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MATHEMATICS AND PHYSICS

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Mathematics and physics are different enterprises: physics is looking for laws of nature, mathematics is trying to invent the structures and prove the theorems of mathematics. Of course these structures are not invented out of thin air but are linked, among other things, to physics.

In this lecture I shall discuss a number of instances where mathematicians have come to the aid of physics and striking examples where physicists have suggested new problems and new subjects for mathematical investigations. Many of these have a bearing on the real world.

The relation of mathematics and physics is a natural topic for a Gibbs Lecture; at least two previous lectures, by Freeman Dyson and by Edward Witten, were devoted to it.

I start with five famous quotes:

"Nature not only suggests to us problems, she suggests their solution."
—Henri Poincaré

"The human mind has never invented a labor-saving device equal to algebra."
—J. Willard Gibbs

"I didn't become a mathematician because mathematics was so full of beautiful and difficult problems that one might waste one's power in pursuing them without finding the central question." —Albert Einstein

"The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve." —Eugene Wigner

"Mathematics is trivial, but I can't do my work without it." —Richard Feynman

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I suspect that most mathematicians disagree with Einstein: for us there is no central question, except the one we happen to be working on.

I will choose most of my examples from the 19th and 20th centuries, but I will start with several from earlier times. First, the greatest scientist of antiquity, Archimedes, who was also its greatest mathematician.

Skipping ahead quite a number of centuries, we come to Galileo, who was mostly an observational and experimental physicist but who had a deep appreciation of mathematics. He wrote: "The great book of Nature lies ever open before our eyes and the true philosophy is written in it....But we cannot read it unless we have first learned the language and the characters in which it is written....It is written in mathematical language and the characters are triangles, circles, and other geometrical figures."

Today we would add the derivative and the integral to that list of characters.

Kepler, Galileo's great contemporary, was the first to describe correctly the laws governing the planetary system. He not only determined accurately the shape of the orbits of planets as elliptical, with the Sun at their focus but also discovered the equal area rule and the relation of the size of the orbits of planets to their period. These planetary laws played a crucial role in validating Newton's laws of motion and gravitation.

Kepler was an outstanding mathematician as well as a physicist. He had interesting notions on the theory of volume. A conjecture he made about the densest packing of balls in three-dimensional space is still the subject of research.

There were two great inventions at the time of Galileo and Kepler: the telescope, which revolutionized observational astronomy; and the base ten logarithm, invented by Briggs, and independently by Burgi, an associate of Kepler, which revolutionized scientific calculation. Kepler immediately recognized that the use of logarithms freed scientists from the painful task of multiplication and division and the pitfall of numerical errors.

The distinguished historian Otto Neugebauer wrote in "Notes on Kepler" that "the number of trivial computing errors in Kepler's writings is enormous." Amazingly, the final answer was always correct.

The younger generation is probably unaware how important base ten logarithms were in the past. Most mathematics books published before 1950 had such a log table appended in the back. Self-respecting engineers always had their slide rules with them. It is therefore understandable but bizarre that until recently most pocket calculators had a built-in program to evaluate base ten logarithms.

Kepler's vision went beyond mere description; he wrote: "My goal is to show that the heavenly machine is not a kind of divine living being but similar to a clockwork insofar as all the manifold motions are taken care of by one single, absolutely simple magnetic bodily force, as in a clockwork all motion is taken care of by a single weight."

Kepler was both the first modern scientist and the last medieval one. His scientific outlook was tinged with a mysticism that is well described in Arthur Koestler's book *The Sleepwalkers.*

Kepler's vision of the solar system as a clockwork was carried to completion by Newton, the greatest physicist and mathematician of all time. Euler, the second greatest mathematician, was also an outstanding physicist; 250 years ago he formulated the equations governing the flow of fluids, both compressible and incompressible. Later I shall describe some of the successes we had and the difficulties we still have in solving them.

It was in the 19th century that the professions of mathematician and physicist took on their distinctive coloring, but the separation was far from absolute. The two greatest mathematicians of the century, Gauss and Riemann, both had a deep interest in physics; one of the magnetic units is named after Gauss.

More than a third of Riemann's publications were in physics; he was the first to suggest that the components of the electric field satisfy the wave equation. His most important contribution to science was laying the foundations for the theory of propagation of waves in compressible gases. The concepts he introduced—Riemann invariants, the Riemann initial-value problem, the law of propagation of shock waves—are still the basic building blocks of the theory. However, he assumed the flow to be isentropic, and so the shock conditions he derived do not conserve energy. The correct form was derived by Rankine and Hugoniot.

Most of the leading French mathematicians of the first half of the 19th century—Lagrange, Laplace, Fourier, Poisson, Cauchy, Liouville—had a deep interest in physics and made significant contributions to it. For the British the distinction between mathematician and physicist was not sharp: Green, Airy, Stokes, W. Thomson, the Irish Hamilton, Lord Rayleigh can be called either. Rayleigh was the first mathematician to be honored by a Nobel Prize, in physics, for work he did in chemistry—the discovery of argon.

The greatest 19th century physicist was Maxwell; he had a thorough training in mathematics. He discovered the equations governing the propagation of electromagnetic waves that are named after him. He observed that the speed of propagation of these waves, based on the values of the physical constants entering the theory, agrees closely with the measured value of the speed of propagation of light. He concluded that light is an electromagnetic phenomenon. Apparently this was a surprise, for he pointed out that the experiments measuring the velocity of light made no use whatsoever of electricity or magnetism and the experiments to measure the electromagnetic constants made no use of light "except to see the instruments."

Another great contribution of Maxwell was the kinetic theory of gases. He and two other great 19th-century figures, Boltzmann and Gibbs, created the science of statistical mechanics, in which mathematics plays an important role. This being a Gibbs Lecture, I will now describe the contributions of Gibbs to mathematics.

The most important of these is the Gibbs phenomenon for Fourier series. The origin of this discovery is very peculiar. Albert Michelson (whose discovery about the speed of light played such an important role in the special theory of relativity) wrote a brief note in *Nature* (1898) about the manner in which a sequence of discontinuous functions approximates a continuous function. He gave as an example the function defined as $f(x) = x$ on the interval $-\pi < x < \pi$ and continued periodically with period 2π for all x and asked how it can be the limit of the partial sums of its Fourier series.

Gibbs perceived that Michelson had mixed up the concepts of the graph of the limit and the limit of the graphs of the approximating functions. In a note in *Nature* (December 29, 1898), he pointed out that the graph of the limit of the partial sums of the Fourier series of f consists of an infinitude of 45-degree lines (see Figure 1), whereas the limit of the graphs contains, in addition to the 45-degree lines, vertical

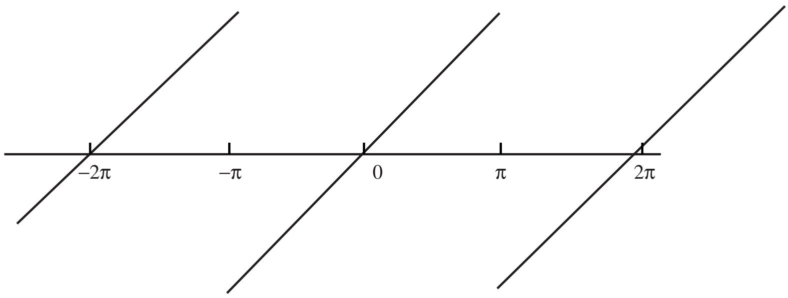

FIGURE 1

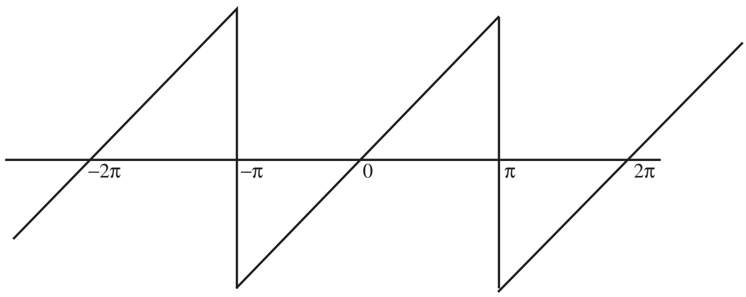

FIGURE 2

lines connecting the points where the function is discontinuous (see Figure 2). In a subsequent note in *Nature* (April 27, 1899), Gibbs wrote:

"I apologize for a careless error which I made in describing the limiting form of the family of curves represented by the equations

$$y = 2(\sin x - 1/2\ \sin 2x \ldots +_{1/n} \sin nx)$$

as a zigzag line consisting of alternate inclined and vertical portions. The inclined portions were correctly given, but the vertical portions which are bisected by the x axis extend beyond the points where they meet the inclined portions, their total length being expressed by four times the definite integral $\sin u/u\, du$."

This is all Gibbs ever published on the subject.

Figure 3 is the graph of the nth partial sum for $n = 10$, 50 and 100; clearly, the vertical portion of the limit of their graphs extends beyond the inclined portions.

The Gibbs phenomenon produces large errors when the so-called spectral method is used to compute solutions of partial differential equations, using the partial sums of their Fourier series, when these solutions contain discontinuities. Fortunately, as David Gottlieb and his collaborators have shown, these errors can be eliminated by a suitable post-processing.

Gibbs's other contribution to mathematics was the development of modern vector analysis. His lectures on the subject were never published but circulated as lecture notes sufficiently widely to draw the fire of the old guard, who based vector algebra on quaternions, as follows:

Given two vectors X and U in three-dimensional space,

$$X = (x, y, z), \qquad U = (u, v, w)$$

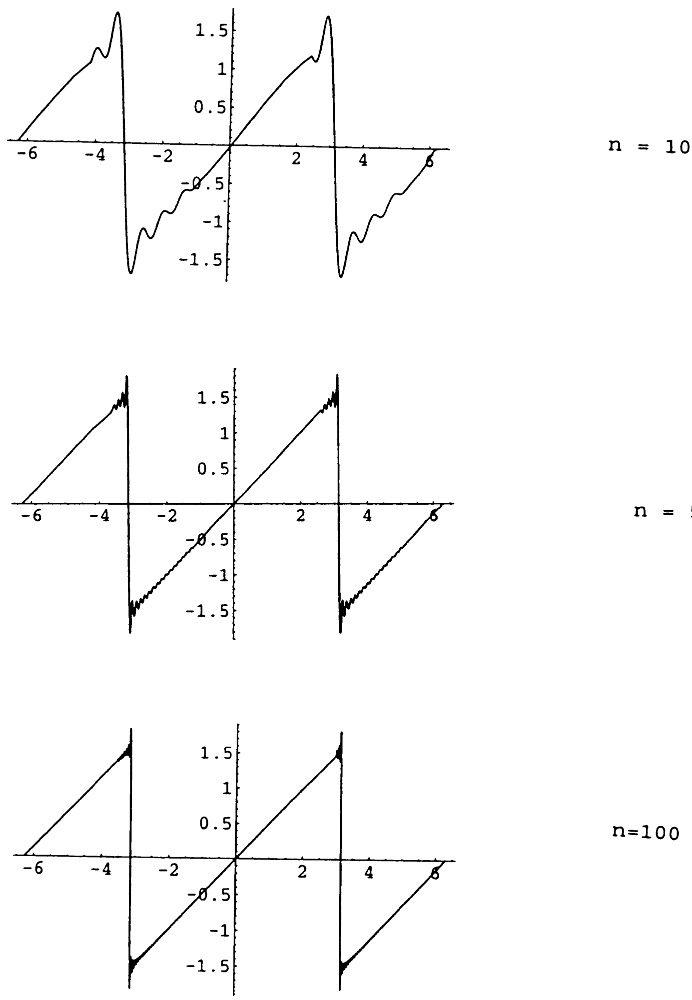

FIGURE 3

form the quaternions $ix + jy + kz$ and $iu + jv + kw$ with zero real part and take their quaternionic product $-s + ip + jq + kr$. Then s is identified as the scalar product of X and U, and (p, q, r) is their vector product $X \times U$.

Gibbs thought quaternions were irrelevant to vector analysis and felt obliged to defend his view against attacks. In a note in *Nature*, pp. 511–513, April 2, 1891, he wrote:

"The following passage, which has recently come to my notice, in the preface to the third edition of Prof. Tait's *Quaternions* seems to call for some reply:

'Even Prof. Willard Gibbs must be ranked as one of the retarders of quaternion progress, in virtue of his pamphlet on Vector Analysis, a sort of hermaphroditic monster, compounded of the notation of Hamilton and Grassmann.'

"It seems to be assumed that a departure from quaternionic usage in the treatment of vectors is an enormity. If this assumption is true, it is an important truth; if not, it would be unfortunate if it should remain un-challenged, especially when supported by so high an authority. The criticism relates particularly to notation, but I believe that there is a deeper question of notions underlying that of notations. Indeed, if my offence had been solely in the matter of notation, it would have been less accurate to describe my production as a monstrosity, than to describe its dress as uncouth."

Gibbs then goes on to give a spirited defense of his presentation of vector analysis. This was not the end, for two years later he felt obliged to publish another note in *Nature*, pp. 364–367, Aug. 17, 1893:

"In a paper by Prof. C. G. Knott on 'Recent Innovations in Vector Theory', of which an abstract has been given in *Nature* (vol. XLVII, pp. 590–593), the doctrine that the quaternion affords the only sufficient and proper basis for vector analysis is maintained by arguments based so largely on the faults and deficiencies which the author has found in my pamphlet, *Elements of Vector Analysis*, as to give these faults an importance which they would not otherwise possess.

"The charge which most requires a reply...is that in the development of dyadic notation, Prof. Gibbs, being forced to bring the quaternion in, logically condemns his own position."

Gibbs then goes on to demolish, politely, Prof C. G. Knott.

In the 20th century statistical mechanics acquired a double life as part of mathematics as well as of physics. For instance, Hilbert studied the derivation of the equations of gas dynamics from the Boltzmann equation. This double life has been extremely fruitful for the subject; I give a few examples.

Boltzmann, in order to justify the equality of time-averages and averages over phase-space, had formulated the ergodic hypothesis. In 1931 von Neumann and G. D. Birkhoff proved the mean and the pointwise ergodic theorems. These theorems turned out to be basic in many parts of analysis; von Neumann regarded it as one of his finest accomplishments. However, Jack Schwartz argues persuasively in a witty article, "The pernicous influence of mathematics on science", that the ergodic theorems are irrelevant for statistical mechanics:

Schwartz points out that both ergodic theorems assume that the flow is "metrically transitive", which means that phase-space cannot be divided into two nontrivial parts that are not connected by the flow. This hypothesis is very difficult to verify and has never been verified for flows that are of interest in statistical mechanics; one of the difficulties is that the hypothesis may very well be false.

On the other hand, Schwartz points out that if true, the ergodic property would show the equality of the time-averages along trajectories with phase-averages of all continuous functions in phase-space. This is much more than what is needed; in statistical mechanics we are not interested in every continuous function, only in those that have thermodynamic significance. These are very special functions, with a high degree of symmetry, and the equality of their time-average and phase-space average is due to their special form.

A striking example where mathematics has helped statistical mechanics had its origin in Onsager's study of the Ising model. Onsager showed that the spin correlation function of an $N \times N$ lattice is given by the determinant of an $N \times N$ Toeplitz matrix. Of interest is the thermodynamic limit as N tends to infinity. One of Onsager's colleagues at Yale, Kakutani, knew that the world's greatest expert on the determinants of Toeplitz matrices was Gabor Szegö at Stanford and communicated the problem to him. Indeed Szegö succeeded in deriving the desired asymptotic result, published in 1952 under the title "On certain Hermitean forms associated with the Fourier series of positive functions". An article by Barry McCoy in the collected works of Szegö has a fascinating discussion of the mathematical and physical ramifications of this subject.

A very recent example of the double life of statistical mechanics is Stochastic Loewner Evolution (SLE), used in two-dimensional statistical physics models. For this work Gregory Lawler, Oded Schramm and Wendelin Werner were honored by a Polya Prize in 2006, and Wendelin Werner received a Fields Medal for his contributions to this subject. The remarkable thing is that Loewner had invented his differential equation describing the deformation of one-parameter conformal maps as a tool for studying the Bieberbach conjecture, a somewhat esoteric subject. He succeeded in proving that the absolute value of the third Taylor coefficient of univalent functions does not exceed 3. The Loewner differential equation was an important ingredient as well in de Branges' proof of the conjecture for all Taylor coefficients. How about this as an example of "the unreasonable effectiveness of mathematics in the physical sciences"?

The 20th century was racked by a series of revolutions in physics, with profound influences on mathematics. Einstein's general theory of relativity was one; in the preface of his basic paper on the subject, Einstein credits Minkowski, as well as Gauss, Riemann, Christoffel, Ricci and Levi Civita, with laying the foundations on which he built his theory. The astrophysical consequences of the general theory were based on special solutions of the Einstein equations. A general existence theory of solutions is the subject of lively research today; see for instance recent work of Demetrios Christodoulou and Sergiu Klainerman.

The most profound upheaval in physics has been the creation of quantum mechanics; it attracted many mathematicians: Hermann Weyl, von Neumann, Friedrichs, later Kato, and Freeman Dyson among others. It was Weyl who helped Schrödinger calculate the eigenvalues of the Schrödinger operator for the hydrogen atom; these agreed with the spectral lines of hydrogen, an important piece of evidence for the validity of Schrödinger's theory.

Friedrichs once told me of a chance encounter with Heisenberg in the sixties. He took the opportunity to express to Heisenberg the profound gratitude of mathematicians for his having created a subject that has led to so much beautiful mathematics. Heisenberg allowed that this was so; Friedrichs then added that mathematics has, to some extent, repaid this debt. Heisenberg was noncommittal, so Friedrichs pointed out that it was a mathematician, von Neumann, who clarified the difference between a selfadjoint operator and one that was merely symmetric. "What's the difference?" said Heisenberg.

The distinguished physicist Eugene Wigner, a close friend of von Neumann since their high school days, had a lifetime love affair with mathematics and made many contributions to it. He made use of group theory to deal with problems of quantum mechanics; this led to interesting problems of group representation. Another of his fruitful ideas was to describe the spectrum of very complicated quantum mechanical systems by statistical considerations as the spectrum of a random matrix. The subject of random matrices is today a very exciting research area, relevant to seemingly different subjects.

An astounding connection of random matrices to number theory was discovered in the seventies when Hugh Montgomery lectured at the Institute for Advanced

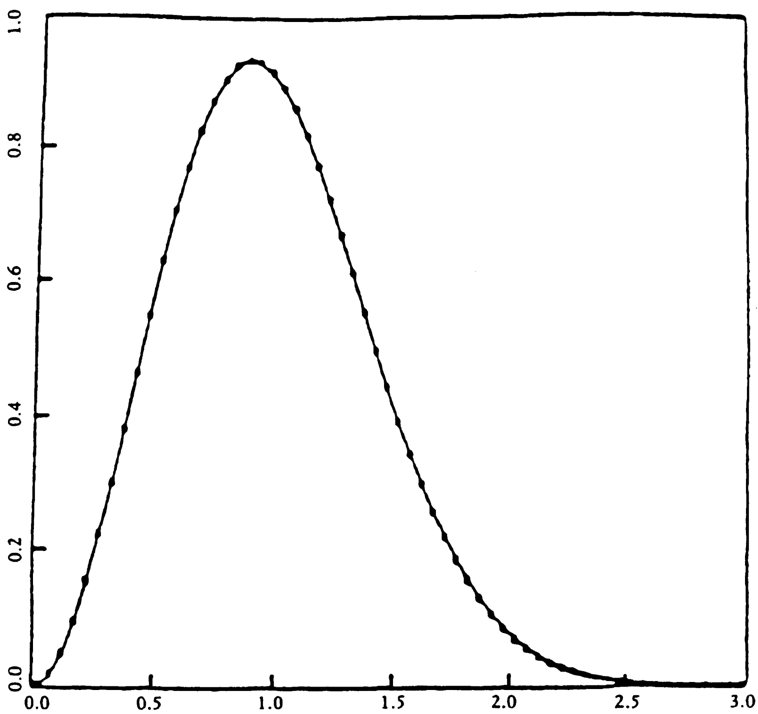

FIGURE 4. Nearest neighbor spacings among 70 million zeroes beyond the $10^{20\text{th}}$ zero of zeta versus μ_1 (GUE).

Study at Princeton on a conjecture of his on the distribution of the normalized zeroes of the Riemann zeta-function. Dyson recognized this distribution as being the same as the distribution of consecutive eigenvalues of random matrices.

That discovery prompted Andrew Odlycko to investigate numerically the distribution of the spacing of the zeroes of the zeta-function. In the region of the first 70 million zeroes beyond the first 10^{20} zeroes he found (see Figures 4 and 5) well nigh perfect agreement with the distribution of the eigenvalues of random matrices!

Early in the 20th century Hilbert and Polya had suggested a way of proving Riemann's hypothesis by identifying the zeroes of the zeta-function with eigenvalues of an operator of the form $1/2I + iS$, where S is a selfadjoint operator. That a random selfadjoint operator would have this property on the average came as a complete surprise.

I return now to the subject of fluid dynamics, whose roots go back 250 years to Euler and 150 years to Navier and Stokes. The first definitive result on the existence of solutions of the initial value problem was due to Jean Leray, who in 1934 obtained sharp results for incompressible Navier-Stokes flows in two dimensions and tantalizingly incomplete results in three dimensions, where solutions exist, but they satisfy the equations only in a generalized sense, and their uniqueness is not assured. Some advances have been made in the intervening 70 years, but they are remarkably modest. The difficulties are not merely technical, but are probably connected with the profound mystery of turbulence.

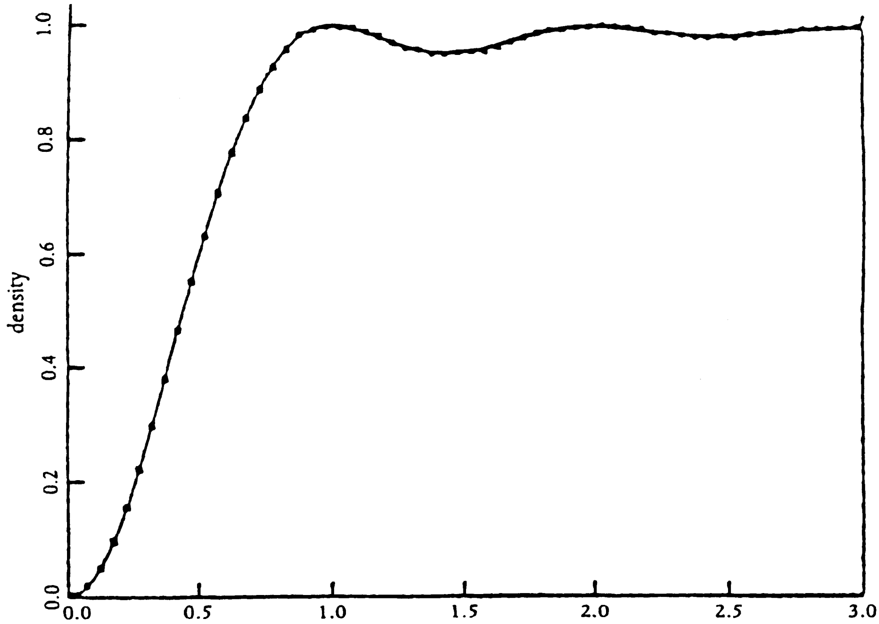

FIGURE 5. Pair correlation for zeroes of zeta based on 8×10^6 zeroes near the $10^{20\text{th}}$ zero versus the GUE conjectured density $1 - \left(\frac{\sin \pi x}{\pi x}\right)^2$.

The theory of compressible flows is sketchier. In 1965 Jim Glimm succeeded in showing that for a large class of equations in one space variable, which includes the Euler equations for compressible flow, the initial-value problem has a solution for all time, provided that the data are not too large. These solutions contain discontinuities, shocks; the unique physical solution is characterised by an entropy condition.

The formation of shocks is the cause of a great deal of information loss. It is a very interesting problem to give a quantitative estimate for this loss of information for compressible flow; not much is known about this question.

There is no theory for the initial value problem for compressible flows in two space dimensions once shocks show up, much less in three space dimensions. This is a scientific scandal and a challenge. Although there is a good theory for steady flows in two dimensions around bodies as long as the flow remains subsonic, once the flow becomes transonic, shocks appear and so do theoretical difficulties.

Just because we cannot prove that compressible flows with prescribed initial values exist doesn't mean that we cannot compute them. Computing, a very much younger sibling of the fraternal twins mathematics and physics, has been flexing its muscles since birth. Its accomplishments are impressive. For instance, all aircraft that have entered service in the last ten years have been designed using computers. The flow at cruising speed around a three-dimensional design can be computed, and lift and drag calculated. Then the design is changed and the flow is recomputed; this process is repeated to increase lift and decrease drag. Once a satisfactory design is reached, but only then, is a model built and tested in a wind tunnel to check the calculations.

How much faith can we put in a numerical calculation when we cannot provide a rigorous proof that the process that produced it converges nor that the flow we are calculating approximately exists? Performing a physical experiment every time is not an option. But there are other ways of testing the reliability of numerical results: use several different numerical methods to calculate approximate solutions and compare the results. If they are in reasonable agreement, we can be reasonably sure of the result.

Figures 6–11 present examples of this procedure on one of the standard test problems, the Riemann initial-value problem in two-dimensional gas dynamics. Here four different constant states are prescribed in the four quadrants of the x,y plane. These are the simplest initial-value problems, yet their solution turns out to be surprisingly complicated. The figures are the density contours of the solution of six Riemann problems calculated by three different methods, developed by three different methods, developed by three different teams [22], [14] and [13]. The numerical results agree to a remarkable extent, down to small details.

This year marks the fiftieth anniversary of the death of von Neumann, without doubt one of the most important and influential scientists of the first half of the 20th century. Some of his great contributions to mathematics and physics were mentioned earlier. He invented game theory, a fundamental idea in economics. He was also one of the creators of the modern computer and, less well known, one of the founders of modern computational science and computational fluid dynamics in particular.

During World War II, when von Neumann was working at Los Alamos, he realized that analytical methods were inadequate for designing weapons and that the only way to deal with such mathematical problems is to discretize the continuum equations and solve the resulting finite system of equations numerically. The tools needed to carry out such calculations effectively are high speed programmable electronic computers, large capacity storage devices, programming languages, a theory of how to discretize partial differential equations, and a variety of algorithms for solving rapidly the discretized equations. It is to these tasks that von Neumann devoted a large part of his energies in the last ten years of his life. He was keenly aware that computational methods are crucial not only for designing weapons but also for solving an enormous variety of scientific and engineering problems; understanding the weather and climate particularly intrigued him.

But he also realized that computing can do much more than grind out by brute force answer to concrete questions. In a lecture delivered in Montreal in 1945, he concluded that "really high speed computing devices, in the field of partial differential equations, as well as in many other fields which are now difficult or entirely denied access, provide us with those heuristic hints which are needed in all parts of mathematics for genuine progress."

In precisely such a way did the numerical experiments of Kruskal and Zabusky suggest the complete integrability of the Korteveg-De Vries equation. There is no doubt that computer experimentation will become a way of life in most parts of mathematical research. And there is no doubt that mathematics and physics will continue to invigorate each other.

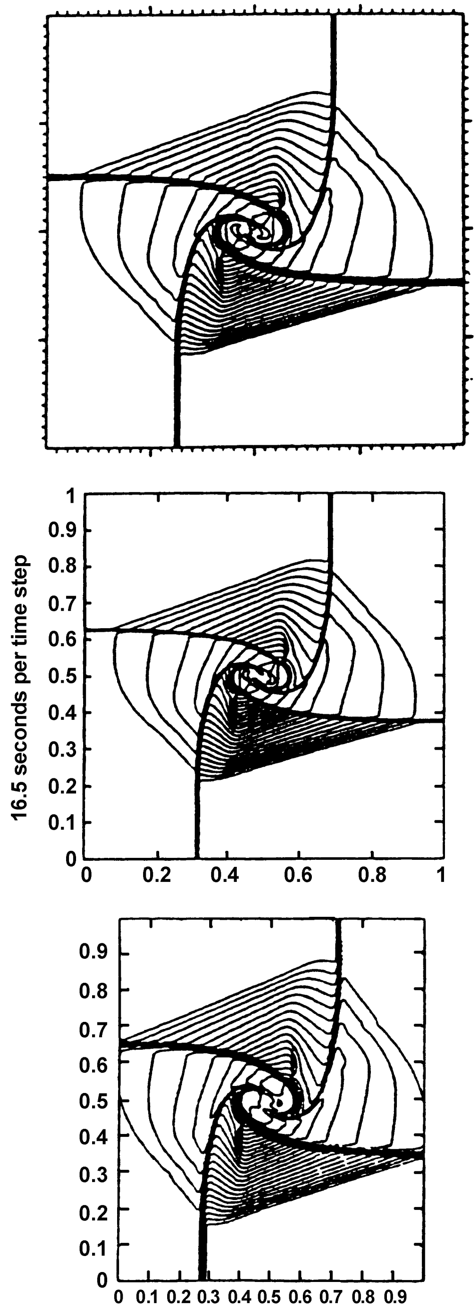
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FIGURE 6

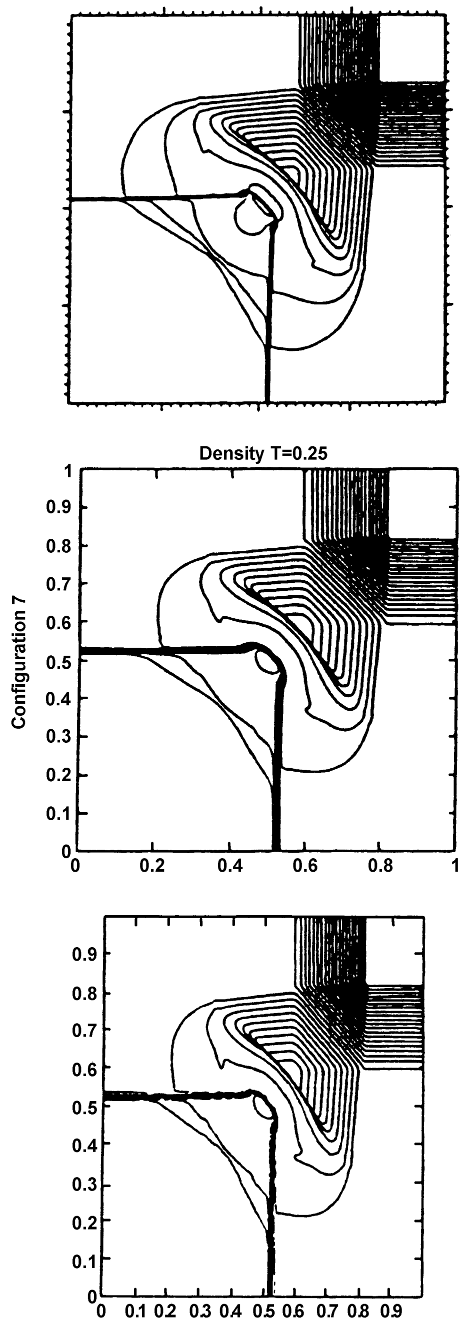
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FIGURE 7

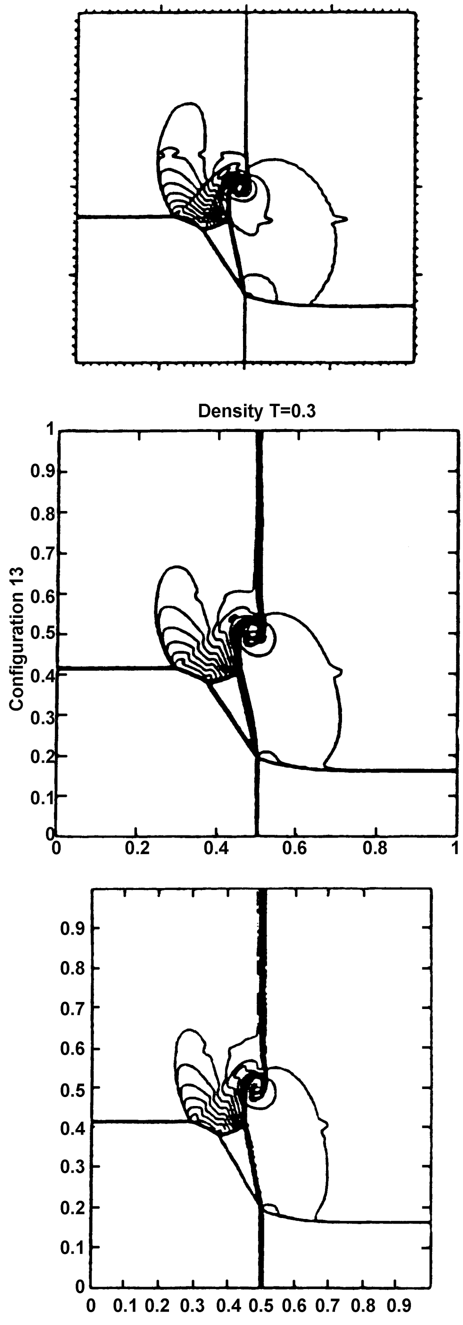
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FIGURE 8

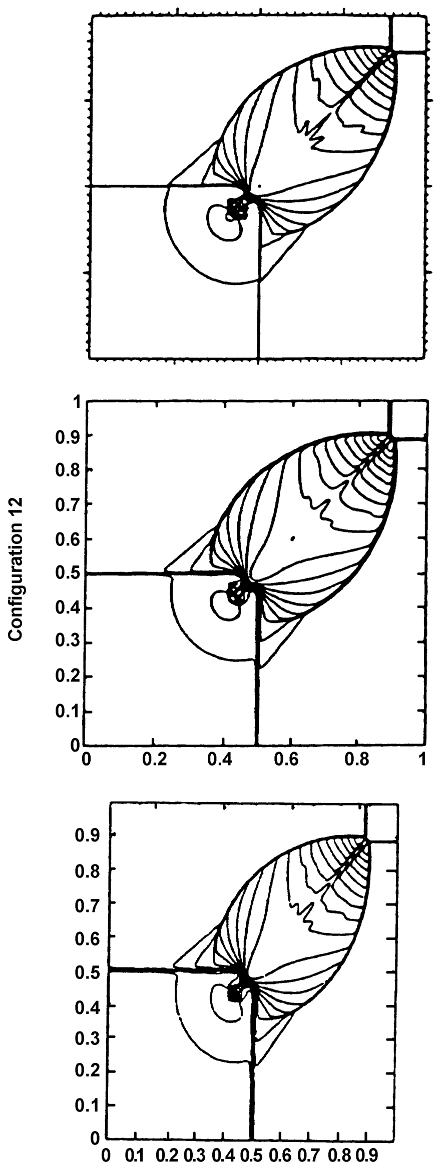
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FIGURE 9

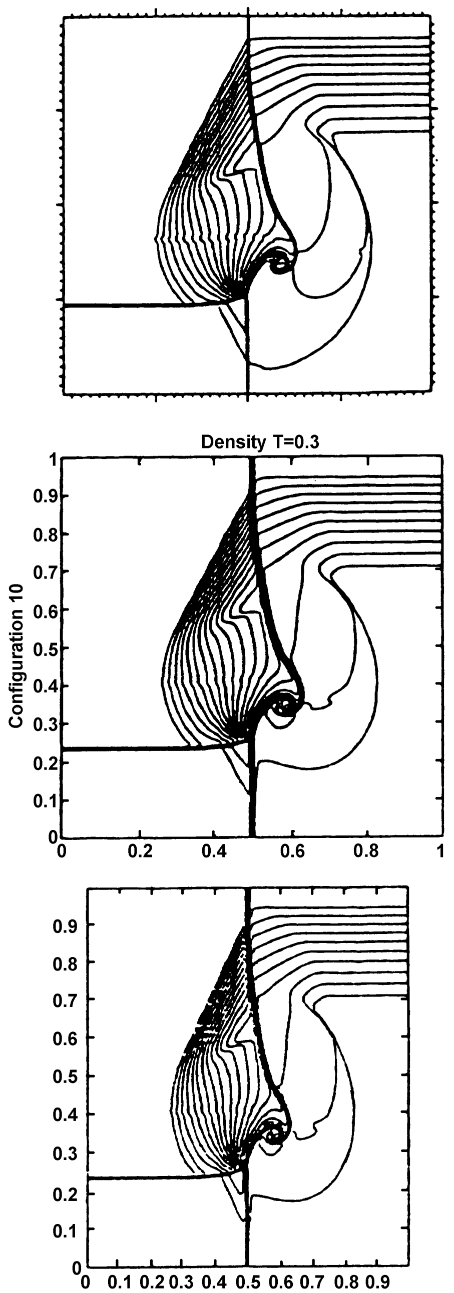
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FIGURE 10

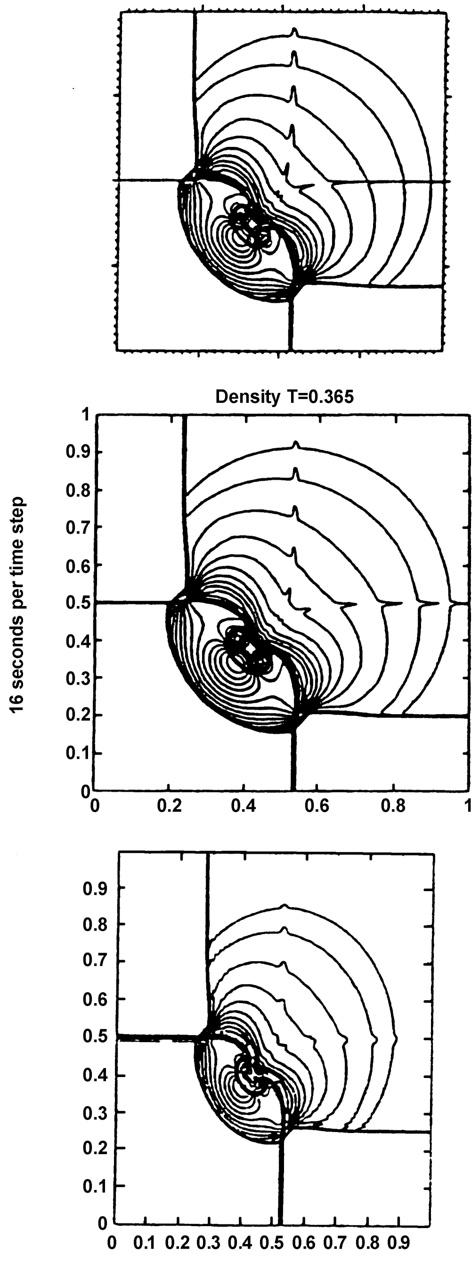

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FIGURE 11

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References

[1] Birkhoff, G. D., What is the ergodic theorem? Amer. Math. Monthly 49 (1942), 222–226. MR0006619 (4:15b)

[2] Boltzmann, L., Wissenschaftliche Abhandlungen, Chelsea Pub. Co., New York, 1968. MR0237281 (38:5571)

[3] Christodoulou, D., and Klainerman, S., The Global Nonlinear Stability of the Minkowski Space, Princeton University Press, Princeton, NJ, 1993. MR1316662 (95k:83006)

[4] de Branges, L., A proof of the Bieberbach conjecture, Acta Math. 154 (1985), 137–152. MR772434 (86h:30026)

[5] Dyson, F., Statistical theory of the energy levels of complex systems, I, II, III, J. Math. Physics 3 (1962), 140–175.

[6] Dyson, F., Missed opportunities, Bull. Amer. Math. Soc. 78 (1972), 635–652. MR0522147 (58:25442)

[7] Gibbs, J. W., The Collected Works of J. Willard Gibbs, Longman's, Green and Co., 1931.

[8] Glimm, J., Solutions in the large for nonlinear hyperbolic systems of equations, CPAM 18 (1965), 697–715. MR0194770 (33:2976)

[9] Gottlieb, D., and Shu, C.-W., On the Gibbs phenomenon and its resolution, SIAM Review 39 (1997), 644–668. MR1491051 (98m:42002)

[10] Hewitt, E., and Hewitt, R. E., The Gibbs-Wilbraham phenomenon: an episode in Fourier analysis, Archive for History of Exact Sciences 21 (1979), 129–160. MR0555102 (81g:01015)

[11] Hilbert, D., Begründung der kinetischen Gastheorie, Math. Ann. 72 (1912), 562–577. MR1511713

[12] Hou, S., and Liu, X.-D., Solutions of multi-dimensional hyperbolic systems of conservation laws by square entropy condition satisfying discontinuous Galerkin method, J. Sci. Computing 31 (2007), 127–151. MR2304273

[13] Koestler, A., and Butterfield, H., The Sleepwalkers, Peregrine Books, 1959.

[14] Leray, J., Sur le mouvement d'un fluide visqueux emplissant l'espace, Acta Math. 63 (1934), 193–248. MR1555394

[15] Liu, X. D., and Lax, P. D., Solution of two-dimensional Riemann problems of gas dynamics by positive schemes, SIAM J. Sci. Comp. 19 (1998), 319–340 (electronic). MR1618863

[16] Loewner, C., Untersuchungen über schlichte konforme Abbildungen des Einheitskreises, I. Math. Ann. 89 (1923), 103–121. MR1512136

[17] Maxwell, J. C., A dynamical theory of the electromagnetic field, Royal Soc. Trans. 155 (1865), The Scientific Papers of James Clerk Maxwell, Scottish Acad. Press, 1982, 526–597. MR0778034 (87i:01032)

[18] McCoy, B. M., Introductory Remarks to Szegö's Paper "On Certain Hermitean Forms Associated with the Fourier Series of a Positive Function", in Gabor Szegö Collected Papers, R. Askey, ed., Vol. 1, 47–51, Birkhäuser, 1982.

[19] Neugebauer, O., Notes on Kepler, CPAM 14 (1961), 593–597. MR0131339 (24:A1191)

[20] Odlyzko, A., On the distribution of the spacing between the zeros of the ζ function, Math. Comp. 48 (1989), 273–308.

[21] Onsager, L., Statistical hydrodynamics, Nuovo Cimento 6, Suppl. 261 (1949). MR0036116 (12:60f)

[22] Riemann, B., Über die Fortpflanzung ebener Luftwellen von endlicher Schwingingsweite, 188–207, Collected Papers, Springer Verlag, 1990.

[23] Schrödinger, E., Collected Papers on Wave Mechanics, Blackie and Son, Ltd., London, 1928. MR0750301 (85k:01057)

[24] Schultz-Rinne, C. W., Collins, J. P., and Glaz, H. M., Numerical solution of the Riemann problem for two-dimensional gas dynamics, SIAM J. Sci. Comp. 14 (1993), 1394–1414. MR1241592 (94j:76062)

[25] Schwartz, J., The pernicious influence of mathematics on science, in Logic, Methodology and Philosophy of Science, Nagel, Suppes and Tarski, eds., Stanford University Press, 1962. MR0166069 (29:3347)

[26] Szegö, G., On certain Hermitean forms associated with Fourier series of a positive function, Comm. Sem. Math. Univ. Lund, Tome Supplementaire, Festshrift Marcel Riesz, Lund (1952), 228-238; also Gabor Szegö Collected Papers, E. Askey, ed., Vol. 3, 270–280, Birkhäuser, 1982. MR0051961 (14:553d), MR0674484 (84d:01082c)

[27] von Neumann, J., Proof of the Quasi-ergodic Hypothesis, NAS Proc. 186 (1932), 70–82; see also Vol. I, Collected Works, Pergamon Press, 1961. MR0157871 (28:1100)

[28] ———, Mathematische Begründung der Quantenmechanic, Goett. Nach. 1927, 1–57; see also Vol. I, Collected Works, Pergamon Press, 1961. MR0157871 (28:1100)

[29] ———, with Burke, A. W., and Goldstine, H. H., Preliminary Discussion of the Logical Design of an Electronic Computing Instrument, Vol. V, Collected Works, Pergamon Press, 1963. MR0157875 (28:1104)

[30] Werner, W., Random planar curves and Schramm-Loewner evolutions, Lectures on Probability Theory and Statistics, 107–195, in Lecture Notes in Math. 1840, Springer Verlag, 2004. MR2079672 (2005m:60020)

[31] Wigner, E., On the statistical distribution of the width and spacing of nuclear resonance levels, Proc. Cambridge Phil. Soc. 47 (1951), 790–798.

[32] Wigner, E., Random Matrices in Physics, SIAM Review 9 (1967), 1–23.

[33] Wilbraham, H., On certain periodic functions, Cambridge Dublin Math. J. 3 (1848), 198–204.

[34] Witten, E., Magic, Mystery and Matrix, 343-352, in Mathematics: Frontiers and Perspectives, Arnold, Atiyah et al., eds., Amer. Math. Soc., Providence, RI, 2000. MR1756796

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