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Monopoles and three-manifolds, by Peter Kronheimer and Tomasz Mrowka, New Mathematical Monographs, 10, Cambridge University Press, Cambridge, 2007, xii+796 pp., ISBN 13: 978-0-521-88022-0

A topological manifold of some given dimension is, by definition, a Hausdorff space whereby every point has an open neighborhood that is homeomorphic to the Euclidean space of the given dimension. This being the case, a neighborhood of any one point looks just like that of another. There is a natural equivalence relation on such manifolds; it identifies two when they are mutually homeomorphic. This is to say that there is a continuous map from one to the other with continuous inverse. One can then ask for a 'list' of the possible equivalence classes. The word 'list' is in quotes because there are countability issues with regard to the fundamental group. Better to fix the homotopy type and ask for the set of homeomorphism classes with the given homotopy type. For example, the issue of classification of compact manifolds with the homotopy type of the n-dimensional sphere is known for all n: there is but one. Smale proved this around 1960 for $n \geq 5$ (as did Stallings at nearly the same time). Freedman proved the case $n = 4$ in 1980. Perelman's recent proof of the Geometrization Conjecture has this as a corollary in the case $n = 3$. The case $n = 2$ follows from the Riemann mapping theorem, and the case $n = 1$ is a nice exercise for an undergraduate. In any event, the classification question for topological manifolds is well understood save for dimension 4 in the cases when the fundamental group is suitably large.

There is a notion of a smooth structure on a topological manifold. To say more, suppose one is given a topological manifold. Each point has a neighborhood with a homeomorphism to Euclidean space. The intersection of two such neighborhoods thus embeds in two ways into a Euclidean space. This being the case, the inverse of one embedding followed by the other defines a homeomorphism from one open set in a Euclidean space to another. Such a map is called a *transition function*. A manifold has a *smooth structure* if each point has a neighborhood with a homeomorphism to Euclidean space such that all transition functions are smooth maps. This is to say that derivatives to any given order exist. This condition is necessary and sufficient to do calculus on the manifold. The point being that such a system of coordinate neighborhoods supplies an unambiguous notion of a differentiable function.

A manifold with a smooth structure is said to be a differentiable manifold. Two such manifolds are deemed to be equivalent if there is a homeomorphism between them that intertwines the corresponding sets of differentiable functions. Such a homeomorphism is said to be a *diffeomorphism*. Here is a natural question: Fix a homeomorphism type of a compact, topological manifold and ask for a list of the set of equivalence classes of diffeomorphism types. In dimensions 2 and 3, each topological manifold has a unique equivalence class of smooth structure. The classification of diffeomorphism equivalence classes for manifolds of dimension greater than 4 is also well understood. This is due to work of people such as Smale, Stallings, Hirsch, Kirby, Seibenmann, Milnor, Kervaire, and then others who I pray forgive me for not naming them explicitly. The justly famous example of the n-dimensional

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sphere supplies a wonderful example; the list stemming from the work of Milnor and Kervaire.

I left out dimension 4 because the story in dimension 4 is not at all understood. The fact is that there are no truly compelling conjectures; the story may well be truly bizarre. What follows is meant to give a sense of where the frontier of ignorance lies. To start, I remark that necessary and sufficient conditions for a compact topological 4-dimensional manifold to have a smooth structure are not known. There are known obstructions; some come from old work of Rochlin and the pair of Kirby and Seibenmann. Then there are those discovered since 1980 that use gauge theories and owe allegiance to the pioneering work of Donaldson (more on gauge theory in a moment). There may be obstructions as yet undiscovered.

To give more of an indication of the divide between known and unknown, start with a smooth, compact, oriented 4-dimensional manifold. Suppose, in addition, that this manifold has an embedded torus inside with a tubular neighborhood that is diffeomorphic to the product of the torus and a disk; the disk giving the directions normal to the surface. A new manifold can be had by taking out this tubular neighborhood and then gluing in something with the same boundary. The boundary is a 3-dimensional torus. What follows gives an example: Take the product of the circle with the complement of a solid tubular neighborhood of a knot in the 3-sphere. The boundary of this product is also a 3-dimensional torus. The result of this excision and regluing is called *knot surgery*. The pioneers of this are Ron Fintushel and Ron Stern. If the torus has certain desirable properties, then the new manifold will be homeomorphic to the original. This begs for the following question:

When is the knot surgery manifold diffeomorphic to the original?

Gauge theories can be used to prove that there are simply connected examples—not terribly complicated—where the new manifold is different if the knot has a non-trivial Alexander polynomial. This Alexander polynomial is a classical invariant of knots that can be used to tell one knot from the other. For example, the Alexander polynomial of the unknotted circle is 1, while that of the trefoil (think of a pretzel) is $t - 1 + t^{-1}$. (In all cases, knot surgery using the unknotted circle gives back the original smooth manifold.)

What follows is now the key point: There are countably many pairwise inequivalent knots with the same Alexander polynomial. This last observation begs the next question:

If two inequivalent knots have the same Alexander polynomial, must the corresponding knot surgeries give diffeomorphic 4*-manifolds*?

At this writing, there is no way to tell. The knot surgery 4-manifolds may well see the more subtle invariants of knots. If this is the case, then the 4-dimensional classification problem is most likely no simpler than the problem of classifying knots in the 3-sphere. Of course, the 4-dimensional classification problem may be way more complicated than this. There may be a zoo of undetected *neutrino manifolds* that pass through ordinary matter, mathematicians in particular, without interaction.

Gauge theory, in the guise of the Seiberg-Witten equations, has supplied invariants of smooth 4-manifolds that ‘see’ the Alexander polynomial of the knot. In fact, all the myriad post-1980 discoveries about smooth 4-dimensional manifolds invoke the Seiberg-Witten equations, or else closely related tools such as the self-dual Yang-Mills equations as originally employed by Donaldson, or else tools recently developed by Peter Ozsvath and Zoltan Szabo that are widely believed to be equivalent to the Seiberg-Witten tools.

The Seiberg-Witten tools give invariants of 4-manifolds that are obtained by counting the solutions to the eponymous equations with suitable algebraic weights. This counting of Seiberg-Witten solutions for a knot surgered manifold (or from other sorts of surgeries) involves what can be loosely described as a multiplicative form of the classical Meyer-Vietoris calculations. The latter allow one to compute the homology groups of a manifold from the homology groups of a given decomposition into subsets and those of the mutual intersections of the various combinations of these subsets. In the case at hand, when the pieces are submanifolds with boundary, the classical Meyer-Vietoris requires knowledge of the homology of the manifolds with boundary, the homology of the boundary, and the manner in which the parts glue together to give the whole.

The gauge theory analogs of the 'homology of the boundary' is deemed *Floer homology* in honor of Andreas Floer. Any given compact, oriented 3-dimensional manifold has various versions of Seiberg-Witten Floer homology. Each version is a diffeomorphism invariant of the 3-manifold; and each is constructed from the solutions to both the 3-dimensional Seiberg-Witten equations as defined on the given manifold, and also the 4-dimensional Seiberg-Witten equations as defined on the product of the manifold with the real line. Here is a very rough idea of what is involved. The chain complex for the Floer homology is taken to be the free Z module that is generated by solutions to the equations on the given 3-manifold. Given this basis for the chain complex, the differential appears as an integer valued matrix whose entries are computed using the solutions to the Seiberg-Witten equations on the product of the manifold with the line. If α and β are used to denote two generators, then the (α, β) matrix element of the differential is a cleverly weighted count of those solutions on the product of the given manifold and the line with the following property: the solution converges at large negative values on the line to the solution on the 3-manifold that defines α, and it converges at large positive values to the solution that defines β.

What was just said raises the following questions: Why not just count solutions outright? Why go through all of this trouble with weighted counts on the manifold and on its product with the line? Here is why. The Seiberg-Witten equations constitute a system of differential equations whose definition requires the a priori choice of a Riemannian metric on the manifold. As it turns out, corresponding solution sets for distinct metrics can be, and often are, vastly different. This is the source of most (but not all) of the evil. Were the solution set metric independent, one could obtain a 3-manifold or 4-manifold invariant by simply counting solutions. Given that the solution sets can vary with the metric, one must come to terms with the manner in which solutions appear and disappear as the metric is varied. Donaldson in the context of 4-manifolds, and Floer in the context of 3-manifolds understood how to compensate for this appearance and disappearance phenomena by assigning suitable algebraic weights to the solutions.

To be sure, there are antecedents for this sort of business in bifurcation theory, Morse theory, algebraic topology, and algebraic geometry. Even so, a stunning breadth of mathematics is needed to tell the full Seiberg-Witten story. Moreover, a string of novel applications has, and is still, increasing this breadth.

For example, there is the relationship to knots. The appearance of the Alexander polynomial was noted almost at the beginning of the Seiberg-Witten era. However, Peter Kronheimer and Tom Mrowka have now found clever variants of 3-dimensional gauge theories that see much more of knot theory. In particular, their new work gives a beautiful geometric perspective to the novel knot invariants that come via the work of Jake Rasmussen, the pair Ozsvath and Szabo, and Mikhail Khovanov. The work of Kronheimer and Mrowka does not speak, as of this writing, to whether smooth 4-manifolds 'see' more than the Alexander polynomial.

Kronheimer and Mrowka uncovered another side of the 3-dimensional Seiberg-Witten story, this its surprise tie-in to the theory of foliations on 3-manifolds. In particular, they find intimate connections to the work of David Gabai and his students. Their most recent discoveries tie in to work of Yi Ni; Ni uses Gabai's notion of a sutured manifold to say things about the invariants of Ozsvath and Szabo. As it turns out, Yi Ni's work and that of Kronheimer and Mrowka is not unrelated to the recent resolution by Stefan Friedl and Stefan Vidussi of a question about symplectic 4-manifolds: When does the product of a circle with a 3-manifold admit a symplectic structure? They find that such is the case if and only if the 3-manifold fibers over the circle. Their proof starts with the Seiberg-Witten/Alexander polynomial connection mentioned above, plus some facts that date to the dawn of the Seiberg-Witten era about the Seiberg-Witten invariants of symplectic 4-manifolds.

A symplectic manifold is one with a closed and non-degenerate 2-form. The odd-dimensional analog of a symplectic manifold is a *contact* manifold. The investigations of contact structures on 3-manifolds is a fast moving and exciting field, due in large part to the pioneering work of Helmut Hofer, Emmanuel Giroux, Yasha Eliashberg, and to many others who I also pray forgive me for not naming them explicitly. Contact structures enter the story through recent work that identifies a version of Seiberg-Witten Floer (co)homology with a novel Floer homology theory for contact structures found by Michael Hutchings.

All of this name dropping is done for a the following reason: it is to indicate that Seiberg-Witten Floer homology has tie-ins to almost all aspects of 3-dimensional differential topology. Here we see knot theory, the theory of Heegard decompositions (via the Ozsvath-Szabo invariants), foliations and sutured manifolds, fiberings over the circle, contact geometry, ... Indeed, what of 3-manifolds is missing?

Ahh, yes something immense is missing: the Geometrization Theorem and its recent proof by Grigory Perelman. Richard Hamilton's Ricci flow and the question of Einstein metrics has not as yet entered. Even so, work by Claude Le Brun and now others uses the 4-dimensional Seiberg-Witten invariants to obtain obstructions to Einstein metrics and complete Ricci flows on compact 4-manifolds. This work hints at a new and possibly very profound chapter in the Seiberg-Witten saga.

I mentioned Tom Mrowka and Peter Kronheimer some paragraphs back not just because their work is so exciting. They wrote the book *Monopoles and three-manifolds*; the book that motivates this essay. To my mind, this book is the definitive bible for anyone wanting to learn the full story of the various Seiberg-Witten Floer homology theories. This bible tells the story in all of its depth and glory. The first eighty pages provides an exceptional outline and sketch of the whole story. The details appear in the subsequent chapter; the analysis, the differential geometry, topology, group theory, algebraic geometry, K-theory, operator theory, . . . ; it is all here, presented completely and in a most elegant way.

I am thinking that there are mathematics books that are classics; these are books that tell a particular story in the right way. As such, they will never go out of date and never be bettered. Kronheimer and Mrowka's book is almost surely such a book. If you want to learn about Floer homology in the Seiberg-Witten context, you will do no better than to read Kronheimer and Mrowka's masterpiece *Monopoles and three-manifolds*.

Some additional reading

Differential topology in dimensions 5 and higher:

- *Lectures on the h-cobordism theorem*, by John Milnor (notes by L. Siebenmann and J. Sondow); Princeton University Press, Princeton, NJ, 1965.
- *Foundational essays on topological manifolds, smoothings, and triangulations*, by Robion C. Kirby and Laurence C. Siebenmann; Annals of Math. Studies **88**, Princeton University Press, Princeton, NJ, 1977.

Differential topology in dimension 3:

- *Three dimensional geometry and topology*, by William P. Thurston and Silvio Levy; Princeton University Press, Princeton, NJ, 1997.
- *Ricci flow and the Poincaré conjecture*, by John Morgan and Gang Tian; Clay Mathematics Monographs, 3, American Mathematical Society, Providence, RI; Clay Mathematics Institute, Cambridge, MA, 2007.

Topological manifolds in dimension 4:

- *The topology of* 4*-manifolds*, by Robion Kirby; Springer Lecture Notes in Mathematics, 1374, Springer-Verlag, Berlin, 1989.
- *The topology of* 4*-manifolds*, Michael H. Freedman and Frank Quinn; Princeton Math. Ser., 39, Princeton University Press, Princeton, NJ, 1990.

Invariants of smooth 4-dimensional manifolds:

- *The geometry of* 4*-manifolds*, by Peter B. Kronheimer and Simon K. Donaldson; Oxford Mathematical Monographs, Oxford Science Publications, The Clarendon Press, Oxford University Press, NY, 1990.
- *Floer homology groups in Yang-Mills theory*, by Simon K. Donaldson with the assistance of Mikio Furuta and Dieter Kotschick; Cambridge Tracts in Mathematics **147**, Cambridge University Press, Cambridge, 2002.
- *The Seiberg-Witten invariants and applications to the topology of smooth four-manifolds*, by John Morgan; Princeton University Press, Princeton, NJ, 1995.
- 4*-manifolds and Kirby calculus*, by Robert E. Gompf and Andras I. Stipsicz; Graduate Studies in Mathematics **20**, American Mathematical Society, Providence, RI, 1999.
- *An introduction to Heegard Floer homology* and *Lectures on Heegard Floer homology*, both by Peter Ozsvath and Zoltan Szabo; pages 3–27 and 29–70 in *Floer homology, gauge theory, and low-dimensional topology*, Clay Mathematics Institute Proceedings 5, American Mathematical Society, Providence, RI, 2006.

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