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Computational aspects of polynomial identities, by Alexei Kanel-Belov and Louis Halle Rowen, Research Notes in Mathematics, vol. 9, A K Peters, Ltd., Wellesley, MA, 2005, xxi + 378 pp., US$69.00, ISBN 1-56881-163-2

Let k be an infinite field, and let $k\langle X\rangle = k\langle x_1, x_2, \dots\rangle$ be a free associative algebra over k in countably many variables. If $f(x_1, \dots, x_n) \in k\langle X\rangle$ and $r_1, \dots, r_n$ are elements of a k-algebra A, then one can make the evaluation $f(r_1, \dots, r_n) \in A$. If all such evaluations on A are zero, then f is called a *polynomial identity* for A. The zero polynomial is a polynomial identity for any k-algebra A, and it may be the only polynomial identity for A. If some nonzero f is a polynomial identity for A, then one says that A is a *polynomial identity algebra over* k or, more simply, a *polynomial identity ring*, or *PI-ring*. Examples of PI-rings are commutative rings, matrix rings over commutative rings, finite-dimensional algebras, algebraic algebras of bounded degree, and exterior algebras. Commutative rings satisfy the polynomial $x_1x_2 - x_2x_1$, and "satisfying a polynomial identity" is a generalization of commutativity. The set of f that vanish on A forms an ideal $\mathcal{T}(A)$ in $k\langle X\rangle$ called the *T-ideal* of identities of A. It is easy to see that the T-ideals of $k\langle X\rangle$ are precisely those ideals which are closed under k-endomorphisms of $k\langle X\rangle$. Polynomial identities are also defined for algebras over arbitrary commutative rings, but this involves a minor technicality which is best avoided in a review.

The theory of polynomial identity rings has two branches, a ring-theoretic or structural and a combinatorial or computational, both of which have well-defined sources. The former, which began with a paper of I. Kaplansky [13] in 1948, studies the ring-theoretic properties of polynomial identity rings. The latter, which began with a paper of W. Specht [31] in 1950, studies T-ideals. A handful of significant earlier papers by M. Dehn [6] in 1922, W. Wagner [32] in 1937, and M. Hall [12] in 1943 involved polynomial identities, but it did not become a well-defined field until 1948. Indeed, the term "polynomial identity" was introduced by Kaplansky, and "T-ideal" was introduced by Specht.

Kaplansky not only began the field of polynomial identities, but he proved the single most important PI-theorem: *A primitive PI-ring is a finite-dimensional central simple algebra over its center, which is a field.* Kaplansky's Theorem is the precursor to two other major structure theorems, Posner's Theorem and Artin's Theorem, both of which retain the PI-hypothesis and replace the hypothesis of primitivity with a less restrictive one. Specht's paper was rather formal and did not contain any groundbreaking result like Kaplansky's Theorem, but it did provide the foundation for further research in the quantitative side of PI-theory. He assumed that the base field k had characteristic zero, which has two advantages. First, over a field of characteristic zero any T-ideal is generated by the multilinear polynomials it contains. Second, the multilinear polynomials in $x_1, \dots, x_n$ are a module over S_n, the symmetric group on n letters, which means that the highly developed representation theory of S_n in characteristic zero is available. He also posed *Specht's Problem: If k has characteristic zero, is every T-ideal in $k\langle X\rangle$ finitely generated as a T-ideal?* (For J to be generated *as a T-ideal* by a set A means that J is

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the smallest ideal which contains A and is closed under endomorphisms of $k\langle X\rangle$.) I will outline the major combinatorial developments in PI-rings since 1950.

The first significant quantitative result was the Amitsur-Levitzki Theorem [2], which appeared in 1950. It says that the *standard polynomial* of degree $2n$,

$$\mathcal{S}_{2n}(x_1,\dots,x_{2n}) = \sum\{sign(\pi)x_{\pi(1)}\cdots x_{\pi(2n)} \mid \pi \in S_{2n}\},$$

is a polynomial identity for $M_n(k)$, and is, up to scalar multiplication, the unique polynomial of minimal degree satisfied by $M_n(k)$. The Amitsur-Levitzki Theorem is the first step toward a description of $\mathcal{T}(M_n(k))$, the T-ideal of identities satisfied by $M_n(k)$. Although a great deal more is now known about this ideal, many natural questions about it remain open.

From 1950 to 1972, nearly all significant PI-results were structural. A partial exception was S. A. Amitsur's theorem [1] that the only nonzero prime T-ideals are the $\mathcal{T}(M_n(k))$, the T-ideal of identities satified by $M_n(k)$. Although it is an important fact about T-ideals, it is not a quantitative one, and its proof is entirely ring-theoretic. The most important quantitative theorem in this period was proved by A. I. Shirshov [28], [29] via the combinatorics of words (i.e., monomials in $x_1, x_2, \dots$) and is more general than the form given here, since it applies to some nonassociative algebras. If C is a commutative ring, an *affine C-algebra* is a ring $A = C\{a_1,\dots,a_m\}$ which is finitely generated as a C-algebra. Equivalently, A is a C-algebra homomorphic image of a free associative algebra $C\langle x_1,\dots,x_m\rangle$, with m finite. His fundamental result is Shirshov's Height Theorem: *Let C be a commutative ring, and let $A = C\{a_1,\dots,a_m\}$ be an affine C-algebra generated by m elements. Suppose that A satisfies a polynomial identity of degree d. Then there is an integer $\mu = \mu(d,m)$ (which depends only on d and m) such that A is generated as a C-module by the set of words $w_1^{i_1}w_2^{i_2}\cdots w_\mu^{i_\mu}$ in which each w_j is a word in $a_1, a_2, \dots, a_m$ of length $\le d$, and $i_1, i_2, \dots, i_\mu$ are nonnegative integers.* Shirshov's Height Theorem has many important ring-theoretic consequences, most of which depend on the following corollary: *Let $A = C\{a_1,\dots,a_m\}$ be an affine algebra over a commutative ring C. Suppose that A satisfies a polynomial identity of degree d and that every word in $a_1,\dots,a_m$ of length $\le d$ is integral over C. Then A is finitely generated as a C-module.*

The next major quantitative result arose with A. Regev's theorem that the tensor product of PI-rings is a PI-ring [24], a structural result. Its proof, however, is combinatorial and required putting some teeth in the formalism introduced by Specht. Recall that Specht studied the k-vector subspace V_n of $k\langle X\rangle$ of dimension $n!$ spanned by the monomials which are permutations of $x_1,\dots,x_n$. As an S_n-module, V_n is isomorphic to the group algebra $k[S_n]$. Moreover, if A is a ring and $\mathcal{T}(A)$ is the T-ideal of identities satisfied by A, then $\mathcal{T}(A)\cap V_n$ is an S_n-submodule of V_n. It had already been proved by Kaplansky [13] that every ring which satisfies a polynomial identity satisfies a multilinear polynomial identity. Thus A is a PI-ring if and only if $\mathcal{T}(A)\cap V_n \neq 0$ for some n. It is natural to ask whether $\mathcal{T}(A)\cap V_n$ is a large or small subspace of V_n as $n \to \infty$. It turns out that it is large, which motivates a definition: *The nth codimension of A is $c_n(A) = dim_k(V_n/(\mathcal{T}(A)\cap V_n))$.* The main lemma of Regev, as sharpened by V. N. Latyshev [19] is: *If A satisfies a polynomial identity of degree d, then $c_n(A) \le (d-1)^{2n}$.* The tensor product theorem then follows from the following easier fact, which was Regev's starting point: *If A and B are PI-rings, then $c_n(A\otimes B) \le c_n(A)c_n(B)$.* For if A and B satisfy PI's of

degrees d and e respectively, then the two results above say that $c_n(A \otimes B) < (de)^{2n}$. Since $(de)^{2n} < n!$ for large enough n, this means that $\mathcal{T}(A \otimes B) \cap V_n \neq 0$ for large n, and hence $A \otimes B$ satisfies a polynomial identity.

Since $M_n(A) = A \otimes_k M_n(k)$, the tensor product theorem of Regev has the immediate corollary that $M_n(A)$ is a PI-ring whenever A is a PI-ring. It was also used by A. R. Kemer [17] to prove that any PI-algebra over a field of characteristic $p > 0$ satisfies a standard identity, and by A. Z. Anan'in [3] to prove that a left Noetherian PI-algebra which is affine over a field can be embedded in $M_n(K)$ for some integer n and some field K. However, the greater part of research inspired by Regev's paper concerns the *codimension sequence*, $\{c_n(A)\}$, and the related *cocharacter sequence*, which is the sequence of characters of the S_n-modules $\mathcal{T}(A) \cap V_n$, assuming k has characteristic zero. It has remained an active area for thirty years and has been the source of a wealth of combinatorial results, not only for T-ideals, but also for representations of the symmetric group and the evaluation of certain multi-integrals. The general results about the codimension and cocharacter sequences are asymptotic estimates, and the sequences are exactly known for only a handful of PI-rings. These asymptotic estimates have often involved the evaluation of multi-integrals, and the evaluation of the Mehta integral in the 1980's—which was eventually accomplished by rediscovering integrals of A. Selberg and L. Euler—began when R. Stanley observed that some integrals which appeared in a talk by Regev in 1978 about codimension sequences were Mehta integrals.

Shortly after Regev's Theorem came the construction of central polynomials for $M_n(k)$ by E. Formanek [8] and Y. P. Razmyslov [21]. A *central polynomial* for A is a polynomial $f(x_1, \ldots, x_s) \in k\langle X \rangle$ such that: (a) f has constant term zero; (b) f is not a polynomial identity for A; and (c) for any $a_1, \ldots, a_s \in A$, $f(a_1, \ldots, a_s)$ lies in the center of A. The constructions are combinatorial, but the main applications of central polynomials are to structure theory. In contrast to the tensor product theorem, the existence of central polynomials was essential to the solutions of many long-standing open problems, especially extensions of basic theorems about affine commutative rings to affine PI-rings. For example: *Prime ideals satisfy the descending chain condition in affine PI-rings* (L. W. Small [30]). *Affine PI-rings are catenary* (*maximal chains of prime ideals have the same length*) (W. Schelter [26]). *The Jacobson radical of an affine PI-ring is nilpotent* (Y. P. Razmyslov - A. R. Kemer - A. Braun [22], [14], [5]). Other than Kaplansky's Theorem, the existence of central polynomials is the most important result in structural PI-theory, where most of its applications lie. However, some quantitative results about the T-ideal of identities of $M_n(k)$ also involve central polynomials.

A couple of years later Razmyslov [22] defined the Capelli polynomial, and Razmyslov [23] and C. Procesi [20] rediscovered and formalized the theory of trace identities for $M_n(k)$. The latter had already been introduced and used by B. Kostant in 1958 [18] to give a new proof of the Amitsur-Levitzki Theorem.

The n-th *Capelli polynomial* is

$$\mathcal{C}_n(x_1, \ldots, x_n, y_1, \ldots, y_n) = \sum \{sign(\pi) x_{\pi(1)} y_1 x_{\pi(2)} y_2 \cdots x_{\pi(n)} y_n \mid \pi \in S_n\}.$$

Note that the standard polynomial is a specialization of the Capelli polynomial: $\mathcal{S}_n(x_1, \ldots, x_n) = \mathcal{C}_n(x_1, \ldots, x_n, 1, \ldots, 1)$. The Capelli polynomial played a role both in the Razmyslov-Kemer-Braun Theorem mentioned above and Kemer's classification of verbally prime T-ideals below.

Trace identities are best illustrated by an example rather than a formal definition. Letting $tr(X)$ denote the trace of a matrix X,

$$tr(X_1)tr(X_2) - tr(X_1X_2) - tr(X_1)X_2 - tr(X_2)X_1 + X_1X_2 + X_2X_1$$

and

$$\begin{aligned} tr(X_1)tr(X_2)tr(X_3) - tr(X_1X_2)tr(X_3) - tr(X_1)tr(X_2X_3) \\ -tr(X_2)tr(X_1X_3) + tr(X_1X_2X_3) + tr(X_2X_1X_3) \end{aligned}$$

are both expressions which vanish when $X_1, X_2, X_3 \in M_2(k)$. The former is called a *mixed trace polynomial* because it involves both traces and monomials, and the latter is called a *pure trace polynomial* because it involves only traces. The latter is obtained from the former by multiplying on the right by X_3 and applying tr. The vanishing of one thus implies the vanishing of the other, because $tr(YZ)$ is a nondegenerate bilinear form on $M_n(k)$: If Y is fixed and $tr(YZ) = 0$ for all $Z \in M_n(k)$, then $Y = 0$.

The main theorem on trace identities describes all multilinear trace identities in r variables satisfied by $M_n(k)$ over a field k of characteristic zero in terms of the group algebra $k[S_r]$. Since ordinary polynomial identities are mixed trace identities in which only monomials and no traces occur, one gets a lot of information about $\mathcal{T}(M_n(k))$.

However, I want to emphasize that there remain quite natural and easy-to-state questions about the identities of $M_n(k)$ (even when the characteristic of k is zero) which remain unanswered in spite of trace identities and the positive solution of Specht's problem. For example, except for very small n, the following are all unknown for $M_n(k)$: the least degree of a central polynomial, the least degree of a two-variable polynomial identity, the exact codimension and cocharacter sequences of $M_n(k)$, and an explicit finite generating set for $\mathcal{T}(M_n(k))$.

By the early 1980's, Specht's problem was the most important open problem in PI-rings. Given that the analogous question in group theory (Are varieties of groups finitely based?) had a negative answer, it was a surprise when Kemer proved that every T-ideal is finitely based over a field of characteristic zero [15], [16]. This theorem, the deepest in PI-theory, was based on a series of major results of greater importance than the Specht problem itself. The single most significant of these results is the classification of verbally prime T-ideals (a T-ideal J is *verbally prime* (or *T-prime*) if $J \supseteq J_1$ or $J \supseteq J_2$ whenever J_1 and J_2 are T-ideals such that $J \supseteq J_1J_2$). It was noted above that Amitsur proved much earlier [1] that the only prime T-ideals are the zero ideal and $\mathcal{T}(M_n(k))$, the T-ideal of identities of $M_n(k)$. Another important theorem of Kemer is the following: *If A is an affine k-algebra over a field of characteristic zero, then there is a finite-dimensional k-algebra B such that $\mathcal{T}(A) = \mathcal{T}(B)$.*

In fact, Kemer proves "super", or $\mathbb{Z}_2$-graded, versions of the above results, which requires introducing free $\mathbb{Z}_2$-graded algebras, $\mathbb{Z}_2$-graded T-ideals, supertraces, etc. The ungraded versions are deduced from the $\mathbb{Z}_2$-graded versions. Their introduction is essential to his classification of verbally prime T-ideals, since the nonzero verbally prime T-ideals are $\mathcal{T}(M_n(k))$, $\mathcal{T}(M_n(E))$, and $\mathcal{T}(M_{k,\ell})$, where E is the exterior or Grassmann algebra, which is $\mathbb{Z}_2$-graded, and the $M_{k,\ell}$ are certain $\mathbb{Z}_2$-graded algebras. It is curious that just when superalgebras became a hot topic in the 1980's—also making a big impact in physics—they turned out to be essential for

the solution of Specht's problem, whose statement makes no reference to them. Kemer has written a monograph which gives a full account of his proof [16].

After Kemer's solution of Specht's problem, interest turned to algebras over fields of characteristic $p > 0$, where the situation is much less well behaved, although parts of Kemer's program remain. Above all, T-ideals are no longer finitely generated, as was shown by A. Belov [4], A. V. Grishin [11] and V. V. Shchigolev [27] in 1999. This remains an area of active research, as does the asymptotics of Regev's codimension and cocharacter series of T-ideals. For example, Giambruno and Zaicev [9] have confirmed a conjecture of Amitsur (later refined by Regev [25]) which gives an estimate for the growth of the codimensions.

The book under review is not intended to be either introductory or encyclopedic. The main structure theorems are stated without proof, and among combinatorial results, the Amitsur-Levitzki Theorem, codimensions and cocharacters of T-ideals, central polynomials, and trace identities are given brief treatments.

Most of the book is devoted to a thorough exposition of Kemer's solution of Specht's problem in characteristic zero and an up-to-date exposition of much, but by no means all, of the research in computational PI-theory since 1990. This includes counterexamples to Specht's problem and other work on T-ideals in characteristic $p > 0$, improvement of the bounds in Shirshov's Height Theorem, and rationality of Hilbert series associated with relatively free algebras.

There is little overlap with the recent books of V. Drensky - E. Formanek [7] and A. Giambruno - M. Zaicev [10]. The former is aimed at nonexperts, and the latter, although aimed at active researchers, is more narrowly focused on the asymptotic study of codimensions and cocharacters.

The exposition of Kemer's theory is very welcome, since it is fundamental to the study of T-ideals and underlies much of the PI-research of the past fifteen years. Except for Kemer's 1991 monograph [16], it has not appeared in a book before. This part of the book will be useful to all algebraists who have contact with PI-rings. The other topics are more recent and still the subject of active research. Their definitive form remains to be determined, and this part of the book will be of greatest interest to active researchers in these topics.

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