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Gesammelte Werke. Vol. II, *Grundzüge der Mengenlehre*, by Felix Hausdorff, Springer, Berlin, 2002, xvi + 884 pp., ISBN 3-540-42224-2; Vol. IV, *Analysis, Algebra, und Zahlentheorie*, Springer, 2002, xix + 554 pp., ISBN 3-540-41760-5; Vol. V, *Astronomie, Optik, und Wahrscheinlichkeitstheorie*, Springer, 2006, xviii + 933 pp., ISBN 3-540-30624-2; Vol. VII, *Philosophisches Werk*, Springer, 2002, xx + 920 pp., ISBN 3-540-20836-4

Four volumes of the planned nine-volume edition of the *Gesammelte Werke* (Collected Works) of Felix Hausdorff have now been published, and we can expect the rest at the rate of about one a year. With their appearance we can for the first time begin to appreciate the life and work of this remarkable man. It is likely that mathematicians will be surprised with the publication of volume VIII, which will be devoted to Hausdorff's literary output, because Hausdorff wrote a play that proved to be quite successful in its day. And if they do not look in volume VII, they will fail to get the measure of the man, because Hausdorff's philosophy was important for his later development as a mathematician.

Hausdorff began his mathematical life as a student of Bruns and wrote a thesis on problems in terrestrial astronomy, reprinted here in volume V, but, for reasons that were not entirely Hausdorff's fault, this was not a truly satisfactory piece of work and he dropped out of mathematics for a time. He also took up an assumed name, Paul Mongré (it can be translated as Paul to my liking) and began to write aphorisms and philosophical works as an upbeat Nietzschean. He became better known outside the world of mathematics under this name. Indeed, the *Handbuch des jüdischen Wissens*, which was a dictionary of Jewish culture published in Berlin in 1936, did not list Hausdorff among the 46 Jewish mathematicians in the article on that subject but as Paul Mongré among the Nietzscheans as an author of philosophy, lyrics, and drama (see Epple [2006]).

His philosophy was generally sceptical of most received positions. He was sympathetic to empiricism, he preferred quantitative to qualitative explanations, but he was reluctant to endorse existence claims for metaphysical entities such as force or energy. So he was particularly critical of the famous Nietzschean doctrine of the recurrence of time. Certainly, Hausdorff said, we make an ordered whole out of a potentially chaotic world; we make what he called a cosmos. But nothing allows us to infer that our constructed picture is how the universe really is. Nietzsche had argued that there are only finitely many states the world can be in, and so after a suitable period of time, a previous state will recur and the universe will cycle. Hausdorff/Mongré initially objected that the world is three-dimensional but time is one-dimensional and so a two-dimensional continuum of 'times' would be necessary before such a cycle could commence. This argument takes no notice of Cantor's 1-1 correspondence between sets of different dimensions, and within a year (1898) Hausdorff came forward with new criticisms that drew on Cantor's theory of point sets. His belief that our experience of time was compatible with many very different versions of 'absolute' time extended to similar thoughts about space. We might only be able to be aware of an everywhere dense subset of space, a topic he

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discussed in his inaugural lecture in 1903 as a professor at Leipzig (to be reproduced in the forthcoming volume VI of his *Gesammelte Werke*).

From 1898, Hausdorff was drawn ever more deeply into the theory of sets and point-set topology, and the result was his celebrated book on the subject. He gave his first lecture course on set theory in 1901 (one of the first such courses ever taught), and in that year he published his first paper on order types of sets. David Hilbert's *Grundlagen der Geometrie* of 1899 had made a profound impression on Hausdorff. A letter he wrote to Hilbert in October 1900 and many sheets in Hausdorff's *Nachlaß* document the attention which Hausdorff gave to this new style of mathematical thinking. But his major paper on set theory, his *Grundzüge einer Theorie der geordneten Mengen*, published in *Mathematische Annalen* in 1908 (and to be reproduced in volume I of the *Gesammelte Werke*), did not give an axiomatic treatment of set theory. Here it is very pleasant to be able to note that several of Hausdorff's earlier papers on order types, including the *Grundzüge*, have recently been translated into English with an introduction and commentaries by J.M. Plotkin; see Plotkin [2005].

Volume II of the *Gesammelte Werke* is devoted to Hausdorff's *Grundzüge der Mengenlehre*, which, as the editors point out, is his most important work. The original edition is reproduced here rather than the better known and more readily accessible second edition. And indeed, the so-called second edition is really a new book, devoted to the subject of descriptive set theory as it emerged in the 1920s. For that reason, it will be reproduced in volume III of the *Gesammelte Werke*. The first edition is the work that introduced a generation of mathematicians to set theory in the broadest sense of the term. It contains several of Hausdorff's pioneering ideas on point-set topology, including his eponymous separation axiom. It also includes his account of measure theory and an example of a non-measurable set.

Topics covered include cardinality, ordered sets and order types, well ordered sets and ordinal numbers; and among the topological ideas are those of compact sets, convergent and divergent sets, connectedness, density, various kinds of metrical spaces, and a study of continuous functions and convergence of sequences of functions.

The edition treats this most attentively. There is a 91-page introduction by Purkert to the work, which starts by looking at Hausdorff's earlier work on ordinal types. This is followed by an analysis of what else had been done: Cantor's famous *Beiträge*, various French monographs, Schoenflies's first report to the *Jahrsbericht der Deutschen Mathematiker-Vereinigung* (1900), his second report (1908) and Sierpinski's book. Then came Schoenflies's book of 1913, which Purkert compares with Hausdorff's. In between came Russell's *The Principles of Mathematics* with its paradoxes about classes that are not members of themselves, and Purkert discusses Hausdorff's review of it. Finally, there is an analysis of the reception of the book, which considers general set theory and point set topology and then the responses of Bourbaki.

After the reprint of the book and some 40 pages of footnotes, we get a further 180 pages of commentary, consisting of 11 chapters written by specialists on particular themes in the work. Not only do these help clarify the mathematical work, they are extremely valuable historical essays. With the introduction, they establish that this volume alone is a major contribution to the history of mathematics.

Although concepts are defined and discussed, the *Grundzüge der Mengenlehre* does not give an axiomatic foundation for set theory. Hausdorff noted that Zermelo had attempted to do this in 1908, but progress was still needed – Zermelo relied on an undefined concept of definiteness that was much criticised. Therefore, he felt, an axiomatic approach was not yet the right way to present higher set theory. Rather, Hilbert's axiomatics was the right way to systematise more elaborate theories, such as geometry or probability theory. Only in the *Nachlaß* did Hausdorff consider providing axioms for topology and the theory of order types.

Volume IV contains Hausdorff's papers in analysis. Generally speaking, these were more minor, but this volume reprints his famous construction of a non-measurable set, which appeared in a paper of 1914 and which shows that, granted the axiom of choice, no definition of measure will be able to assign a measure to every bounded set in $\mathbb{R}^3$ that is invariant under Euclidean congruences. There is also his remarkable, and influential, paper on the fractional dimension of spaces, his paper on the moment problem, and his work on what is today called the Hausdorff-Young inequality. A further 100 pages on papers from the *Nachlaß* document, among other things, that Hausdorff had discovered the long line several years before Alexandroff. The volume also carries his few papers in algebra, one of which is his contribution to the Baker-Campbell-Hausdorff formula for the exponential map of a Lie algebra. Many of these papers have helpful commentaries by S.D. Chatterji, R. Remmert, and W. Scharlau. Chatterji's lucid and careful commentary, for example, on Hausdorff's fractional dimension paper greatly assists the reader both mathematically and historically by pointing out that Hausdorff's ideas led to Bouligand's work in the 1920s and then to Frostman, who made the connection to potential theory and capacity, then to Besicovitch and his associates, and ultimately to Federer.

Volume V carries his work on astronomy and optics, with which he began his mathematical career and which, as noted above, was something of a false start. It also contains his work on probability theory. Hausdorff lectured on this subject in 1923, and Chatterji observes that Hausdorff came very early to the idea that the right approach to the fundamentals of probability theory was to use measure theory, several years before Kolmogorov's decisive axiomatisation of probability theory along just those lines in 1933. This is not to say that Hausdorff preceded Kolmogorov in every respect, and Chatterji observes that precise definitions of random variables and suitable probability spaces are lacking in Hausdorff's work. 'Hausdorff', he writes (p. 742), 'sees clearly that mathematical probability is a branch of measure and integration theory but fails to make any decisive steps beyond this recognition.' Hausdorff also had a deep insight into insurance mathematics, as Purkert, in another of his very helpful commentaries, points out.

Volume VII contains his philosophical books: *Sant' Ilario, Gedanken aus der Landschaft Zarathustras* and *Das Chaos in kosmischer Auslese*, and his reviews of books by Nietzsche. Here we are given another extensive introduction, which looks at Hausdorff's philosophical orientation and the influences of Nietzsche and the neo-Kantians, but which ranges back to Spinoza and Leibniz and as far forward as Lotze, Helmholtz, and Otto Liebmann.

It is worth noting that all these volumes contain very helpful and erudite commentaries, many of which are in English, but that these are generally not listed among the contents. A casual perusal by monoglot English readers will not lead them to these instructive pages. Also, the present half-finished stage of the edition does not carry an index showing which works and notes by Hausdorff are to be included and where. This will be put right with the publication of the final volume.

Hausdorff is undoubtedly one of the finest mathematicians whose reputation has suffered from a lack of proper scholarship, and the mathematical community owes a great debt of gratitude to the editors and publishers for this series of volumes. The editors are Egbert Brieskorn, who is to be thanked for having started the project and got it off the ground, Friedrich Hirzebruch, Walter Purkert, Reinhold Remmert, and Erhard Scholz; and on the evidence so far they have done a magnificent job of ordering the material, selecting the commentators, and producing the volumes. In particular, Purkert has managed the details of the editorial job with a high degree of accuracy and tact. Springer is producing an extremely handsome set of books that should be a permanent monument, one which is all the more welcome in the era of the Web. Ease of reference is one thing, but study is another; these are volumes to be studied. I wish to thank Walter Purkert and Moritz Epple for their help with the preparation of this review.

References

[2006] Epple, M., 2006, Felix Hausdorff's Considered Empiricism, pp. 263-290, in *The Architecture of Modern Mathematics*, J. Ferreirós and J.J. Gray (eds), Oxford University Press.

[2005] Plotkin, J.M., 2005, *Hausdorff on Ordered Sets*, American and London Mathematical Societies, History of Mathematics Sources 25, Providence, Rhode Island. MR2187098 (2007a:01016)

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