BOOK REVIEWS

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Resolution of curve and surface singularities in characteristic zero, by K. Kiyek and J. L. Vicente, Kluwer Academic Publishers, Dordrecht/Boston/London, 2004, xxii + 483 pp., US\$129.00, ISBN 1-4020-2028-7 (hardcover), ISBN 1-4020-2029-5 (e-book)

Should we try to resolve singularities?

Not necessarily! You may rather wish to count rational points on pseudo-elliptic curves, motivate motives by the motion of locomotives, understand the Langlands program from scratch or try out the Erdös-Szekeres conjecture on convex polygons.

Singularities were born when people started to squeeze things into tiny boxes (e.g., sardines in cans). Lack of space destroyed smoothness. Resolution of singularities intends to give back to algebraic varieties the space they need to live and feel comfortable.

Situations

Let $\mathbb{A}^3$ denote affine three-space and consider the map $\pi : \mathbb{A}^3 \to \mathbb{A}^3$ given by $(x, y, z) \to (x, xy, xz)$. This map contracts the vertical yz-plane $x = 0$ to one point, the origin 0. Points close to this plane are mapped vertically to points near the x-axis. Horizontal lines $y = z = c$ are mapped to the lines through the origin whose equations are $y = cx$ and $z = cx$ (cf. fig. 1).

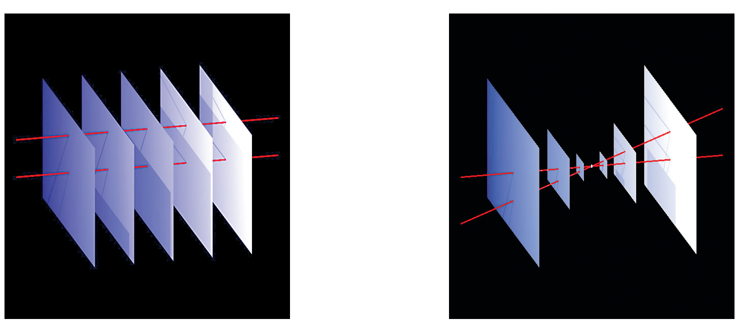

Figure 1. The contraction of three-space by the map $(x, y, z) \to (x, xy, xz)$.

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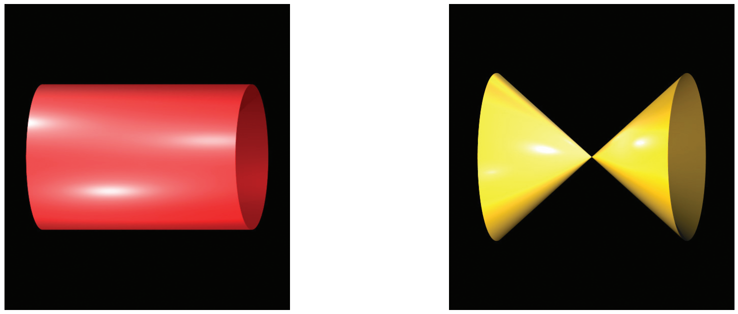

FIGURE 2. The contraction of the cylinder to a cone.

We conclude that π contracts (for $c < 1$) the vertical planes $x = c$ to small vertical planes by multiplying each point by c. Take now the horizontal cylinder $y^2 + z^2 = 1$. Its image under π is the cone $y^2 + z^2 = x^2$. We have created a singularity at 0 (cf. fig. 2).

Conversely, forget about the preceding contraction π and consider the cone alone. What is the correct way to reconstruct the cylinder and the map τ : *cylinder* $\rightarrow$ *cone*? How do we detect that the cone was obtained from the cylinder by contracting a circle to a point? Another example will show that it is by no means obvious how to answer this last question systematically and in a mathematically satisfactory manner.

We consider the Calyx surface in $\mathbb{A}^3$ whose equation is $x^2 + y^2z^3 - z^4 = 0$. Before we attack our singularity it is certainly appropriate to clarify the geometric structure of this surface. The singular locus S is given by equating the partial derivatives $2x$, $2yz^3$ and $3yz^2 - 4z^3$ to 0. So the singular locus is the y-axis. Taking transversal hyperplane sections $y = c$ produces curves of the type $x^2 + z^3(c^2 - z) = 0$, which, for $c \neq 0$, are cusps along S. For $c = 0$, however, we get $x^2 = z^4$, which is the union of two parabolas tangent to the vertical z-axis at 0. Further hyperplane sections reveal the geometry of the Calyx (cf. fig. 3).

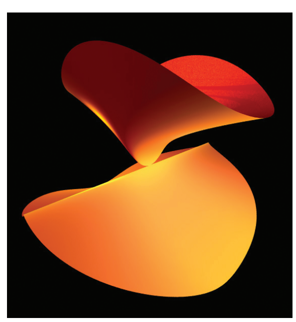

FIGURE 3. The Calyx surface whose equation is $x^2 + y^2z^3 = z^4$.

Here, we can no longer guess a smooth surface C projecting onto the calyx. Even though we have a very precise picture of the geometry of the Calyx along its singular locus, we lack information about how such a configuration is created. And if you are told that Calyx is the image of the cylinder $y^2 + z^2 = 1$ under the projection map $(x, y, z) \to (4xy^2z^4(1+y), 2yz, 2yz^2(1+y))$ from $\mathbb{A}^3$ to $\mathbb{A}^3$, you will have no problem verifying that this is correct, but probably a hard time discovering how the parametrization was found (except if you are familiar with the notion of blowups of varieties in smooth centers, a basic technique of algebraic geometry used to modify and simplify varieties).

Views

Resolution of singularities is concerned with describing a conceptual way to construct for any singular variety X a smooth variety Y and a map ε from the latter to the former which reveals the singularities of X as a contraction of Y along certain subvarieties (cf. fig. 4).

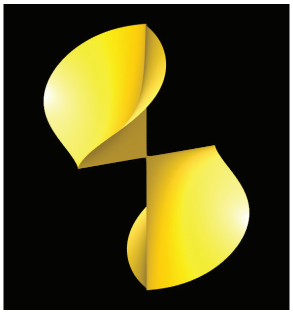

FIGURE 4. The contraction of the sphere over a cross.

Here is another viewpoint. Let $f(x, y, z)$ be an arbitrary polynomial in three variables. Solve the equation $f = 0$ for x in terms of the remaining variables y and z. Take again the Calyx $x^2 + y^2z^3 - z^4 = 0$. The solution $x = \pm(z^4 - y^2z^3)^{1/2}$ may please (although it involves a square-root) but makes immediately clear that this will not be possible in general. In the sequel, we allow the following substitutions of the variables: Polynomial coordinate changes and monomial substitutions (with certain restrictions). Is it then possible to transform the equation into a new equation which can easily be solved?

Replace the variable x by xz and get $x^2z^2 + y^2z^3 - z^4 = 0$, say $z^2(x^2 + y^2z - z^2) = 0$, with essential part $x^2 + y^2z - z^2 = 0$. Now replace the variables x and z by xy and zy. The new equation is $x^2y^2 + y^3z - y^2z^2 = 0$ and thus (after deletion of the monomial factor y^2) we get $x^2 + (y - z)z = 0$, which is seen by the coordinate change $(x, 2y, z + y)$ to be the cone $x^2 + y^2 - z^2 = 0$. Recalling our first example, we conclude that the equation of the Calyx stems from the equation of the cylinder via a sequence of permissible coordinate transformations. (If you compute the composition, you will get precisely the contraction map mentioned earlier.)

Here is a third viewpoint. Let R be a local integral domain, and let $Q = \mathrm{Quot}(R)$ be its field of fractions. Does there exist an intermediate ring $R \subseteq R' \subseteq Q$ which is a *regular* local ring? And is there an explicit construction of such a ring?

Again, let us look at an example. Take $R = \mathbb{C}[[t^3, t^4, t^5]]$ with quotient field $\mathbb{C}((t))$. It is unavoidable to guess that $R' = \mathbb{C}[[t]]$ should be a good candidate. And indeed, it is. Observe that R' is the integral closure of R. So now take the rings $R = \mathbb{C}[[x, y, z]]/(x^2 - y^2z)$ and $S = \mathbb{C}[[x, y, z]]/(x^3 - y^2 - z^2)$ (cf. fig. 5). Would

FIGURE 5. The Whitney-umbrella $x^2 = y^2z$ and the Fanfare $x^3 = y^2 + z^2$.

you have seen that the integral closure $R' = \overline{R} = \mathbb{C}[[\frac{x}{y}, y, z]]/((\frac{x}{y})^2 - z)$ of R again does the job, whereas the integral closure $\overline{S}$ of S fails?

All this leads to the question: What is the correct strategy to resolve singularities?

Recipes

Geometrically, the most natural approach to resolution is via Nash modifications. Associate to the regular part X° of X its tangent bundle $\mathrm{T}X^\circ$. Then take its Zariski-closure. This boils down to attaching to regular points one element of a suitable Grassmannian and to singular points all the limits of tangent spaces at regular points nearby. This construction loosens the way the regular part of X moves in the singularities, and one would hope that a finite iteration yields a smooth variety. Unfortunately, this is not the case, and already for surfaces the method has to be complemented with normalizations (see below). In higher dimensions, aside from the restriction to real or complex numbers, no substantial results on how to use Nash modifications for resolution seem to be known.

Ring-theoretically, the passage to the integral closure is known in algebraic geometry as normalization. Introduced in the 19th century, Zariski exploited it systematically to improve singularities. Essentially, normalization succeeds in removing all singularities in codimension 1; i.e., it allows us to reduce to a singular locus of codimension at least 2. This is fine for curves, where codimension 2 means smooth, and halfway okay for surfaces, where it implies having isolated singular points. It remains unknown how to refine the concept of integral closure so as to eliminate singularities of higher codimension.

Algebraically, the principal device for resolution purposes is blowups. These are birational maps which allow us to unfold the singular structure. Their effect, however, is minimal, so that one needs lots of blowups to get a smooth variety. This, in turn, suggests using induction: Each blowup improves the singularities in a measurable sense exhibited by a resolution invariant belonging to a well-ordered set. The invariant drops after each blowup, and induction applies.

This inductive strategy dominates the work on resolution in the last seventy years, starting with Zariski in the forties with various treatments of surface and threefold resolution (in characteristic zero), continued by his students Abhyankar (resolution of surfaces and threefolds in positive characteristic) and Hironaka (resolution in arbitrary dimension but characteristic zero) in the fifties and sixties, and conceptualized and made constructive since the eighties by Giraud, Villamayor, Bierstone-Milman, Bodnár-Schicho and Encinas-Hauser. We also mention the important contributions of Lipman (resolution of excellent two-dimensional schemes); Cossart and Moh (threefolds in positive characteristic), Cutkosky, Włodarzcyk, Abramovich, Karu, Matsuki (monomialization and factorization of morphisms); Kuhlmann, Teissier (local uniformization); de Jong (weak resolution in positive characteristic and arbitrary dimension via alterations); and Seidenberg, Cano, Aroca, Panazzolo (resolution of vector fields and foliations). (We apologize for any omissions.)

Despite these tremendous efforts, resolution of singularities keeps its flavor as a difficult and hardly accessible topic within algebraic geometry. This is not totally justified: first, because the ideas and techniques are extremely rich and structured, yielding an amazing building of mutual interactions, and, secondly, because there have appeared lately papers which are very readable and present a completely purified exposition of the main arguments.

Moreover, resolution is an extremely interesting and challenging field for future research. It can be done both on a very elementary level or with big machinery. In characteristic zero, we still have a very vague understanding of how to resolve singularities (we just have a method). More particularly, there should exist a natural construction of an ideal supporting the singular locus of the variety such that blowing up the variety with this (non-reduced and possibly singular) center resolves all singularities in *one stroke*. Find it!

And there remains the still unsolved case of positive characteristic p and the arithmetic situation over the integers $\mathbb{Z}$. Work to be done!

The book

The book under review gives a thorough presentation of resolution of singularities of curves and surfaces (the latter over an algebraically closed field of characteristic zero). It is organized in eight chapters and two appendices, where the authors carefully work out every single detail in the intricate puzzle that these resolution processes require. The dedicated reader will meet one by one all the pieces of the puzzle in the first seven chapters and spend an instructive time putting them together in the last one.

While resolution of curves can easily be achieved by normalizing, for many applications it is useful to have some control of the process: blow up singular points repeatedly until normalization is reached, or use local algebra arguments to compare the singularities in a sequence of blowups. The latter method leads to *embedded desingularization* of curves, which plays an important role in induction arguments. Both techniques are carefully described in the first section of Chapter VIII.

Resolution of surfaces is a major task since, as mentioned above, normalization will often not suffice. Section 2 of Chapter VIII is dedicated to an exposition of resolution of surface singularities "à la Jung": Given an irreducible surface, project it to a suitable smooth surface and then *force* the image of the singular points to be contained in a divisor with normal crossings support. Once this situation is achieved, the normalization of the surface has only *toric singularities*, which can easily be resolved after a finite number of blowups in points.

The final sections of Chapter VIII are dedicated to Zariski's approach to desingularization of surfaces using valuation theory. In this case the procedure can be quickly described as follows: Repeatedly alternate blowups of singular points and normalization. Here the *Uniformization Theorem* (i.e., the existence of a suitable regular local ring that dominates a sequence of blowups and normalizations over a singular point) guarantees that smoothness is reached after a finite number of steps.

Not a single detail of any of these proofs is left out. The authors have made a major effort to make this book as self-contained as possible: The setup needed for the Uniformization Theorem is presented in Chapter I (Valuation Theory) and Chapter III (Ramification Theory). Chapter II, on one-dimensional Cohen-Macaulay rings, is needed for desingularization of curves; and Chapter VII, on two-dimensional regular local rings, is needed for embedded resolution of curves and desingularization of surfaces. Chapters IV, V and VI on formal and convergent power series rings, quasi-ordinary singularities and resolution of two-dimensional toric singularities, respectively, develop the machinery needed for the proof of resolution of surface singularities following Jung's strategy. The book ends with two appendices containing generalities and classical results in both algebraic geometry and commutative algebra.

Upshot: This book invites the reader on a beautiful journey through some classical results in resolution of singularities, a topic frequently used but seldom understood in complete detail.

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