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Malliavin calculus with applications to stochastic partial differential equations, by Marta Sanz-Solé, Fundamental Sciences, EPFL Press, Lausanne, distributed by CRC Press, Boca Raton, FL, 2005, viii+162 pp., US$84.95, ISBN 978-0849340307

1. Introduction

Stochastic analysis is a thriving area of mathematics initiated by Wiener's introduction in 1923 [28] of a probability measure on $C[0,\infty)$ providing a trajectorial description of the Brownian motion introduced in the 1900 thesis of Bachelier and the celebrated paper of Einstein in 1905. We begin with a thumbnail sketch of the history of these developments. The early development involved integration with respect to Wiener measure and the analysis of Wiener functionals. The expansion of functionals which are L^2 with respect to Wiener measure in terms of multiple Wiener integrals was published by Wiener in 1938 [29]. Integration with respect to Wiener measure and the characterization of measures absolutely continuous with respect to Wiener measure were developed by Cameron and Martin in the 1940's [4]. Another important aspect was the determination of almost sure properties of the trajectories such as nowhere-differentiability. During the 1940's Itô [11], [12] developed his theory of stochastic integration with respect to the Wiener process and celebrated chain rule. Itô also established the existence and uniqueness of solutions to nonlinear stochastic differential equations (SDE) with Lipschitz coefficients and established the relation between multiple Wiener integrals and iterated stochastic integrals. Itô's theory of stochastic integration and SDE together with the powerful tools of martingale theory introduced by Doob were developed into a rich theory of semimartingales and stochastic analysis based on them by Paul Andre Meyer and the French school in the 1960's and 1970's. Around the mid-1970's two major new developments emerged: the theory of stochastic partial differential equations and the Malliavin calculus. The relation between these two developments is the focus of the book under review.

The study of infinite dimensional Wiener processes and stochastic partial differential equations (SPDEs) began in the 1970's and has developed into a mature subject over the past 30 years. It continues to be a challenging and important research area. SPDEs arise naturally as partial differential equations perturbed by noise as well as in numerous applications including nonlinear filtering theory, quantum field theory, population biology, wave propagation in random media, etc.

An important new tool of stochastic analysis, Malliavin calculus, was introduced by Malliavin in the mid-1970's; it overcame the difficulty of developing calculus over $C[0,\infty)$ and produced a number of remarkable results including a probabilistic proof of Hörmander's theorem on hypoelliptic differential operators. The objective of this book is to give an exposition of the application of Malliavin calculus to a class of stochastic partial differential equations which was developed by the author and her coworkers during the past few years (e.g. [21]).

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2. Gaussian noises and martingale measures

We begin by introducing the basic notions of Gaussian noises, Itô-Wiener chaos expansions and martingale measures.

Let H be a real separable Hilbert space and $(W(h), h \in H)$ be a random linear functional on H defined on a probability space $(\Omega, \mathcal{G}, \mu)$. $\{W(h)\}$ is also assumed to be a zero mean Gaussian system with covariance $\langle h_1, h_2 \rangle_H$. Given a separable Banach space B, injection from $i : H \to B$ with dense image and a Gaussian probability measure on B such that the above linear functionals have the stated Gaussian distributions, the triple (i, H, B) is called an abstract Wiener space.

For the present purposes we consider $H = L^2(A, \mathcal{A}, m)$ where $(A, \mathcal{A}, m)$ is a separable σ-finite atomless measure space. The corresponding Gaussian linear functional, W, is a white noise if $W(F), W(G)$ are independent if $F \cap G = \emptyset$ where $F, G \in \mathcal{A},\ m(F) < \infty, m(G) < \infty$. In this case we can work with the canonical probability space $(\Omega, \mathcal{G}, \mu)$ where $\Omega = \mathbb{R}^{\otimes\mathbb{N}},\ \mathcal{G} = \mathcal{B}^{\otimes\mathbb{N}},\ \mu = \mu_1^{\otimes\mathbb{N}}$ where μ_1 is a standard Gaussian measure on $\mathbb{R}^1$ and $\mathcal{B}$ denotes the Borel σ-algebra.

A basis for $L^2(\Omega, \mathcal{G}, \mu)$ is obtained as follows. First let $\{e_n\}$ be a complete orthonormal basis for H and g_n the coordinate functions. Then

$$W(h) = \sum \langle h, e_n \rangle g_n.$$

Next, let $a = (a_1, a_2, \dots),\ a_i \in \mathbb{Z}_+,\ a_i = 0,\ a.a.\ i$ and

$$H_a = \sqrt{a!} \prod_{i=1}^{\infty} H_{a_i}(W(e_i))$$

where $H_n(\cdot)$ is the Hermite function

$$H_n(x) = \frac{(-1)^n}{n!} e^{\frac{x^2}{2}} \frac{d^n}{dx^n} e^{-\frac{x^2}{2}}.$$

Then (H_a) is an orthonormal basis for $L^2(\Omega, \mathcal{G}, \mu)$. Let $\mathcal{H}_n$ denote the closed subspace generated by $(H_a, |a| = n)$ and J_n the orthogonal projection onto $\mathcal{H}_n$.

Example 1. Wiener process.

The case $A = \mathbb{R}_+$ with Lebesgue measure λ and $W_t = W([0,t))$, corresponds to the standard Wiener process.

To obtain a representation of $F \in L^2(\Omega, \mathcal{G}, \mu)$ in this case in terms of multiple Itô-Wiener integrals, we begin with

$$f(t_1, \dots, t_n) = \sum_{j_1, \dots, j_n}^{k} a_{j_1, \dots, j_n = 1} 1_{A_{j_1} \times \dots \times A_{j_n}}(t_1, \dots, t_n)$$

where the $A_{j_1}, \dots, A_{j_n}$ are pairwise disjoint elements of $\mathcal{A}$ and the $a_{j_1, \dots, j_n} = 0$ if any pair of $j_1, \dots, j_n$ coincide. Then

$$I_n(f) = \sum_{j_1, \dots, j_n}^{k} a_{j_1, \dots, j_n = 1} W(A_{j_1}) \dots W(A_{j_n}).$$

For any function f on A^n $I_n(f) = I_n(\tilde{f})$ where $\tilde{f}$ is the symmetrization of f and I_n extends to a continuous linear functional on $L^2(A^n)$ with values in $L^2(\Omega)$. Then $I_n(f)$ is also given by the iterated Itô integral

$$I_n(f) = n! \int_0^{\infty} \int_0^{t_{n-1}} \dots \int_0^{t_2} f_n(t_1, \dots, t_n) dW_{t_1} \dots dW_{t_n}.$$

We then have the Wiener chaos decomposition

$$F = E(F) + \sum_{n=1}^{\infty} I_n(f)$$

with $f_n \in L^2(A^n)$ symmetric and uniquely determined by F.

The fact that W_t and stochastic integrals with respect to W_t are continuous martingales plays a central role in Itô's theory.

Example 2. Spatially homogeneous Gaussian noise

In the formulation of SPDE in $\mathbb{R}^d$ it is natural to consider noises on $A = \mathbb{R}_+ \times \mathbb{R}^d$ whose law is invariant under spatial translation but not necessarily *white* in space. To make this precise let Γ be a non-negative definite tempered measure and define the covariance functional

$$J(\varphi, \psi) = \int_{\mathbb{R}_+} ds \int_{\mathbb{R}^d} \Gamma(dx)(\varphi(s) * \tilde{\psi}(s))(x)$$

where $\varphi,\ \psi \in \mathcal{S}(\mathbb{R}^d)$, $\tilde{\psi}(s,x) = \psi(s,-x)$ and $*$ denotes convolution. In the spectral domain there then exists a non-negative tempered measure μ on $\mathbb{R}^d$ such that

$$J(\varphi, \psi) = \int_{\mathbb{R}_=} ds \int_{\mathbb{R}^d} \mu(d\xi)\mathcal{F}\varphi(s)(\xi)\overline{\mathcal{F}\psi(s)(\xi)}$$

where $\mathcal{F}$ denotes the Fourier transform

$$\mathcal{F}\varphi(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i\langle x,\xi\rangle}\varphi(x)dx.$$

Consider the pre-Hilbert space with inner product

$$\langle \varphi, \psi \rangle_{\mathcal{E}} = \int_{\mathbb{R}^d} \Gamma(dx)(\varphi * \tilde{\psi})(x),$$

and let $\mathcal{H}$ denote the completion of $\mathcal{E}$. Then we set

$$H_T = L^2([0,T];\mathcal{H}).$$

If $\Gamma(dx) = \text{const} \cdot \delta_0(dx)$, then the resulting Gaussian noise is *space-time white noise*. Otherwise, it is said to be *coloured noise*.

For any $t \geq 0$ and $A \in \mathcal{B}_b(\mathbb{R}^d)$ let $M_t(A) = W([0,t] \times A)$ and $\mathcal{G}_t$ the σ-field generated by $M_s(A),\ 0 \leq s \leq t,\ A \in \mathcal{B}_b(\mathbb{R}^d)$. Then $(M_t(A),\ t \geq 0,\ A \in \mathcal{B}_b(\mathbb{R}^d))$ is a *martingale measure* as introduced by Walsh (cf. [26]). Walsh also developed stochastic integration with respect to martingale measures that is used to provide a precise formulation of the SPDE considered in the book under review.

3. Malliavin calculus

A major development in stochastic analysis was initiated by Malliavin in a series of papers around 1976 (see [17] and [18] for a systematic exposition of the theory). A number of alternate approaches and extensions were developed during the following decade due to Kusuoka and Stroock [14], [15], Bismut [3] [16], Watanabe [27] and others. Several other books on the subject have appeared, including [22] and [2].

We briefly sketch the elements of Malliavin calculus along the lines developed in the book under review. Let $\mathcal{S}$ be the class of random variables of the form

$$F = f(W(h_1), \ldots, W(h_n))$$

where $f \in C^\infty(\mathbb{R}^n)$ with polynomial growth and $h_i \in H$. $\mathcal{S}$ is dense in $L^2(\Omega, \mathcal{G}, \mu)$. Let $\mathbb{D}^{k,p}$ be the completion of $\mathcal{S}$ with respect to the norm $\|\cdot\|_{k,p}$ defined by

$$\|F\|_{k,p} = \|(I-L)^{\frac{k}{2}} F\|_p.$$

Here $\|\cdot\|_p$ denotes the L^p norm and L is the infinitesimal generator of the Ornstein-Uhlenbeck semigroup $P_t F = \sum_{n=0}^\infty e^{-nt} J_n(F)$, that is,

$$LF = \sum(-n) J_n(F).$$

Let

$$\mathbb{D}^\infty := \cap_{p\geq 1} \cap_{k \geq 0} \mathbb{D}^{k,p}.$$

Given $F \in \mathcal{S}$ the *Malliavin derivative* is defined by

$$DF = \sum_{i=1}^n \partial_i f(W(h_1), \ldots, W(h_n)) h_i.$$

The adjoint of D is the *Skorohod integral* $\delta(\cdot)$ with domain $Dom(\delta)$ given by the set of $u \in L^2(\Omega; H)$ such that for any $F \in \mathbb{D}^{1,2}$

$$|E(\langle DF, u\rangle_H)| \leq c\|F\|_2.$$

For u in the domain, $\delta(u)$ is characterized through the integration by parts formula

$$E(F\delta(u)) = E(\langle DF, u\rangle_H) \; \forall \; F \in \mathbb{D}^{1,2}.$$

For adapted processes, the Skorohod integral coincides with the Itô integral. If $u \in L^2(\Omega \times A)$ has Wiener chaos decomposition

$$u(t) = \sum_{n=0}^\infty I_n(f_n(\cdot, t))$$

where $f_n \in L^2(A^{n+1})$ is a symmetric function in the first n variables, then $u \in Dom(\delta)$ if and only if

$$\sum_{n=0}^\infty I_{n+1}(f_n)$$

converges in $L^2(\Omega)$. For $u \in Dom(\delta)$,

$$\delta(u) = \sum_{n=0}^\infty I_{n+1}(\tilde{f}_n)$$

where $\tilde{f}_n$ denotes the symmetrization of f_n in its $n+1$ variables.

In the case $H = L^2([0,T], \mathcal{B}([0,T]), \lambda)$, DF is a function of t, and we can represent it by

$$D_t F = \sum_n n I_{n-1}(f_n(\cdot, t)).$$

One of the key results of Malliavin calculus is the following representation of Wiener functionals, $F \in \mathbb{D}^{1,2}$, due to Clark [6] and Ocone [23] (and generalized to Itô processes by Haussmann):

$$F = E(F) + \int_0^T E(D_t F | \mathcal{F}_t) dW_t$$

where $E(\cdot|\mathcal{F}_t)$ denotes conditional expectation and $\mathcal{F}_t$ is the canonical filtration. The Clark-Ocone formula has applications in mathematical finance, in particular, to obtain replicating hedging strategies for options (see [22]).

A central result of Malliavin calculus provides criteria for the existence of a density for the law of a functional on Gaussian spaces as well as for the regularity of the density. Malliavin's original work [17] used this criteria to give a probabilistic proof of Hörmander's theorem on the hypoellipticity of second order differential operators. This criterion, which is also essential for the analysis of the above class of SPDE, is based on the Malliavin matrix, which is defined as follows.

Given $F : \Omega \to \mathbb{R}^n$, with $F^j \in \mathbb{D}^{1,2}$, the *Malliavin matrix* is given by

$$\gamma = (\gamma)_{i,j} = \langle DF^i, DF^j \rangle_H.$$

The result is the following.

Theorem 3. *Let $F : \Omega \to \mathbb{R}^n$ with $F^j \in \mathbb{D}^\infty$.*

(a) *If $F^j \in \mathbb{D}^{2,4}$ and γ is invertible a.s., then the law of F has a density with respect to Lebesgue measure.*

(b) *If $\det \gamma^{-1} \in \cap_{p\in[1,\infty)} L^p(\Omega)$, then the density in (a) is infinitely differentiable.*

4. Stochastic partial differential equations

As in the case of PDE the structural properties of an SPDE depend on the type of differential operator (parabolic, hyperbolic or elliptic) as well as on the nature of the noise (white Gaussian noise, coloured Gaussian noise, Lévy noise, etc.). The study of different classes of SPDE has been developing over the past 30 years (see e.g. [8], [20], [26]). An overview of results on some of these different classes is given in Carmona and Rozovskii [5].

The class of SPDE under consideration in the book under review has the form

$$Lu(t,x) = \sigma(u(t,x))\dot{W}(t,x) + b(u(t,x)) \tag{1}$$

where L is a differential operator of the form

$$L_1 = \partial_t - \Delta_d, \quad \text{or} \quad L_2 = \partial_{tt}^2 - \Delta_d$$

where Δ_d is the d-dimensional Laplacian and W is the space-time white noise or spatially homogeneous coloured noise process.

A precise formulation of (1) in terms of the corresponding martingale measure $M(dt, dx)$ is given by

$$u(t,x) = \int_0^t \int_{\mathbb{R}^d} \Lambda(t-s, x-y)\sigma(u(s,x))M(ds,dy) + \int_0^t \int_{\mathbb{R}^d} b(u(t-s,x-y))\Lambda(s,dy)$$

where $\Lambda(t, dx)$ is the fundamental solution for $Lu = 0$.

The stochastic heat equation, that is, $L = L_1$, driven by space-time white noise has function-valued solutions in one dimension but in higher dimensions has solutions only as random (Schwartz) distributions, and there is a fundamental problem in formulating nonlinear equations in these dimensions. Some natural nonlinear SPDE have solutions that are singular random measures (cf. [9]). Some nonlinear equations can be reformulated in terms of generalized functionals such as Wick powers on Ω extending $L^2(\Omega)$ based on Wiener chaos expansions (see e.g. [13], [10]).

The approach followed in this book is to restrict consideration to coloured noises with associated spectral measure μ such that function-valued solutions exist and to analyse these using Itô-Wiener expansions and Malliavin calculus.

5. Application of Malliavin calculus to SPDE

The last part of the book under review is devoted to the regularity properties of solutions of nonlinear stochastic wave and heat equations driven by coloured noises using the tools of Malliavin calculus. Another approach to the study of these equations is developed in the work of Peszat and Zabczyk [24]. Bally and Pardoux [1] also obtained regularity results using Malliavin calculus for a nonlinear stochastic heat equation in one dimension driven by space-time noise.

The main results of the last three chapters establish the existence of densities for the laws of solutions to the SPDE as well as the regularity of the densities. A typical result is the following.

Theorem 4. *Assume that $L = L_1$, (t, x) is a fixed point in $(0, T] \times \mathbb{R}^d$ and*

(a) *the coefficients σ, b are C^∞ functions with bounded derivatives of any order greater than or equal to one,*

(b) *there exists $\sigma_0 > 0$ such that*

$$\inf\{|\sigma(z)|, z \in \mathbb{R}\} \geq \sigma_0,$$

(c) *there exists $\eta \in (0, \frac{1}{3})$ such that the spectral measure μ satisfies*

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{(1 + |\xi|^2)^\eta} < \infty.$$

Then the law of the solution of (1)*, $u(t, x)$, has an infinitely differentiable density with respect to Lebesgue measure on $\mathbb{R}$.*

A similar result for the stochastic wave equation, $L = L_2$, is stated and proved in dimensions $d = 1, 2$. The key technical ingredient in the proof is the verification of Malliavin's criterion for the associated Malliavin matrices. Intuitively, the condition (c) on the spectral measure μ is imposed to control the high spatial frequency behavior of the coloured noise.

The theory of the wave equation in $d = 3$ is technically more challenging because in this case the fundamental solution is no longer a function but is a (nonnegative) Schwartz distribution. An extension of Walsh's stochastic integration with respect to martingale measures to cover this case and the solution of the nonlinear wave equation in dimension 3 was developed by Dalang [7]. Using Dalang's results, we also obtain an analogue of Theorem 4.

A natural question is the extension of the above results to the joint law of $(u(t, x_1), \ldots, u(t, x_n))$ where $x_1, \ldots, x_n$ are distinct points. Some partial results are presented in the last section of Chapter 8, but the completion of this program is left as an open problem.

6. Structure of the book

The book by Sanz provides an introduction to the essentials of Malliavin calculus and to stochastic partial differential equations driven by coloured noise. The main objective of the book is to bring the reader up-to-date on recent developments due to the author and coworkers on the application of Malliavin calculus to establish the Malliavin regularity of solutions to a class of SPDE. The book begins with the finite dimensional Malliavin calculus which introduces some basic concepts such as the Ornstein-Uhlenbeck operator in a simple setting. The basic elements of Malliavin calculus are then developed in Chapters 3-5. Stochastic partial differential equations driven by spatially homogeneous Gaussian noises are introduced in Chapter 6 and then an in-depth analysis of the Malliavin regularity and analysis of the Malliavin matrix of solutions is given in Chapters 7 and 8.

In spite of the specialized nature of the subject, the book is remarkably readable and fully succeeds in achieving these objectives. The book is highly recommended for readers interested in Malliavin calculus, SPDE, or both. For readers interested in other applications of Malliavin calculus such as those in mathematical finance, it would be necessary to supplement this with other books such as [19], [22].

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