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Stochastic calculus of variations in mathematical finance, by Paul Malliavin and Anton Thalmaier, Springer-Verlag, Berlin, 2006, xii+142 pp., US\$59.95, ISBN 978-3-540-43431-3

The stochastic calculus of variations or Malliavin calculus is a differential calculus on a Gaussian space, introduced by Malliavin in [6] in order to provide a probabilistic proof of Hörmander's hypoellipticity theorem. The basic ideas and results of Malliavin calculus can be presented in the framework of a Gaussian measure on a finite dimensional space as follows.

Suppose that γ_n is the standard normal distribution on $\mathbb{R}^n$ with density

$$p(x) = (2\pi)^{-\frac{n}{2}} \exp\left(-\frac{|x|^2}{2}\right).$$

Denote by $C_b^k(\mathbb{R}^n)$ the space of k-times continuously differentiable functions on $\mathbb{R}^n$ which are bounded together with all their first k partial derivatives. Given $F \in C_b^1(\mathbb{R}^n)$ we denote its gradient by $\nabla F = (\partial_1 F, \dots, \partial_n F)$, where $\partial_i = \frac{\partial}{\partial x_i}$. The adjoint of ∇ with respect to the measure γ_n has the following expression:

$$\delta(u) = \sum_{i=1}^{n} (u_i x_i - \partial_i u_i) = \langle u, x \rangle - \text{div} u,$$

where $u \in C_b^1(\mathbb{R}^n; \mathbb{R}^n)$. That is, the following duality formula holds:

$$\int_{\mathbb{R}^n} \langle \nabla F(x), u(x) \rangle \, d\gamma_n(x) = \int_{\mathbb{R}^n} F(x) \delta(u)(x) d\gamma_n(x). \tag{0.1}$$

For any $p \geq 1$ we can define a seminorm on $C_b^k(\mathbb{R}^n)$ by

$$\|F\|_{k,p}^p = \int_{\mathbb{R}^d} \left(|F(x)|^p + \sum_{j=1}^{k} \sum_{\alpha_i \in \{1,\dots,n\}} |\partial_{\alpha_1} \cdots \partial_{\alpha_j} F(x)|^p \right) d\gamma_n(x).$$

Then, the completion of $C_b^k(\mathbb{R}^n)$ with respect to the seminorm $\|\cdot\|_{k,p}$, denoted by $\mathbb{D}^{k,p}(\mathbb{R}^n)$, is the Banach space of functions on $\mathbb{R}^n$ for which all derivatives up to order k in the distribution sense belong to $L^p(\gamma_n)$. The following result is a basic tool in the development of Malliavin calculus.

Theorem 0.1. *For any $p > 1$ and any integer $k \geq 1$, the operator δ is continuous from $\mathbb{D}^{k,p}(\mathbb{R}^n; \mathbb{R}^n)$ into $\mathbb{D}^{k-1,p}(\mathbb{R}^n)$, and there exists a constant $c_{k,p}$, not depending on the dimension n, such that*

$$\|\delta(u)\|_{k,p} \leq c_{k,p} \|u\|_{k-1,p} .$$

This theorem was first proved by P. A. Meyer using the Littlewood-Paley inequality, and Pisier gave in [10] an analytic proof based on the boundedness in L^p of the Hilbert transform.

These notions and results can be extended to infinite dimensional Gaussian spaces, and the most interesting example is the Wiener space. Denote by $\mathcal{W}$ the space of continuous functions $\omega : [0,1] \to \mathbb{R}$ vanishing at $t = 0$. Let $\mathcal{F}$ be the

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Borel σ-field on $\mathcal{W}$, and let γ be the *Wiener measure* on $(\mathcal{W}, \mathcal{F})$. This measure is characterized by its values on cylindrical sets: For all $0 \leq t_1 < \cdots < t_m$, and $a_i < b_i$, $1 \leq i \leq m$,

$$\gamma\left(\omega : a_i \leq \omega(t_i) \leq b_i, 1 \leq i \leq m\right)$$
$$= \int_{a_m}^{b_m} \cdots \int_{a_1}^{b_1} \prod_{i=1}^{m} \frac{1}{\sqrt{2\pi(t_i - t_{i-1})}} e^{-\frac{(x_i - x_{i-1})^2}{2(t_i - t_{i-1})}} dx_1 \cdots dx_m,$$

with the convention $x_0 = 0$. This is equivalent to saying that the canonical stochastic process $W = \{W_t, t \in [0,1]\}$ on the probability space $(\mathcal{W}, \mathcal{F}, \gamma)$ defined by the evaluations $W_t(\omega) = \omega(t)$ is a *Brownian motion*; that is, it satisfies the following properties:

(1) $W_0 = 0$.
(2) For all $s < t$, $W_t - W_s$ has the normal distribution $N(0, t-s)$.
(3) W has independent increments.

It turns out that the Wiener measure γ on $\mathcal{W}$ is carried by the Hölder continuous functions with exponent $\alpha < 1/2$.

Consider a random variable $F : \mathcal{W} \to \mathbb{R}$. The derivative DF of F is introduced as a random element in the Hilbert space $H := L^2([0,1])$ such that, for any $h \in L^2([0,1])$, the scalar product $\langle DF, h\rangle_H$ equals the directional derivative of F along $\int_0^\cdot h_s ds$:

$$\int_0^1 D_t h_t F dt = \frac{d}{d\varepsilon} F\left(\omega + \varepsilon \int_0^\cdot h_s ds\right)|_{\varepsilon=0}.$$

This definition leads to the following formula for any $f \in C_b^1(\mathbb{R}^m)$:

$$D_t(f(W_{t_1}, \ldots, W_{t_m})) = \sum_{i=1}^{m} \partial_i f(W_{t_1}, \ldots, W_{t_m}) \mathbf{1}_{[0,t_i]}(t).$$

Roughly speaking, the coordinates $i = 1, \ldots, d$ are replaced here by a continuous parameter $t \in [0,1]$ representing time.

The adjoint of the derivative, called the *divergence*, acts on H-valued random variables, that is, stochastic processes $u = \{u_t, t \in [0,1]\}$, and the duality relationship (0.1) between D and δ reads

$$E(F\delta(u)) = E(\langle DF, u\rangle_H), \tag{0.2}$$

where E denotes the mathematical expectation with respect to the Wiener measure.

The divergence operator coincides with the Itô stochastic integral; that is,

$$\delta(u) = \int_0^1 u_t dW_t$$

if u is an adapted and square integrable stochastic process. This property has been the starting point for the development of anticipating stochastic calculus by Nualart and Pardoux (see [8]).

On the other hand, the stochastic calculus of variations provides an explicit formula for the integral stochastic representation of a random variable $F \in \mathbb{D}^{1,2}$, by means of the so-called Clark-Ocone formula:

$$F = E(F) + \int_0^1 E(D_t F | \mathcal{F}_t) dW_t,$$

where $\mathcal{F}_t$ is the σ-field generated by the random variables $\{W_s, 0 \leq s \leq t\}$.

The duality formula (0.2) leads to integration-by-parts formulas which constitute the main ingredient in the applications of Malliavin calculus. For example, consider the case of a one-dimensional random variable $F \in \mathbb{D}^{1,p}$, $p > 1$. Using formula (0.2), we obtain, for any test function $\varphi \in C_b^\infty(\mathbb{R})$ and random variable G,

$$E(\varphi'(F)G) = E\left(\left\langle D(\varphi(F)), \frac{GDF}{\|DF\|_H^2} \right\rangle_H\right) = E\left(\varphi(F)\delta\left(\frac{GDF}{\|DF\|_H^2}\right)\right), \tag{0.3}$$

provided $\delta(GDF/\|DF\|_H^2)$ exists. In particular, this implies that if $\frac{DF}{\|DF\|_H^2}$ belongs to $\mathbb{D}^{1,q}(H)$ for some $q > 1$, then F has a continuous density given by

$$p(x) = E\left(\mathbf{1}_{\{F>x\}}\delta\left(\frac{DF}{\|DF\|_H^2}\right)\right).$$

Consider the case of a d-dimensional random vector $F : \Omega \to \mathbb{R}^d$ such that $F^i \in \mathbb{D}^{1,2}$ for $i = 1, \dots, d$. The *Malliavin covariance matrix* of F is the symmetric nonnegative definite matrix defined by

$$\sigma_F^{ij} = \langle DF^i, DF^j \rangle_H.$$

If the Malliavin matrix is nonsingular a.s., then the probability law of F (that is, the image measure $\gamma \circ F^{-1}$ on $\mathbb{R}^d$) is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$. This result was proved by Bouleau and Hirsch using the theory of Dirichlet forms (see [2]).

Under additional stronger conditions one can obtain the regularity of the density. Set $\mathbb{D}^\infty = \cap_{k,p}\mathbb{D}^{k,p}$. Then F is said to be *nondegenerate* if $F^i \in \mathbb{D}^\infty$ for $i = 1, \dots, d$ and $E[(\det(\sigma_F))^{-p}] < \infty$ for all $p \geq 1$. The main result on the regularity of the density is the following.

Theorem 0.2. *Let F be a nondegenerate map. Then, the law of F has a density with respect to the Lebesgue measure, which is infinitely differentiable and belongs to the Schwartz space $\mathcal{S}(\mathbb{R}^d)$.*

This theorem can be viewed as a consequence of Watanabe's theory of Wiener distributions (see [11]). Denote by $\mathbb{D}^{-\infty}$ the space of continuous linear forms on $\mathbb{D}^\infty$. Then, the composition $T(F)$ of a tempered distribution $T \in \mathcal{S}'(\mathbb{R}^d)$ with a nondegenerate random variable F is well defined as an element of $\mathbb{D}^{-\infty}$, and the mapping $T \to T(F)$ is continuous. In particular, the density of F can be expressed as $p(x) = \delta_x(F)$.

Consider the particular case $F = S_t$, where S_t is the diffusion process on $\mathbb{R}^d$:

$$dS_t = A_0(S_t)dt + \sum_{k=1}^n A_k(S_t)dW_t^k. \tag{0.4}$$

The functions $A_k : \mathbb{R}^d \to \mathbb{R}^d$, $k = 0, \dots, n$, are infinitely differentiable with bounded derivatives, and W^k are independent Brownian motions. From Itô's formula it follows that the density of S_t satisfies the Fokker-Planck equation $\left(-\frac{\partial}{\partial t} + \mathcal{L}^*\right)p_t = 0$, where

$$\mathcal{L} = \frac{1}{2}\sum_{i,j=1}^n \left(AA^T\right)^{i,j}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^n A_0^i(x)\frac{\partial}{\partial x_i}$$

is the infinitesimal generator of the diffusion process. In this case, $p_t \in C^\infty$ means that the operator $\frac{\partial}{\partial t} - \mathcal{L}^*$ is hypoelliptic and Theorem 0.2 implies Hörmander's hypoellipticity theorem. That is, if the Lie algebra generated by the vector fields $A_1, \ldots, A_n$ at the initial condition x_0 has full rank, then $E((\det \sigma_{S_t})^{-p}) < \infty$ for all $p \geq 1$ and $t > 0$.

The Malliavin calculus has recently been applied to different problems in mathematical finance. Consider first the pricing and hedging of derivatives in a diffusion-type model. Suppose that the diffusion process S_t defined in (0.4) represents the price of a d-dimensional asset in a stock market under the risk-neutral probability measure. This implies that $A_0 = rS_t$, where r is the interest rate.

Consider a *European option* with *payoff* $\phi(S_T)$ at the expiration time T. A formula for the fair price of this option at time $t_0 < T$ is given by

$$\Phi(t,x) = e^{-rt} E\left(\phi(S_T) | S_{t_0} = x\right), \tag{0.5}$$

where $t = T - t_0$. From Feynman-Kac's formula, it follows that the price function satisfies the following backward heat equation:

$$\left.\begin{array}{rcl} \left(\frac{\partial}{\partial t} + \mathcal{L} - r\right)\Phi & = & 0 \\ \Phi(T,x) & = & \phi(x) \end{array}\right\}.$$

There are then two complementary approaches to handle the pricing problem: one uses PDE, and another one, more probabilistic, is based on formula (0.5) together with a Monte-Carlo simulation.

The Greeks are infinitesimal first or second order variations of the price functional of an option with respect to the corresponding infinitesimal variations of econometric data such as the actual price of the asset x. A methodology based on the integration by parts of Malliavin calculus has been developed from the papers by Fournié et al. [3, 4] in order to provide formulas for the Greeks better adapted to Monte-Carlo simulations.

For example, suppose $d = 1$ and consider the Black-Scholes model for the stock price S_t, $dS_t = \sigma S_t dW_t$, under the risk neutral probability, where σ is the *volatility*. That is, S_t is the geometric Brownian motion

$$S_t = S_0 e^{\sigma W_t + (r - \frac{\sigma^2}{2})t}.$$

Consider the price $\Phi(T,x)$ of a European option at time 0 given by (0.5). Then, using the integration by parts formula (0.3), we can derive the following formula for the Delta defined as $\Delta = \frac{d\Phi}{dx}$:

$$\Delta(T,x) = e^{-rT} E\left(\phi(S_T)\frac{W_T}{x\sigma T}\right).$$

In fact, $\Delta = \frac{e^{-rT}}{x} E(\phi'(S_T)S_T)$, and it suffices to apply (0.3) with $G = S_T$.

This analysis can be extended to other Greeks like the Vega, which is the derivative of the price function with respect to the volatility.

On the other hand, Itô's stochastic representation theorem implies that the Black-Scholes model is complete in the sense that any square integrable payoff H can be replicated, and Clark-Ocone's formula can be used to compute the replicating portfolio:

$$e^{-rT}H - e^{-rT}E(H) = \int_0^T u_t dW_t = \int_0^T \beta_t dS_t,$$

where

$$\beta_t = \frac{u_t}{\sigma S_t} = \frac{e^{-r(T-t)}}{\sigma S_t} E(D_t H | \mathcal{F}_t).$$

In the particular case $H = \phi(S_T)$ we get $\beta_t = \Delta(t, S_t)$; see Karatzas and Ocone [9] for the application of Malliavin calculus in hedging derivatives in a more general setting.

This monograph is devoted to an updated presentation, in a rigorous mathematical framework, of the applications of the stochastic calculus of variations in mathematical finance. There is an emphasis on the geometric point of view, and the authors are able to describe the economic meaning of the different notions in Malliavin calculus.

The first two chapters contain the basic material on the Malliavin calculus and its applications in finance. These chapters are prerequisites for the remaining parts of the book which deal with specific topics.

Chapter 1 presents the basic elements of the Malliavin calculus. The Wiener space is approximated by a dyadic discretization scheme in time. This approximation is motivated by the finite dimensionality of any financial data and for the need of fast numerical Monte-Carlo simulations.

Chapter 2 deals with the applications of Malliavin calculus in hedging and pricing of derivatives. The authors make use of the complementary PDE and probability approaches to handle these problems. For instance, the computation of Greeks can be carried over using PDE weights or pathwise weights. As an example, the Delta and Vega are evaluated, and the particular case of barrier options is discussed.

Chaper 3 is based on the paper [1]. Suppose that the one-dimensional price process with respect to the risk-free measure is given by the stochastic differential equation

$$dS_t = \sigma(S_t)dW_t - \mu(S_t)dt.$$

Consider the variation of the price during a short period of the order of a few days. The *price-volatility feedback rate* $\lambda(t)$ is introduced as the logarithmic derivative of the rescaled variation $\frac{1}{\sigma(S_t)}\frac{dS_t}{dx}$. Different formulas for $\lambda(t)$ are given. This quantity is supposed to describe the facility for the market to absorb small variations.

In Chapter 4 the authors develop the applications of the Malliavin calculus to the regularity of probability laws to obtain continuous versions of conditional expectations. New formulas for the densities are established using Riesz transforms instead of the Heaviside function.

Hypoelliptic models are motivated by the high dimensionality of the state space in interest rate models or by the low dimensionality of the variance. In Chapter 5 these types of models are analyzed. The regularity of the transition densities of hypoelliptic diffusions plays an important role here.

Chapter 6 reviews the results by Imkeller [5] on the application of the anticipative stochastic calculus to the modelization of insider traders.

In Chapter 7 the authors explain the results by Watanabe [12] on the methodology of projecting asymptotic expansions through nondegenetare maps in order to get asymptotic expansions of densities. The last two sections deal with specific problems of convergence related to the Euler scheme obtained by the authors in [7].

The last chapter reviews some approaches to the stochastic calculus of variations for processes with jumps, based on pathwise instantaneous derivatives.

In conclusion, this book aims to explain the role played by the stochastic calculus of variations in mathematical finance, and it will be useful for researchers working in these fields.

References

1. Emilio Barucci, Paul Malliavin, Maria Elvira Mancino, Roberto Renò, and Anton Thalmaier, *The price-volatility feedback rate: an implementable mathematical indicator of market stability*, Math. Finance **13** (2003), no. 1, 17–35, Conference on Applications of Malliavin Calculus in Finance (Rocquencourt, 2001). MR1968094 (2004d:91099)
2. Nicolas Bouleau and Francis Hirsch, *Dirichlet forms and analysis on Wiener space*, de Gruyter Studies in Mathematics, vol. 14, Walter de Gruyter & Co., Berlin, 1991. MR1133391 (93e:60107)
3. Eric Fournié, Jean-Michel Lasry, Jérôme Lebuchoux, and Pierre-Louis Lions, *Applications of Malliavin calculus to Monte-Carlo methods in finance. II*, Finance Stoch. **5** (2001), no. 2, 201–236. MR1841717 (2002e:91063)
4. Eric Fournié, Jean-Michel Lasry, Jérôme Lebuchoux, Pierre-Louis Lions, and Nizar Touzi, *Applications of Malliavin calculus to Monte Carlo methods in finance*, Finance Stoch. **3** (1999), no. 4, 391–412. MR1842285 (2002e:91062)
5. Peter Imkeller, *Malliavin's calculus in insider models: additional utility and free lunches*, Math. Finance **13** (2003), no. 1, 153–169, Conference on Applications of Malliavin Calculus in Finance (Rocquencourt, 2001). MR1968102 (2004b:91079)
6. Paul Malliavin, *Stochastic calculus of variation and hypoelliptic operators*, Proceedings of the International Symposium on Stochastic Differential Equations (Res. Inst. Math. Sci., Kyoto Univ., Kyoto, 1976) (New York-Chichester-Brisbane), Wiley, 1978, pp. 195–263.
7. Paul Malliavin and Anton Thalmaier, *Numerical error for SDE: asymptotic expansion and hyperdistributions*, C. R. Math. Acad. Sci. Paris **336** (2003), no. 10, 851–856. MR1990027 (2004d:60142)
8. D. Nualart and É. Pardoux, *Stochastic calculus with anticipating integrands*, Probab. Theory Related Fields **78** (1988), no. 4, 535–581. MR950346 (89h:60089)
9. Daniel L. Ocone and Ioannis Karatzas, *A generalized Clark representation formula, with application to optimal portfolios*, Stochastics Stochastics Rep. **34** (1991), no. 3-4, 187–220. MR1124835 (93b:60098)
10. Gilles Pisier, *Riesz transforms: a simpler analytic proof of P.-A. Meyer's inequality*, Séminaire de Probabilités, XXII, Lecture Notes in Math., vol. 1321, Springer, Berlin, 1988, pp. 485–501. MR960544 (89m:60178)
11. S. Watanabe, *Lectures on stochastic differential equations and Malliavin calculus*, Tata Institute of Fundamental Research Lectures on Mathematics and Physics, vol. 73, published for the Tata Institute of Fundamental Research, Bombay, 1984; notes by M. Gopalan Nair and B. Rajeev. MR742628 (86b:60113)
12. Shinzo Watanabe, *Analysis of Wiener functionals (Malliavin calculus) and its applications to heat kernels*, Ann. Probab. **15** (1987), no. 1, 1–39. MR877589 (88h:60111)

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