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Advanced analytic number theory: L-functions, by Carlos Moreno, Mathematical Surveys and Monographs, vol. 115, American Mathematical Society, Providence, RI, 2005, xx+291 pp., US$76.00, ISBN 978-0-8218-3641-5

This book is devoted to the study of two problems concerning general automorphic L-functions together with some of their applications to number theory. One problem is to find non-vanishing theorems for the L-function at the edge of the critical strip; the other is to describe explicit formulas which express prime numbers in terms of functions summed over the set of all the zeros of the L-function. These problems were first considered for Riemann's zeta function and then more generally for certain Langlands automorphic L-functions.

Some history

The Riemann zeta-function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

with

$$Re(s) > 1.$$

In his 1859 memoir, *On the Number of Prime Numbers Less Than a Given Quantity*, Riemann proved that $\zeta(s)$ had a meromorphic continuation to all complex s and satisfied the functional equation

$$\begin{aligned}\xi(s) &= \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) \\ &= \pi^{-\frac{1-s}{2}}\Gamma(\frac{1-s}{2})\zeta(1-s) = \xi(1-s).\end{aligned}$$

One way he found to establish this was to express $\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$ as the Mellin transform of the theta function

$$\sum_{n=1}^{\infty} e^{\pi i n^2 z} = \theta(z)$$

studied by Jacobi. By exhibiting $\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$ as the Mellin transform of $\theta(z)$, Riemann became the father of the theory of automorphic forms. This was about fifty years before Hecke and others realized how important this Mellin transform was. But Riemann went on to open many more questions in number theory using this zeta function. First of all, he derived (or really outlined the proof of) the first "explicit formula" for the zeta-function:

$$\begin{aligned}\Pi(x) = Li(x) - \sum_{\rho}\{Li(x^{\rho}) + Li(x^{1-\rho})\} \\ + \log\xi(0) + \int_x^{\infty} \frac{dt}{t(t^2-1)\log t}\end{aligned}$$

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where $\Pi(x) = \sum_{n\geq 1} \ \pi(x^{1/n})\frac{1}{n}, \pi(x)$ = the number of primes less than x (with the proviso that when x is prime, we put $\pi(x) = \frac{\pi(x+0)+\pi(x-0)}{2}$), ρ runs through the complex zeros of $\zeta(s)$, and $Li(x^\beta)$ (almost equals) $\int_0^{x^\beta} \frac{du}{\log u}$.

This explicit formula equates the prime numbers (the left side of the equation) with the complex zeros ρ of the zeta function (the right-hand side of the equation). Its depth is profound. It comes from Riemann's combining the Euler product representation

$$\zeta(s) = \prod_p \frac{1}{1-p^s}$$

with the (1859 unproved) Hadamard identity

$$\xi(s) = e^{A+Bs} \prod_\rho (1 - \frac{s}{\rho}) e^{s/\rho}.$$

Riemann's work took almost thirty-five years before it was exactly proved, thanks to Hadamard, von Mangoldt, and others. In 1893, von Mangoldt actually proved a more transparent formula that implied Riemann's: if

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \\ 0 & \text{otherwise,} \end{cases}$$

then

$$\psi(x) = \sum_{n\leq x} \Lambda(n) = x - \sum_\rho \frac{x^\rho}{\rho} - \frac{\zeta'}{\zeta}(0) - \frac{1}{2}\log(1-x^{-2}).$$

With this formula the prime number theorem was proved in 1896 by Hadamard and (independently) by de la Vallée Poussin:

$$\pi(x) \sim \frac{x}{\log x} \sim Li(x)$$

as x approaches ∞; in fact, this was just equivalent to proving that

$$Re(s) = 1 \Rightarrow \zeta(s) \neq 0.$$

Three years later, de la Vallée Poussin extended his method to proving there exists a positive constant c such that $\zeta(s)$ has no zeroes in the region of $s = \sigma + it$ such that

$$\sigma > 1 - \frac{c}{log(|t|+2)}.$$

Riemann, in his investigations, came to suspect that

$$Re(s) = 1/2$$

for any complex s such that $\zeta(s) = 0$; in fact, he had numerically computed the first few zeros. This hypothesis of course remains unproved today. The prime number theorem, i.e., that $Re(s) = 1 \Rightarrow \zeta(s) \neq 0$, implies

$$\psi(x) = x + o(x);$$

the Riemann Hypothesis, i.e., that $Re(s) > \frac{1}{2} \Rightarrow \zeta(s) \neq 0$, implies

$$\psi(x) = x + O(x^{1/2+\varepsilon})$$

for all $\epsilon > 0$.

From the turn of the century after Riemann to the 1950's came a wide sampling of results that began to give a more modern approach to number theory, all of it under the future cloak of automorphic forms á là Langlands. Especially important was the development of class field theory, building on the work of Fermat, Euler, Gauss, etc. On one side, E. Artin introduced his (not necessarily abelian) L-functions using polynomials of degrees equal to the dimension of representations of the Galois group (the abelian ones he equated with Hecke's, which were - like Riemann's - a transform of some theta function), and Chebotarev also demonstrated the importance of non-vanishing theorems for these L-functions for the study of the distribution of Frobenius conjugacy classes in the Galois group. On another (slightly later) date, Hecke introduced the space of "automorphic forms". "Automorphic forms" were connected again with Dirichlet series, but now their Euler product had factors of degree 2 instead of 1. For example, in place of $\zeta(s)$ we have $F(s) = \prod_p (1 - \tau(p)p^{-s} + p^{11-2s})^{-1}$, where τ is the Ramanujan function. In 1927 Hardy had observed that the localization of the zeros of $F(s)$ gives rise to problems similar to those of $\zeta(s)$; in particular, a prime number theorem for the Ramanujan function

$$\sum_{p \leq x} \tau(p) \log p = o(x^{13/2})$$

would follow if $F(s) \neq 0$ in the right edge of the "critical strip" $11/2 \leq \sigma \leq 13/2$; i.e., $Re(s) = 13/2$, exactly what Rankin showed in 1939. One of the problems left open (until its proof by Deligne) was Ramanujan's Conjecture:

$$|\tau(p)| \leq 2p^{\frac{11}{2}}.$$

Of course, another was a greater "Riemann's Hypothesis"; i.e., $Re(s) > 6$ implied $F(s) \neq 0$.

Although Artin and Hecke were at the same university, neither of them suspected that they were actually working towards the same thing. True, Artin had proved his abelian L-series were the same as special Hecke L-functions generalizing the Riemann zeta-function. But what about two- (and higher) dimensional Artin L-series? Do Hecke's L-functions of modular forms bear the same relation to Artin L-series of dimensional two as Hecke's L-functions of degree one bear to Artin's abelian L-series? That the answer was "yes" came thirty or so years later. In 1967, R.P. Langlands predicted that *all* of Artin's L-series (associated to irreducible n-dimensional representations of the Galois group) were "automorphic" as part of a much more general "Langlands Program" [Lan].

Let us first review the notions of the 1950's and early 1960's. By 1950, Hecke's *grossencharakters* were viewed by Tate and also Iwasawa (see his talk at the 1950 ICM) as functions on the *idele* group of a number field. In fact, Iwasawa influenced many Japanese mathematicians, such as Shimura, Taniyama, Satake, Tamagawa, Ono, and others, to extend these adelic ideas to other simple algebraic groups. (Godement and Kneser did the same in Europe.) However, *grossencharakters* were not readily viewed as "automorphic forms on $GL(1)$". As far as $GL(2)$ was concerned, by the late 1950's the eigenvalues of an automorphic form with respect to the Hecke operators T_p were being realized as eigenvalues for certain convolution operators of the p-adic group $GL(2, \mathbf{Q_p})$. Most of the Japanese mathematicians just mentioned understood this (Tamagawa has told me that Taniyama lectured on some of this in 1958). Then in the early 1960's, ideas of Langlands, Mautner and others in the West converged to these facts, and Satake's report from the 1965 Boulder Summer Conference spelled out exactly how an automorphic form fixed representations of $GL(2, \mathbf{Q_p})$. Furthermore, Harish-Chandra produced his monumental work on general real (and later p-adic) groups, A. Selberg had done his beautiful work on Eisenstein series and the trace formula, and Gelfand and Piatetski-Shapiro et al. had begun their study of adele group representations. These historic developments helped give birth to the Langlands' Program. See [Art] for further discussion of this background.

An "automorphic form on $GL(n)$" of Langlands is an irreducible subspace of $L^2(GL(n,\mathbf{Q}) \setminus \mathbf{GL(n,A)})$. It is an infinite dimensional (when $n > 1$) representation $\pi = \otimes\pi_p$ of $GL(n,\mathbf{A})$ with corresponding automorphic L-function $L(s,\pi) = \prod L(s,\pi_p)$. Each π_p is a representation of the *p-adic* group $GL(n,\mathbf{Q_p})$. When $n = 1$ the notion of automorphic L-function is a generalization of the Riemann zeta function which appears as the one attached to the trivial representation of $GL(1,\mathbf{A}) = \mathbf{A}_\mathbf{Q}^\mathbf{x}$. When $n = 2$ an automorphic L-function generalizes the Ramanujan-Dirichlet series $F(s)$.

A similar notation can be introduced for an arbitrary reductive algebraic group. Actually, automorphic L-functions $L(s,\Pi,r)$ are attached more generally by Langlands to certain representations Π of an arbitrary reductive group G and arbitrary finite dimensional representations r of the (Langlands) "L-group" of G. For example, the Langlands-Shahidi L-functions [Sha] belonging to certain reductive groups G and certain representations r of the L-group of G produce such examples. But so do Artin's Galois representations and even elliptic curves! The conjecture known as "Langlands' Functoriality" then predicts that the L-functions of all these general $L(s,\Pi,r)$'s should coincide with some product of the L-functions of "standard" representation π of some $GL(n)$.

For exactly how Artin's Galois theory and elliptic curves define examples of Langlands Functoriality (and much more), the reader is referred to the survey lectures in *An Introduction to the Langlands Program*, Birkhäuser, 2003 (see its review by M. Harris in the 2004 *Bulletin* [Har]), and the reviewer's 1984 *Bulletin* article "An elementary introduction to the Langlands program".

This book

With this short summary of such historical events, let us look at what Moreno does with these L-functions. One day, given the functoriality principle of Langlands, the two problems considered in the opening paragraph of this review for standard $GL(n)$ L-functions will suffice for all automorphic L-functions.

In Chapter 1, Hecke's L-functions are treated from the point of view of Tate's thesis (1950); more precisely, Moreno gives Weil's treatment of it using distribution theory (*Sem. Bourbaki*, Exp 312). Moreno closes the chapter with a quick discussion of the theory for $GL(n)$, i.e., "standard" Langlands automorphic L-functions $L(s,\pi)$ associated to automorphic representations of $GL(n)$. In Chapter 2 is the background material for Artin-Hecke L-functions through the functoriality principle of Langlands. The newer parts of Moreno's theory are saved for the remaining chapters of this book (roughly half of the text). Chapter 2 also reviews the mathematics surrounding Langlands' important theorem on the decomposition of the root number into local factors.

In Chapter 3 the author focuses on Riemann's work and proves the product formula for the zeta function without appealing to the standard theory of Hadamard. Moreno applies this proof to a much broader class of L-functions, which he uses later in Chapter 5. The author next describes Rademacher's theorem giving a bound for the Hecke L-function in a vertical strip, a useful result for the study of the distribution of zeros within the critical strip. Moreno also states the basic analogous estimates for Artin-Hecke L-functions (which generalize Rademacher's Theorem) and for standard automorphic cuspidal L-functions for $GL(n)$. Such theorems are refinements of weaker statements. First of all, we have in mind the classical "convexity bound" for $GL(n)$ type automorphic L-functions which generalizes the one for $\zeta(s)$; i.e., for $\epsilon > 0$,

$$\zeta(\frac{1}{2} + it) \ll |s|^{\frac{1}{4}+\epsilon}.$$

For $GL(n)$ (see [IwK], Chapter 5, or [Mic], Lecture 1), it reads as follows: for $\epsilon > 0$, $L(s,\pi) \ll q(s,\pi)^{\frac{1}{4}+\epsilon}$ for (the analytic conductor) $q(s,\pi)$ the product of the conductor of π times the analytic conductor at infinity. Its proof still remains so eloquent for $GL(n)$ that I quote its steps here: (1) for $Re(s) > 1$ use the absolute convergence of $L(s,\pi)$ (which is a main conclusion of Jacquet and Shalika [JSh]), (2) apply the functional equation of $L(s,\pi)$, (3) use Stirling's estimate for the gamma functions, and (4) apply the Phragman-Lindelof principle (with $Re(s) = \frac{1}{2}$). Secondly, this Rademacher result obviously implies that $L(s,\pi)$ is bounded in vertical strips. So, as the author points out, the recent work [GeS] of Shahidi and the reviewer extending this boundedness to $L(s,\Pi,r)$ of Langlands-Shahidi type is relevant, though we don't know much about the analyticity of $L(s,\Pi,r)$ to the right of $s = 1$.

Chapter 4 is called "Explicit Formulas". After first reviewing the formulas of Riemann and von Mangoldt already mentioned, the author discusses the formula of Delsarte as a forerunner to the work of Andre Weil. Each of Weil's formulas, like its predecessors, equates two *distributions*: one associated to the zeros of an L-function, the other to its Fourier dual associated to the primes. This identity of distributions is still an "explicit" formula like Riemann's (or the more transparent one of von Mangoldt). However, unlike Riemann's, this smooth version deals with series which are absolutely convergent. Weil goes on to use it for L-series attached to n-dimensional Artin-Hecke representations. In describing the formula for automorphic cuspidal representations π of $GL(n)$, Moreno uses the Langlands correspondence to identify each local representation π_p as a ϕ_p. There is an oversight here: the word "irreducible" should not appear (twice) in the second sentence on page 175. A simple proof is given by Proposition 2.1 of Rudnick-Sarnak [RuS].

Chapter 5 includes the work of Stark and Odlyzko on finding lower bounds for the discriminant of a number field using explicit formulas, namely Weil's. Looking at the explicit formula as an equality of distributions, the approach is to apply the formula to well-chosen test functions; here the presentation follows that of G. Poitou and J.-P. Mestre.

In Chapter 6, the author shows us (among other things) two detailed methods for proving $L(s,\pi) \neq 1$ for $Re(s) = 1$. They are: the generalized method of Hadamard and de la Vallée Poussin, and the method of Eisenstein series. For the first, Moreno follows Deligne's arguments, mixed with some representation theory; for the second, he uses arguments due to Jacquet and Shalika [JSh]. There are also applications to Langlands' ideas concerning Ramanujan's Conjecture ([Lan]). What surprises me is that the author does not go on to discuss *lower bounds* for $L(s,\pi)$ in a neighborhood of the line $Re(s) = 1$ (roughly the generalization of de le Vallée Poissin's 1899 work cited above). Indeed, this generalization is one part of the author's most famous works; cf. [Mor1] and his $GL(n)$ paper [Mor2].

Relevant recent papers include Brumely [Bru], Bump-Friedberg-Hoffstein [BFH], the reviewer with Lapid and Sarnak [GLS], R. and M. Murty [Mur], Sarnak [Sar], Selberg [Sel], Ye (see [LWY]) and many others.

Let me mention some tiny mistakes which make the reading of this book less pleasant. First is the fact that typographical errors abound. Names are often misspelled ("Kashdan" on page 59, "Jaquet" on page 134, etc.), and there are many other typographical mistakes. Secondly, a notational guide is missing, and a fuller and more thorough index would have been much more useful than the single page index that is provided.

For over thirty years, Moreno has been thinking about L-functions as general as $L(s, \Pi, r)$, not just $\zeta(s)$ or abelian L-functions. Although it is now popular for students of number theory to do so, it definitely was not true for other number theorists in the mid 1970's. Moreno worked virtually alone. I regret that I was not more familiar with his work then. As Iwaniec and Kowalski point out on page 135 of their broad survey book [IwK], the first general result for modular forms beyond the classical content of Dirichlet characters is a "prime number theorem" due to Moreno in 1972. In addition to discussing the fundamental basics of L-functions from Riemann to Langlands, this book gives glimpses of the mathematical work that Moreno has developed over the last thirty-odd years.

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