BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 43, Number 1, Pages 127–132
S 0273-0979(05)01074-8
Article electronically published on July 8, 2005

Discrete and continuous nonlinear Schrödinger systems, by M. J. Ablowitz, B. Prinari, and A. D. Trubatch, Cambridge Univ. Press, Cambridge, 2004, ix+257 pp., £38.00, ISBN 0-521-53437-2

1. Nonlinear Schrödinger equation: setup and applications

The nonlinear Schrödinger (NLS) equation [1, 2, 3] is a prototypical dispersive nonlinear partial differential equation (PDE) that has been central for almost four decades now to a variety of areas in Mathematical Physics. It is an equation for a complex field $u(x,t)$ of the form:

$$iu_t = -\Delta u + \sigma |u|^2 u. \tag{1.1}$$

u is the complex field whose spatio-temporal evolution we are interested in probing. The subscript t denotes temporal partial derivative.

The PDE of Eq. (1.1) is a ubiquitous envelope wave equation [3] which arises in a variety of diverse physical contexts. The relevant fields of application may vary from optics and propagation of the (envelope of) the electric field in optical fibers [4, 5], to the self-focusing and collapse of Langmuir waves in plasma physics [6, 7] and the behavior of deep water waves and freak waves (the so-called rogue waves) in the ocean [8, 9].

While there has been a vast amount of literature on the NLS over the past forty years, both on the physical and the mathematical aspects of the equation, the subject has been the focus of renewed attention in the past few years, predominantly due to the developments in nonlinear optics and soft-condensed matter physics. In the optical context, the experimental developments on arrays of coupled optical waveguides [10, 11] have drawn a lot of attention to the discrete NLS model (where the Laplacian operator above is substituted by its difference analog– see also Eq. (2.7) below). On the other hand, the experimental realization of Bose-Einstein condensates (BECs) and their mean field modeling by the so-called Gross-Pitaevskii equation [12], which is an NLS equation with external potentials, has opened new avenues for the study of NLS-type equations. In the optical context, the NLS arises when using an envelope wave approximation to Maxwell's equation (see e.g. [2, 3]), assuming that the medium where light propagates is subject to the so-called Kerr effect (where the refractive index depends on the intensity of the optical pulse). In BECs, the cubic nonlinearity arises from the particle-particle interaction in dilute alkali atoms. In the latter case, the external potentials mentioned above are, typically, in the form of either a parabolic in space magnetic trapping [12, 13] or a periodic in space, the so-called optical lattice trapping [14]. For periodic potentials either in BECs or in optical waveguide arrays, discrete models can be derived self-consistently, using an expansion in the basis of the underlying linear periodic problem [15, 16]. Additionally, other types of potentials such as linear potentials [17], expulsive potentials [18] and more complicated polynomial potentials [19] are now experimentally feasible, giving rise to the need for a mathematical analysis of

---

2000 *Mathematics Subject Classification.* Primary 35Q55.
Key words and phrases. Nonlinear Schrödinger equation, integrable systems, solitons.
The author was supported in part by NSF-DMS-0204585 and NSF-CAREER.

these settings. Both in the BEC and in the optical setting, continuum and discrete solitons have been monitored in numerous experiments in one spatial dimension. Moreover, experimental capabilities are already allowing the observation of two-dimensional discrete solitons and vortices in optical materials [20], as well as in BECs [13].

2. Some mathematical background

Eq. (1.1) is a Hamiltonian system with infinite degrees of freedom under the Hamiltonian flow

$$H = \int \left(|\nabla u|^2 + \frac{\sigma}{2}|u|^4 \right) dx, \tag{2.1}$$

obtained through:

$$iu_t = \frac{\delta H}{\delta u^\star} = \{H, u\}, \tag{2.2}$$

where the standard Poisson brackets have been used.

The dynamics of Eq. (1.1) have a variety of particularly interesting features. For example, except for the conservation of the energy of Eq. (2.1) above, there is a number of additional invariances:

- if we use the transformation $u \to v = ue^{is}$ where s is space and time independent, then the equation for v is the same as the one for u. This is a phase degeneracy/invariance in the system. The generator of the corresponding invariance is found as $v \approx u + \delta u$, with $\delta u = isu$ (the leading order expansion of the exponential). Using Noether's theorem (see e.g. sec. 2.2 in [3]), it can be obtained that

$$N = ||u||_{L^2}^2 = \int |u|^2 dx \tag{2.3}$$

  is conserved, i.e., that the L^2 norm is conserved by the dynamics of Eq. (1.1). This has a meaningful physical interpetation in the applications of the model such as e.g. in optics or Bose-Einstein condensation (BEC) since in the former it means that the power of the beam is conserved, while in the latter it denotes the physically relevant conservation of the number of atoms in the condensate. This invariance is often referred to as the phase or gauge invariance of the NLS.
- Spatial translation $x \to x + \delta x$ also leaves Eq. (1.1) invariant. This leads to the conservation of linear momentum of the form:

$$P = i \int (uu_x^\star - u^\star u_x)\, dx. \tag{2.4}$$

  Hence, translational invariance results in momentum conservation. For higher dimensional problems, there is one such conservation law per translationally invariant direction.
- In two spatial dimensions, there is an additional symmetry, namely the so-called conformal invariance of the NLS equation according to which if we use $l(t) = (t^\star - t)/t_0$, $\xi = x/l(t)$, $\tau = \int^t l^{-2}(s)ds$ and

$$\tilde{u} = l^{d/2} u e^{i\frac{ax^2}{4l^2}} \tag{2.5}$$

with $a(t) = -l dl/dt$, then the equation remains invariant. This invariance with respect to rescalings of the solution leads to the very interesting phenomena of focusing and self-similar wave collapse in higher dimensional NLS equations studied in detail in [3].

We now turn to the special case of one-dimensional NLS which is well-known to be integrable. The integrability of this particular case means that apart from the above integrals of motion, there are infinitely many others. The unusual feature of such an integrable nonlinear partial differential equation (PDE) is that once the initial data is prescribed, we can solve the PDE for all times [1, 2]. The nonlinear wave solutions to such PDEs are often referred to as solitons, because they are solitary coherent structures (i.e., nonlinear waves) which emerge unscathed from their interaction with other such structures.

Such standing wave solutions can be straightforwardly obtained in an explicit form (e.g. for $\sigma = -1$):

$$u = (2\Lambda)^{1/2}\text{sech}(\Lambda^{1/2}(x - ct - x_0))e^{i\left(\frac{c}{2}x+(\Lambda-\frac{c^2}{4})t\right)}, \tag{2.6}$$

where Λ is the frequency of the wave, x_0 the initial position of its center and c its speed. Notice that for these solutions the additional invariance of Eq. (1.1) under a Galilean transformation has been used. This allows us to boost a stationary solution to one with any given speed c. The solutions with $c = 0$ are often referred to as standing waves or as breathers (because of their periodicity in time and exponential localization in space).

As discussed above, oftentimes the physical applications where the NLS equation arises impose either an explicitly discrete or “effectively discrete” (i.e., continuum but with a spatially periodic potential) setting. Therefore it is relevant to study discrete (i.e., differential-difference) forms of the NLS equation. The most direct such example is of the form:

$$i\dot{u}_n = -\Delta_2 u_n + \sigma|u_n|^2 u_n, \tag{2.7}$$

where the overdot denotes temporal derivative and n is now the index of the spatial lattice. However, this discrete NLS equation (DNLS) is not integrable and hence, in some sense, less straightforwardly amenable to mathematical analysis; in Eq. (2.7), $\Delta_2 u_n = (u_{n+1} + u_{n-1} - 2u_n)/h^2$, where h is the spacing of the discrete lattice. A notable exception to the above comment is the development of the anti-continuum limit (as $h \to \infty$) by Mackay and Aubry [21], near which perturbation theoretic arguments and implicit function theorem type considerations are applicable.

However, an alternative integrable type of discretization of the NLS equation was proposed by one of the authors of the present book in [22] and is accordingly often referred to as the Ablowitz-Ladik NLS (AL-NLS) equation of the form:

$$i\dot{u}_n = -\Delta_2 u_n + \frac{\sigma}{2}|u_n|^2(u_{n+1} + u_{n-1}). \tag{2.8}$$

Eq. (2.8) also has explicit standing as well as travelling soliton solutions. We do not give their detailed form here, but they can be found on p. 84 of [2]. In the limit as $h \to 0$, naturally these approach the continuum limit of Eq. (2.6) above. The integrability of Eq. (2.8) makes it a fertile starting point for developing relevant perturbation theory from this well-understood limit for the existence and stability of the solitary waves [23].

Finally, another variant of the NLS equation of both physical and mathematical interest is the case of vector such equations, which was first analyzed by Manakov in [24] (and hence is often referred to as the Manakov system) in the form:

$$iu_t = -\Delta u + \sigma(|u|^2 + |v|^2)u \tag{2.9}$$

$$iv_t = -\Delta v + \sigma(|u|^2 + |v|^2)v. \tag{2.10}$$

The one-dimensional dynamics of Eqs. (2.9)-(2.10) is also integrable, and there also exist integrable discretizations thereof (developed by two of the authors of the present book) [25].

3. The role of the present book

The above developments have, to a considerable extent, been mirrored in a variety of texts addressed primarily to the physics community, such as [26] in the optics context and [27, 28] in the soft condensed-matter physics framework. However, what was missing was a more mathematical text that discusses the underlying partial differential and differential-difference equations and the mathematical theory that allows us to construct and examine solutions of such models.

The present book fills in a substantial part of that gap. Written by experts on continuum and discrete integrable equations (in fact, its authors have contributed to a considerable extent to the development of such equations), it covers a significant volume of mathematical theory formulated on the basis of the inverse scattering transform.

The text starts with a brief introduction on the physical applications of the continuum and discrete NLS equation, as well as the vector NLS equation. While by no means extensive (since this is out of the scope of the text), the introduction will serve as a very useful reference to the reader, for further bibliography on (both mathematical and physical aspects of) the subject. It also gives a flavor of how some of the relevant models are derived through appropriate asymptotic expansions.

The rest of the book follows a very well structured pattern for each one of the equations of interest, namely the continuum NLS equation in one spatial dimension, its discrete AL-NLS analog, then the vector continuum NLS equation (Manakov system) and finally its discrete vector analog. In particular, to make it pedagogically accessible, the equation is first presented, then the Lax pair (whose compatibility condition yields the equation) is derived; subsequently the direct (derivation of the so-called Jost functions) and inverse scattering problem are analyzed, obtaining the explicit form of the regular soliton solutions. Higher order soliton solutions and soliton interactions are also analyzed in this context. Finally, the Hamiltonian structure of the model and some of its physically important conserved quantities are discussed.

Finally, a number of appendices exist which provide supplementary material on issues that may be less directly accessible to the reader, such as the scattering theory for discrete equations with a number of integrable equation examples. Another direction covered in the appendices resonates well with the program whose importance we have stressed above (namely the study of NLS equations for physical applications with external potentials). In particular, the NLS equation in the presence of a linear potential is studied both in the continuum and the discrete case, and its integrability is illustrated.

Overall, we believe that the text will provide a significant addition to the NLS literature and will become a standard reference for the study of discrete and continuum, one-component or multi-component such equations. The development of the inverse scattering formalism based on the Riemann-Hilbert approach presented in the text can be traced for the individual models in various research papers, but has not previously been compiled and included in a single, self-contained text from a more pedagogical point of view (and including numerous details). The latter, we feel, will make the text a substantial study guide and reference for graduate students and young mathematicians, as well as for more seasoned researchers in this field.

References

1. M.J. Ablowitz and H. Segur, *Solitons and the Inverse Scattering Transform*, SIAM (Philadelphia, 1981). M.J. Ablowitz and P.A. Clarkson, *Solitons, Nonlinear Evolution Equations and Inverse Scattering*, Cambridge University Press (Cambridge, 1991). MR0642018 (84a:35251), MR1149378 (93g:35108)
2. M.J. Ablowitz, B. Prinari and A.D. Trubatch, *Discrete and Continuous Nonlinear Schrödinger Systems*, Cambridge University Press (Cambridge, 2004). MR2040621 (2005c:37117)
3. C. Sulem and P.L. Sulem, *The Nonlinear Schrödinger Equation*, Springer-Verlag (New York, 1999). MR1696311 (2000f:35139)
4. A. Hasegawa, *Solitons in Optical Communications*, Clarendon Press (Oxford, NY, 1995).
5. B.A. Malomed, *Variational methods in nonlinear fiber optics and related fields*, Progress in Optics **43** (2002) 69-191.
6. V.E. Zakharov, *Collapse and Self-focusing of Langmuir Waves*, Handbook of Plasma Physics, (M.N. Rosenbluth and R.Z. Sagdeev, eds.), vol. 2 (A.A. Galeev and R.N. Sudan, eds.) 81-121, Elsevier (1984).
7. V.E. Zakharov, *Collapse of Langmuir waves*, Sov. Phys. JETP **35** (1972) 908-914.
8. T.B. Benjamin and J.E. Feir, *The disintegration of wavetrains in deep water, Part 1*, J. Fluid Mech. **27** (1967) 417-430.
9. M. Onorato, A.R. Osborne, M. Serio, and S. Bertone, *Freak waves in random oceanic sea states*, Phys. Rev. Lett. **86** (2001) 5831-5834.
10. H. Eisenberg, Y. Silberberg, R. Morandotti, A. Boyd and J.S. Aitchinson, *Discrete Spatial Optical Solitons in Waveguide Arrays*, Phys. Rev. Lett. **81** (1998) 3383-3386; H. Eisenberg, Y. Silberberg, R. Morandotti and J.S. Aitchinson, *Diffraction Management* **85** (2000) 1863-1866.
11. J. Meier, G.I. Stegeman, Y. Silberberg, R. Morandotti, and J. S. Aitchison, *Nonlinear Optical Beam Interactions in Waveguide Arrays*, Phys. Rev. Lett. **93** (2004) 093903.
12. F. Dalfovo, S. Giorgini, L.P. Pitaveskii and S. Stringari, Rev. Mod. Phys. **71** (1999) 463-512.
13. P.G. Kevrekidis and D.J. Frantzeskakis, *Pattern forming dynamical instabilities of Bose-Einstein condensates*, Mod. Phys. Lett. B **18** (2004) 173-202; P.G. Kevrekidis, R. Carretero-González, D.J. Frantzeskakis, I.G. Kevrekidis, *Vortices in Bose-Einstein condensates: some recent developments*, Mod. Phys. Lett. B **18** (2004) 1481-1505.
14. V.A. Brazhnyi and V.V. Konotop, *Theory of nonlinear matter waves in optical lattices*, Mod. Phys. Lett. B **18** (2004) 627-651.
15. G. L. Alfimov, P. G. Kevrekidis, V. V. Konotop, and M. Salerno, *Wannier functions analysis of the nonlinear Schrödinger equation with a periodic potential*, Phys. Rev. E **66** (2002) 046608. MR1935204 (2003j:81003)
16. M.J. Ablowitz and Z.H. Musslimani, *Discrete spatial solitons in a diffraction-managed nonlinear waveguide array: a unified approach*, Physica D **184** (2003) 276-303. MR2030690 (2004k:78020)
17. B.P. Anderson and M.A. Kasevich, *Macroscopic quantum interference from atomic tunnel arrays*, Science **282** (1998) 1686-1689.
18. K.S. Strecker, G.B. Partridge, A.G. Truscott, and R.G Hulet, *Formation and propagation of matter wave soliton trains*, Nature **417** (2002) 150-153.

19. H. Ott, J. Fortágh, S. Kraft, A. Günther, D. Komma, and C. Zimmermann, *Nonlinear dynamics of a Bose-Einstein condensate in a magnetic waveguide*, Phys. Rev. Lett. **91** (2003) 040402.
20. A.S. Desyatnikov, L. Torner and Yu.S. Kivshar, *Optical vortices and vortex solitons*, nlin.PS/0501026.
21. R.S. MacKay and S. Aubry, *Proof of existence of breathers for time-reversible or Hamiltonian networks of weakly coupled oscillators*, Nonlinearity **7** (1994) 1623-1643. MR1304442 (96a:34085)
22. M.J. Ablowitz and J.F. Ladik, *Nonlinear differential-difference equations*, J. Math. Phys. **16** (1975) 598-603; M.J. Ablowitz and J.F. Ladik, *Nonlinear differential-difference equations and Fourier analysis*, J. Math. Phys. **17** (1976) 1011-1018. MR0377223 (51:13396), MR0427867 (55:897)
23. T. Kapitula and P. Kevrekidis, *Stability of waves in discrete systems*, Nonlinearity **14** (2001) 533-566. MR1830906 (2002f:37116)
24. S.V. Manakov, *On the theory of two-dimensional stationary self-focusing of electromagnetic waves*, Sov. Phys. JETP **38** (1974) 248-253.
25. M.J. Ablowitz, Y. Ohta and A.D. Trubatch, Phys. Lett. A **253** (1999) 287-304. MR1681692 (99m:35215)
26. Yu.S. Kivshar and G.P. Agrawal, *Optical Solitons: From Fibers to Photonic Crystals*, Academic Press (San Diego, 2003).
27. L.P. Pitaevskii and S. Stringari, *Bose-Einstein Condensation*, Oxford University Press (Oxford, 2003). MR2012737 (2004i:82038)
28. C.J. Pethick and H. Smith, *Bose-Einstein Condensation in Dilute Gases*, Cambridge University Press (Cambridge, 2002).

P. G. Kevrekidis
University of Massachusetts
E-mail address: kevrekid@math.umass.edu