomology

$$H^i(M, M_0)$$

vanishes for $i < n$. In the remainder of Volume I and in Volume II, Voisin brings together for the first time a truly comprehensive treatment of the modern theory of variations of Hodge structures, of mixed Hodge structures, and of applications to cycle theory. As such, it has already become the definitive reference.

The remainder of this review will be a record of the reviewer's own attempt to navigate and organize a principal thread in that treatment, namely Nori's Connectedness Theorem for hypersurfaces and its consequences for cycle theory.

4. Variations of Hodge structures

The slicing of a projective complex manifold

$$M \subseteq \mathbb{P}^N$$

by a $\mathbb{P}^1$ of hyperplane sections gives rise to an incidence manifold

$$\hat{M} \subseteq \mathbb{P}^1 \times M.$$

The proper morphism

$$\hat{M} \to \mathbb{P}^1$$

is smooth except over the finite set Δ where the corresponding fiber has a simple node. Letting T denote the unit disk considered as the universal cover

$$\tau : T \to \left(\mathbb{P}^1 - \Delta\right),$$

one considers the pull-back family

$$\tilde{M} := \hat{M} \times_{(\mathbb{P}^1 - \Delta)} T \to T \tag{2}$$

as a deformation of the complex structure on the fiber M_0 over $0 \in T$. For $t \in T$ denote the corresponding fiber of (2) as M_t.

The family (2) is differentiably trivial, so one fixes a smooth family of C^∞-isomorphisms

$$\varphi_t : M_0 \to M_t$$

which are perturbations of the identity on M_0.

The gain achieved by reducing the dimension of the object of study by one had to be repaid by developing an understanding of the opportunities afforded by a continuously varying family of Kähler manifolds as opposed to a single one standing alone. This work was pioneered by Griffiths and Deligne, but only after the theory of variation of complex structure appeared on the scene through seminal work of Kodaira-Spencer and Kuranishi. In particular, much information about the M itself is ultimately derivable from the knowledge of how $H^{p,q}(M_t)$ varies within the (flat) deformation of $H^{p+q}(M_t)$ determined by (2). In essence, this is what is meant by a *variation of Hodge structure*.

Using the identification

$$A^*_{M_0} \leftrightarrow A^*_{M_t} \tag{3}$$

induced by φ_t above, let

$$A^{*,*}_t := \varphi^*_t A^{*,*}_{M_t} \subseteq A^*_{M_0}.$$

Under (3) a section of

$$A^*_{\tilde{M}/T}$$

becomes a smooth function

$$T \to A^*_{M_0}$$

on which the *Gauss-Manin connection* ∇ is given by differentiating this map with respect to t. *Griffiths Transversality* says that

$$\nabla_{\frac{\partial}{\partial t}} \left(F^p H^* \left(\tilde{M}/T\right)\right) \subseteq F^{p-1} H^* \left(\tilde{M}/T\right)$$

where

$$F^p H^* = \bigoplus\nolimits_{p' \geq p} H^{p',q}.$$

5. The Leray spectral sequence

Let Y be any smooth quasiprojective variety and

$$\pi : X \to Y$$

a projective smooth morphism. (In the above case we have $\tilde{M}$ and $Y = T$.) As above the family of fibers is locally differentiably trivial, and we have the induced Gauss-Manin connection ∇. The fundamental instrument for connecting the cohomology of a total space X to the cohomology of its slices $\pi^{-1}(y)$ is called *the Leray spectral sequence*, derived from the filtration

$$F(r) = A_X^{*-r} \wedge \pi^* A_Y^r$$

of A_X^*. Indeed the Leray spectral sequence and a "holomorphic" variant of it are central to Voisin's two volumes and will be the centerpiece of the remainder of this review. First of all

$$\begin{aligned} E_0^{i,j} &= A_Y^i\left(A_{X/Y}^j\right),\ d_0 = d_{X/Y} \\ E_1^{i,j} &= A_Y^i\left(R^j\pi_*\mathbb{C}\right),\ d_1 = \nabla \\ E_2^{i,j} &= H^i\left(Y; R^j\pi_*\mathbb{C}\right). \end{aligned}$$

Thus $E_2^{i,j}$ is realized as the j-th deRham cohomology group of the complex of differential forms on Y with values in the local system $R^j\pi_*\mathbb{C}$ or, what is the same thing, the i-th hypercohomology of the holomorphic deRham complex

$$\mathcal{O}_Y \otimes R^j\pi_*\mathbb{C} \xrightarrow{\nabla} \Omega_Y^1 \otimes R^j\pi_*\mathbb{C} \xrightarrow{\nabla} \dots \tag{4}$$

where ∇ is given by the (flat) Gauss-Manin connection.

Deligne proved the fundamental result that the Leray spectral sequence $\left\{E_r^{i,j}\right\}$ degenerates at E_2 so that

$$H^k(X) = \bigoplus_{i=0}^{k} E_2^{i,k-i}.$$

The idea is as follows. There is a distinguished element $\mathfrak{h} \in H^0\left(R^2\pi_*\mathbb{C}\right)$ given by the cohomology class of the hyperplane section, and the cup product map

$$L : R^i\pi_*\mathbb{C} \xrightarrow{\cup\mathfrak{h}} R^{i+2}\pi_*\mathbb{C} \tag{5}$$

with the flat section $\mathfrak{h}$ of $R^2\pi_*\mathbb{C}$ commutes with the Gauss-Manin connection ∇, and so ∇ passes to the kernel of the adjoint of (5) with respect to (fiberwise) Poincaré duality, the so-called primitive cycles. Thus the decomposition of $R^*\pi_*\mathbb{C}$ into $SL_2(\mathbb{C})$-weight spaces is inherited by $E_1^{i,j}$, and it is compatible with the action of d_r for $r \geq 1$. But on a primitive k-class α

$$\left(L^{n-k+1} \circ d_2\right)(\alpha)$$

is non-zero whenever $d_2(\alpha)$ is, whereas $L^{n-k+1}(\alpha) = 0$. It is this incompatibility of the action of the cup product commuting with a non-trivial d_r for $r > 2$ that gives the degeneration at E_2. That is

$$H^r(X) = \bigoplus_{i+j=r} H^j\left(R^i\pi_*\mathbb{C}\right),$$

expressing the cohomology of X as a direct sum of cohomologies of holomorphic bundles on Y. In the Leray spectral sequence

$$E_r^{i,j} = H^i\left(Y; R^j\pi_*\mathbb{C}\right),$$

we have that $d_r = 0$ for $r \geq 2$.

6. Holomorphic filtration of $R\pi_*\mathbb{C}$

Suppose, in the previous section, we replace the constant sheaf $\mathbb{C}$ on X with its (quasi-isomorphic) holomorphic deRham resolution

$$\mathcal{O}_X \xrightarrow{\partial_X} \Omega_X^1 \xrightarrow{\partial_X} \ldots . \tag{6}$$

Then we can filter by subcomplexes

$$\Omega_Y^l \wedge \left\{\mathcal{O}_X \xrightarrow{\partial} \Omega_X^1 \xrightarrow{\partial} \ldots\right\},$$

and the associated spectral sequence $\left\{E_r^{l,r}\right\}$ abutting on $R^*\pi_*\mathbb{C}$ has $E_1^{l,r}$-term given by $R^{l+r}\pi_*$ applied to the (total) complex

$$\Omega_Y^l \otimes \left\{\mathcal{O}_X \xrightarrow{\partial_{X/Y}} \Omega_{X/Y}^1 \xrightarrow{\partial_{X/Y}} \ldots\right\},$$

that is,

$$\Omega_Y^l \otimes \left\{R^r\pi_*\mathcal{O}_X \oplus R^{r-1}\pi_*\Omega_{X/Y}^1 \oplus \ldots\right\}.$$

Furthermore by Griffiths Transversality

$$d_1 : E_1^{l,r} \to E_1^{l+1,r}$$

is given by the morphisms

$$\bigoplus\nolimits_{p+q=r}\left(\Omega_Y^l \otimes R^q\pi_*\Omega_{X/Y}^{p-l} \xrightarrow{\nabla} \Omega_Y^{l+1} \otimes R^{q+1}\pi_*\Omega_{X/Y}^{p-l-1}\right) \tag{7}$$

of sheaves on Y, where ∇ is induced by the Gauss-Manin connection as in the maps (3) above. (Compare this with (4) above.) So we are led to the study of the relative holomorphic deRham resolution

$$\mathcal{O}_{X/Y} \xrightarrow{\partial_X} \Omega_{X/Y}^1 \xrightarrow{\partial_X} \ldots$$

of the constant sheaf $\mathbb{C}$ and the morphisms (7).

This "holomorphic" Leray spectral sequence for a projective smooth morphism

$$\pi : X \to Y$$

is especially powerful when Y parametrizes smooth intersections

$$F_{y_1} \cap \ldots \cap F_{y_r} \cap M$$

for a fixed $(n+r)$-manifold

$$M \subseteq \mathbb{P}^N$$

and hypersurfaces F_{y_i} of sufficiently high degree. Arguments based on induction on r lead to consideration of the case $r = 1$.

7. Koszul cohomology

Let

$$\tau : Y \rightarrow H^0\left(\mathcal{O}_M\left(d\right)\right)$$

have image inside the set of sections with smooth zero-sets and $X \rightarrow Y$ be the universal zero set. Let L now denote the pullback of $\mathcal{O}_M\left(d\right)$ to $M \times Y$ with universal section s. Following Green and Müller-Stach ([GMV], pp. 41-43), let

$$\mathfrak{D}^1_{M\times Y}\left(L\right)$$

denote the sheaf of first-order differential operators on sections of L, which sits in an exact (Euler) sequence

$$0 \rightarrow \mathcal{O}_{M\times Y} \rightarrow \mathfrak{D}^1_{M\times Y}\left(L\right) \xrightarrow{\Sigma} T_{M\times Y} \rightarrow 0 \tag{8}$$

given by the symbol map Σ. Taking exterior powers of the dual sequence

$$0 \rightarrow \Omega^1_{M\times Y} \rightarrow \mathfrak{J}^1_{M\times Y}\left(L\right) \rightarrow \mathcal{O}_{M\times Y} \rightarrow 0$$

(with respect to the left $\mathcal{O}_{M\times Y}$ module structure on $\mathfrak{D}_1\left(L\right)$) yields short exact sequences

$$0 \rightarrow \Omega^p_{M\times Y} \rightarrow \bigwedge\nolimits^p \mathfrak{J}^1_{M\times Y}\left(L\right) \rightarrow \Omega^{p-1}_{M\times Y} \rightarrow 0.$$

The evaluation map

$$\begin{array}{rcl} \mathfrak{D}^1_{M\times Y}\left(L\right) & \rightarrow & L \\ D & \mapsto & D\left(s\right) \end{array}$$

considered as a holomorphic section σ of

$$\mathfrak{J}^1_{M\times Y}\left(L\right) \otimes L$$

has the property that

$$L^{-1} \otimes \bigwedge\nolimits^p \mathfrak{J}^1_{M\times Y}\left(L\right) \xrightarrow{\wedge\sigma} \Omega^p_{M\times Y} \rightarrow \Omega^p_X$$

is exact and so gives a Koszul resolution

$$\ldots \xrightarrow{\wedge\sigma} L^{-2} \otimes \bigwedge\nolimits^{*-1} \mathfrak{J}^1_{M\times Y}\left(L\right) \xrightarrow{\wedge\sigma} L^{-1} \otimes \bigwedge\nolimits^{*} \mathfrak{J}^1_{M\times Y}\left(L\right) \rightarrow \Omega^*_{M\times Y,X}. \tag{9}$$

As above, filter $\Omega^*_{M\times Y,X}$ by the holomorphic Leray filtration

$$\Omega^l_Y \wedge \Omega^{*-l}_{M\times Y,X}$$

so that, as above,

$$d_1 : E_1^{l,r} \rightarrow E_1^{l+1,r} \tag{10}$$

of the resulting spectral sequence for the filtration of $\Omega^*_{M\times Y,X}$ is given by the morphisms

$$\bigoplus\nolimits_{p+q=r}\left(\Omega^l_Y \otimes R^q\pi_*\Omega^{*-l}_{M\times Y/Y,X/Y} \xrightarrow{\nabla} \Omega^{l+1}_Y \otimes R^{q+1}\pi_*\Omega^{*-l-1}_{M\times Y/Y,X/Y}\right). \tag{11}$$

Similarly the filtration of

$$\bigwedge\nolimits^* \mathfrak{J}^1_{M\times Y}\left(L\right)$$

induced by the subspace

$$\pi^*\Omega^1_Y \subseteq \mathfrak{J}^1_{M\times Y}\left(L\right)$$

induces a filtration on (9). The E_0-term of the resulting spectral sequence for the induced filtration of (9) is given by the tensor product of Ω_Y^l with the resolution

$$\ldots \xrightarrow{\wedge\sigma} L^{-2} \otimes \bigwedge^{*-l-1} \mathfrak{J}^1_{M\times Y/Y}(L) \xrightarrow{\wedge\sigma} L^{-1} \otimes \bigwedge^{*-l} \mathfrak{J}^1_{M\times Y/Y}(L) \to \Omega^{*-l}_{M\times Y/Y, X/Y}$$

of $\Omega^{*-l}_{M\times Y/Y,X/Y}$, where, referring to (8), $\mathfrak{J}^1_{M\times Y/Y}(L)$ is constructed from

$$\mathfrak{D}^1_{M\times Y/Y}(L) = \Sigma^{-1}\left(T_{M\times Y/Y}\right)$$

as above. Furthermore

$$d_1 : E_1^{l,r} \to E_1^{l+1,r}$$

is

$$\begin{array}{c}\Omega^l_Y \otimes R^q\pi_*\left(\ldots \xrightarrow{\wedge\sigma} L^{-2} \otimes \bigwedge^{*-l-1} \mathfrak{J}^1_{M\times Y/Y}(L) \xrightarrow{\wedge\sigma} L^{-1} \otimes \bigwedge^{*-l} \mathfrak{J}^1_{M\times Y/Y}(L)\right)\\ \downarrow \\ \Omega^{l+1}_Y \otimes R^{q+1}\pi_*\left(\ldots \xrightarrow{\wedge\sigma} L^{-2} \otimes \bigwedge^{*-l-2} \mathfrak{J}^1_{M\times Y/Y}(L) \xrightarrow{\wedge\sigma} L^{-1} \otimes \bigwedge^{*-l-1} \mathfrak{J}^1_{M\times Y/Y}(L)\right).\end{array}$$

But, for $d >> 0$, if we filter "to the right" to compute $R^q\pi_*$ in this last diagram, we get

$$\begin{array}{c}\Omega^l_Y \otimes H_{n+1-q}\left(\ldots \xrightarrow{\wedge\sigma} H^{n+1}\left(L^{-2} \otimes \bigwedge^{*-l-1} \mathfrak{J}^1_M(L)\right) \xrightarrow{\wedge\sigma} H^{n+1}\left(L^{-1} \otimes \bigwedge^{*-l} \mathfrak{J}^1_M(L)\right)\right)\\ \downarrow \\ \Omega^{l+1}_Y \otimes H_{n-q}\left(\ldots \xrightarrow{\wedge\sigma} H^{n+1}\left(L^{-2} \otimes \bigwedge^{*-l-2} \mathfrak{J}^1_M(L)\right) \xrightarrow{\wedge\sigma} H^{n+1}\left(L^{-1} \otimes \bigwedge^{*-l-1} \mathfrak{J}^1_M(L)\right)\right).\end{array}$$

To continue the computation, apply Serre duality on M to obtain the diagram (12)

$$\begin{array}{c}\bigwedge^l T_Y \otimes H^{n-q+1}\left(\ldots \leftarrow H^0\left(L^2 \otimes \Omega^{n+1}_M \otimes \bigwedge^{*-l-1} \mathfrak{D}^1_M(L)\right) \leftarrow H^0\left(L \otimes \Omega^{n+1}_M \otimes \bigwedge^{*-l} \mathfrak{D}^1_M(L)\right)\right)\\ \uparrow \\ \bigwedge^{l+1} T_Y \otimes H^{n-q}\left(\ldots \leftarrow H^0\left(L^2 \otimes \Omega^{n+1}_M \otimes \bigwedge^{*-l-2} \mathfrak{D}^1_M(L)\right) \leftarrow H^0\left(L \otimes \Omega^{n+1}_M \otimes \bigwedge^{*-l-1} \mathfrak{D}^1_M(L)\right)\right),\end{array}$$

where the horizontal arrows are induced by differentiating the universal section s. To understand the vertical map, notice that the morphism

$$T_Y \to H^0(L) \tag{13}$$

associated to the universal section s gives a tautological section

$$\tau \in \Omega^1_Y \otimes H^0(L),$$

and the vertical map in (12) is induced by contraction with this section.

8. The case of hypersurfaces

In the case in which $M = \mathbb{P}^{n+1}$ the computation of (12) becomes more transparent using the isomorphism

$$\mathfrak{D}^1_M\left(\mathcal{O}_{\mathbb{P}^{n+1}}(d)\right) = \sum_{i=0}^{n+1} \mathcal{O}_{\mathbb{P}^{n+1}}(1)\frac{\partial}{\partial X_i}$$

so that, writing

$$\begin{aligned} S_k &= H^0\left(\mathcal{O}_{\mathbb{P}^{n+1}}(k)\right), \\ V &= \sum_{i=0}^{n+1} \mathbb{C}\frac{\partial}{\partial X_i} \end{aligned}$$

(12) becomes

(14)

$$\begin{array}{c}\bigwedge^l T_Y \otimes H^{n-q+1}\left(\ldots \leftarrow S_{2d-(n+2)+(*-l-1)} \otimes \bigwedge^{*-l-1} V \leftarrow S_{d-(n+2)+(*-l)} \otimes \bigwedge^{*-l} V\right)\\ \uparrow \\ \bigwedge^{l+1} T_Y \otimes H^{n-q}\left(\ldots \leftarrow S_{2d-(n+2)+(*-l-2)} \otimes \bigwedge^{*-l-2} V \leftarrow S_{d-(n+2)+(*-l-1)} \otimes \bigwedge^{*-l-1} V\right)\end{array}$$

with horizontal maps given by contraction with

$$\sum_{i=0}^{n+1} \frac{\partial s}{\partial X_i} dX_i$$

and vertical maps given by contracting with τ. In fact in this case d does not have to be large to get (14). Since $\mathfrak{J}^1_{\mathbb{P}^{n+1}}(L)$ is itself semi-negative, the only requirement is that $d > 0$.

At this point we assume

$$* - l \leq n$$

so that $S_{d-(n+2)+(*-l)-(d-1)} = S_{*-n-1-l} = 0$. Thus

$$\ldots \leftarrow S_{2d-(n+2)+(*-l-1)} \otimes \bigwedge^{*-l-1} V \leftarrow S_{d-(n+2)+(*-l)} \otimes \bigwedge^{*-l} V \tag{15}$$

is a full Koszul resolution, so that $H^{n-q+1} = 0$ unless

$$* - l = n - q + 1.$$

In this last case, if we let

$$R_k = \frac{\mathcal{O}_Y \otimes S_k}{\left\{\frac{\partial s}{\partial X_i}\right\}_k}$$

where $\left\{\frac{\partial s}{\partial X_i}\right\}$ is the (homogeneous) Jacobian ideal of s in the graded ring S^*, then when $q = n + 1 - (* - l)$ we have

$$\bigwedge^l T_Y \otimes H^{n-q+1} = \bigwedge^l T_Y \otimes R_{d(*-l+1)-(n+2)}$$

and

$$\bigwedge^{l+1} T_Y \otimes H^{n-q} = \bigwedge^{l+1} T_Y \otimes R_{d(*-l)-(n+2)}.$$

Furthermore the dual of d_1 is

$$\left(\bigwedge^{l+1} T_Y\right) \otimes R_{d(*-l)-(n+2)} \xrightarrow{\tau} \left(\bigwedge^l T_Y\right) \otimes R_{d(*-l+1)-(n+2)}.$$

Notice that in the case $\dim Y = 0$ the spectral sequence degenerates at E_1 giving rise to Griffiths' residue theory for the (primitive) Hodge structure of hypersurfaces in $\mathbb{P}^{n+1}$. In the case in which $\dim Y = 1$, the spectral sequence degenerates at E_2, giving rise to Griffiths' residue theory for (one-parameter) variations.

9. Nori connectedness

Now consider the general case of a smooth projective morphism

$$\pi : X \to Y$$

where Y parametrizes smooth intersections

$$F_{y_1} \cap \ldots \cap F_{y_r} \cap M$$

for a fixed $(n + r)$-manifold

$$M \subseteq \mathbb{P}^N$$

and F_{y_i} hypersurfaces of sufficiently high degree d_i. A fundamental result of Nori is that, if Y is an *open* subset of the smooth complete-intersections in

$$\prod_{i=1}^{r} H^0\left(\mathcal{O}_M\left(d_i\right)\right)$$

for $d_i >> 0$, then

$$H^k\left(M\times Y\right)\to H^k\left(X\right) \tag{16}$$

is an isomorphism for

$$k<2n$$

and an injection for

$$k=2n;$$

that is, for $k\leq 2n$,

$$H^k\left(M\times Y,X\right)=0.$$

Furthermore the result continues to hold if Y is replaced by a submersive base extension

$$T\to Y.$$

Using that (16) is a morphism of mixed Hodge structures and applying conjugate symmetry to the associated graded morphism of Hodge structures, the proof of Nori's theorem is reduced to the corresponding assertions for

$$H^q\left(\Omega^p_{M\times Y}\right)\to H^q\left(\Omega^p_X\right) \tag{17}$$

where

$$\begin{aligned} p+q &< 2n, \leq 2n \\ p &< n, \leq n. \end{aligned}$$

In turn these assertions are implied by the corresponding ones for

$$R^q\pi_*\left(\Omega^p_{M\times Y}\right)\to R^q\pi_*\left(\Omega^p_X\right).$$

Thus one must show that, for $p<n$ and $p+q\leq 2n$,

$$R^q\pi_*\left(\Omega^p_{M\times Y,X}\right)=0 \tag{18}$$

where $\Omega^*_{M\times Y,X}$ denotes the kernel of

$$\Omega^*_{M\times Y}\to\Omega^*_X.$$

If $p=n$, one must also show injectivity of

$$R^q\pi_*\left(\Omega^p_{M\times Y}\right)\to R^q\pi_*\left(\Omega^p_X\right)$$

for $q\leq 2n-p$, but we will restrict our attention to (18).

As above the spectral sequence for the filtration

$$\Omega^l_Y\wedge\Omega^{*-l}_{M\times Y,X}$$

of $\Omega^*_{M\times Y,X}$ yields

$$E_1^{l,r}=\Omega^l_Y\otimes R^{l+r}\pi_*\left(\Omega^{*-l}_{M\times Y/Y,X/Y}\right)$$

with

$$d_1=\nabla:\Omega^l_Y\otimes R^{l+r}\pi_*\left(\Omega^{*-l}_{M\times Y/Y,X/Y}\right)\to\Omega^{l+1}_Y\otimes R^{l+1+r}\pi_*\left(\Omega^{*-l-1}_{M\times Y/Y,X/Y}\right).$$

To prove (18) it will suffice to show that

(19)
$$R^{l+q-1}\pi_*\left(\Omega^{p-l+1}_{M\times Y/Y,X/Y}\right)\to R^{l+q}\pi_*\left(\Omega^{p-l}_{M\times Y/Y,X/Y}\right)\to R^{l+q+1}\pi_*\left(\Omega^{p-l-1}_{M\times Y/Y,X/Y}\right)$$

is exact in the middle whenever $p+q\leq 2n$.

We illustrate the exactness of (19) in the case of hypersurfaces in $\mathbb{P}^{n+1}$. Thus

$$T_Y=\mathcal{O}_Y\otimes H^0\left(\mathcal{O}_{\mathbb{P}^{n+1}}(d)\right)=\mathcal{O}_Y\otimes S^d.$$

In this case we have seen above that, if $p-l<n$, then

$$R^{l+q}\pi_*\left(\Omega^{p-l}_{\mathbb{P}^{n+1}\times Y/Y,X/Y}\right)=0$$

unless

$$p-l=n+1-q,$$

in which case

$$R^{l+q}\pi_*\left(\Omega^{p-l}_{\mathbb{P}^{n+1}\times Y,X/Y}\right)=\mathcal{O}_Y\otimes\left(\bigwedge^{l}S^d\right)\otimes R_{d(p-l+1)-(n+2)}.$$

So the dual of (19) becomes

$$\begin{aligned}(20)\qquad \left(\bigwedge^{l+1}S^d\right)\otimes R_{d(p-l)-(n+2)}&\xrightarrow{\tau}\left(\bigwedge^{l}S^d\right)\otimes R_{d(p-l+1)-(n+2)}\\&\xrightarrow{\tau}\left(\bigwedge^{l-1}S^d\right)\otimes R_{d(p-l+2)-(n+2)},\end{aligned}$$

which we must study for $l\leq p$ and

$$\begin{array}{rcl}p&<&n\\ q&=&n+1-(p-l)\leq 2n-p,\end{array}$$

that is, for

$$l\leq p<n.$$

If $p-l=0$, we are studying

$$R^{p+n+1}\pi_*\left(\Omega^{0}_{\mathbb{P}^{n+1}\times Y/Y,X/Y}\right),$$

so only have to worry about the case $p=0$.

But the Koszul cohomology theorem of Green ([GMV], p. 74) gives exactness of

$$\begin{aligned}(21)\qquad \left(\bigwedge^{l+1}S^d\right)\otimes S_{d(p-l)-(n+2)}&\xrightarrow{\tau}\left(\bigwedge^{l}S^d\right)\otimes S_{d(p-l+1)-(n+2)}\\&\xrightarrow{\tau}\left(\bigwedge^{l-1}S^d\right)\otimes S_{d(p-l+2)-(n+2)}\end{aligned}$$

in the middle when

$$d\,(p-l+1)-(n+2)\geq l+d,$$

that is, when

$$(22)\qquad d\,(p-l)\geq l+n+2.$$

So, using that $p<n$, sequence (21) is exact in the middle for $p-l>0$ as long as $d>2n$.

Also we have the free graded resolution (15) of R_* as an S_*-module. Use it to resolve (20) by free S_*-modules. If one arranges the resolution from below and uses Green's theorem to compute the E_1-term of the spectral sequence associated to the "to the top" filtration, one concludes the exactness of (20) at the middle as long as (22) holds, which in turn is satisfied as long as $p-l>0$ and $d>2n$. On the other hand, if $p-l=0$, we have seen above that the only case is $p=0$ where we have

$$R^{n+1}\pi_*\left(\Omega^0_{\mathbb{P}^{n+1}\times Y/Y,X/Y}\right)^\vee = R_{d-(n+2)}$$

mapping to

$$\left(S^d\right)^\vee \otimes R^n\pi_*\left(\Omega^1_{\mathbb{P}^{n+1}\times Y/Y,X/Y}\right)^\vee = Hom\left(S^d, R_{2d-(n+2)}\right)$$

by multiplication. But this map is injective by Macaulay's theorem ([V] II, p. 172). Thus the first assertion of Nori's theorem is proved for hypersurfaces whenever $d>2n$.

10. Cycle theory

To study subvarieties of a quasi-projective manifold X, one builds a free abelian group on the set of closed irreducible subvarieties of codimension k and then divides out by the equivalence relation induced from the linear equivalence of divisors in a codimension-$(k-1)$ subvariety. The resulting *Chow group*, denoted $CH^k(X)$ or $CH_{\dim M-k}(X)$ has the property that a correspondence

$$\begin{array}{ccccc}\Gamma & \subseteq & X\times T & \xrightarrow{p} & X\\ & & \downarrow^{q} & & \\ & & T & & \end{array}$$

in $CH^{k+r}(X\times T)$ induces a morphism

$$p_*\circ q^*: CH_r(T)\to CH^k(X).$$

Let $CH_r(T)_{\text{hom}.}$ denote the group of homologically trivial r-cycles. Nori filters $CH^k(X)$ by the increasing sequence of subgroups

$$\mathcal{N}_r\left(CH^k(X)\right) = image\left(CH^{k+r}(X\times T)\times CH_r(T)_{\text{hom}.}\overset{p_*\circ q^*}{\longrightarrow} CH^k(X)\right).$$

For example, $\mathcal{N}_0\left(CH^k(X)\right)$ is the space of cycles which are called *algebraically equivalent to zero*.

Again consider the case of a smooth projective morphism

$$\pi: X\to Y$$

with Y open in $\prod_{i=1}^r H^0\left(\mathcal{O}_M(d_i)\right)$ parametrizing smooth intersections

$$F_{y_1}\cap\ldots\cap F_{y_r}\cap M$$

for a fixed $(n+r)$-manifold

$$M\subseteq\mathbb{P}^N$$

with $d_i >> 0$. Let $Z\in CH^k(M)$ be such that $0\neq\{Z\}\in H^{2k}(M)_{primitive}$ for $n-k$. Nori used his connectivity result to show that, for $s<n-k$ and general y, no multiple of

$$Z\cdot X_y$$

lies in

$$\mathcal{N}_s\left(CH^k(X_y)\right).$$

11. Algebraic equivalence on hypersurfaces of a quadric

We finish our excursion through Voisin's Volume II by showing how connectivity implies the last result in the simplest non-trivial example, namely when $r = 1$ and Z is the difference of the two rulings of an even-dimensional quadric in complex projective space. For n odd, let

$$M \subseteq \mathbb{P}^{n+2}$$

be a smooth hypersurface of degree 2 and let

$$W \to H^0\left(\mathcal{O}_M\left(d\right)\right)^*$$

be the universal hypersurface of M of degree $d >> 0$. Let

$$Z = P - Q \in CH^{\frac{n+1}{2}}\left(M\right) \to H^{n+1}\left(M\right)_{primitive}$$

be the difference of the two rulings of M. Suppose over some

$$V^{open} \subseteq (\textit{smooth locus}) \subseteq H^0\left(\mathcal{O}_M\left(d\right)\right)$$

that some multiple of $P \cdot W_v$ and $Q \cdot W_v$ are algebraically equivalent in W_v. Then by a classical argument, shrinking V and passing to a smooth base extension T, there must exist a proper family of smooth curves

$$C \to T$$

parametrizing the algebraic equivalence. That is, for

$$W_T = T \times_V W$$

there are sections

$$c_P, c_Q : T \to C$$

and a cycle Γ of codimension $\frac{n+1}{2}$ in the fibered product

$$\begin{array}{ccc} W_C & \xrightarrow{q} & W_T \subseteq M \times T \\ \downarrow^p & & \downarrow \\ C & \to & T \end{array}$$

such that the fiber C_t parametrizes an effective algebraic equivalence between

$$q_*\left(p^*\left(c_P\left(t\right)\right) \cdot \Gamma\right) = m\left(W_t \cdot P\right) + N(t)$$

and

$$q_*\left(p^*\left(c_Q\left(t\right)\right) \cdot \Gamma\right) = m\left(W_t \cdot Q\right) + N(t)$$

in W_t for each $t \in T$.

If we let

$$\begin{aligned} \Gamma_P &= q_*\left(\Gamma \cdot p^*\left(c_P\left(T\right)\right)\right) \\ \Gamma_Q &= q_*\left(\Gamma \cdot p^*\left(c_Q\left(T\right)\right)\right), \end{aligned}$$

then by construction

$$\begin{aligned} \Gamma_P &= m\left(P \times C\right) \cdot W_C + N \\ \Gamma_Q &= m\left(Q \times C\right) \cdot W_C + N. \end{aligned}$$

Since the codimension of Γ in X is $\frac{n+1}{2}$ and $2\left(\frac{n+1}{2}\right) < 2n$ for odd $n \geq 3$, we have by Nori's connectedness that there exists $\delta \in H^{n+1}\left(M \times C\right)$ whose image in

$$H^{n+1}\left(W_C\right)$$

is the class of Γ. Then writing

$$\begin{array}{ccc} M\times C & \xrightarrow{\tilde{q}} & M\times T \\ \downarrow^{\tilde{p}} & & \downarrow \\ C & \rightarrow & T \end{array}$$

we have that

$$\begin{aligned} q_*\left(\{\Gamma_P-\Gamma_Q\}\right) &= \tilde{q}_*\left(\delta\cdot\tilde{p}^*\left\{c_P\left(T\right)-c_Q\left(T\right)\right\}\right)|_{W_T} \\ &= m\left\{\left(\left(P\times T\right)-\left(Q\times T\right)\right)\cdot W_T\right\} \end{aligned}$$

on W_T. So by Nori connectivity for the family $W_T\subseteq M\times T$, we have on $M\times T$ that

$$\tilde{q}_*\left(\delta\cdot\tilde{p}^*\left\{c_P\left(T\right)-c_Q\left(T\right)\right\}\right)=m\left\{\left(P\times T\right)-\left(Q\times T\right)\right\}.$$

But restricting this last equality to a fiber C_t of C/T gives the contradiction in $H^{\frac{n+1}{2}}\left(M\times\{t\}\right)$ that

$$m\left\{\left(P\times\{t\}\right)-\left(Q\times\{t\}\right)\right\}=\tilde{q}_*\left(\delta\cdot\tilde{p}^*\left\{c_P\left(t\right)-c_Q\left(t\right)\right\}\right)=0.$$

12. Afterword

And so, to readers with the interest and fortitude to have arrived at the end of this review, my wish is that the tour just presented gives at least a bit of an idea of the mathematical rewards awaiting those who invest their mathematical energies in this beautiful pair of volumes.

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