

## LINEAR WAVES IN THE KERR GEOMETRY: A MATHEMATICAL VOYAGE TO BLACK HOLE PHYSICS

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ABSTRACT. This paper gives a survey of wave dynamics in the Kerr space-time geometry, the mathematical model of a rotating black hole in equilibrium. After a brief introduction to the Kerr metric, we review the separability properties of linear wave equations for fields of general spin  $s = 0, \frac{1}{2}, 1, 2$ , corresponding to scalar, Dirac, electromagnetic fields and linearized gravitational waves. We give results on the long-time dynamics of Dirac and scalar waves, including decay rates for massive Dirac fields. For scalar waves, we give a rigorous treatment of superradiance and describe rigorously a mechanism of energy extraction from a rotating black hole. Finally, we discuss the open problem of linear stability of the Kerr metric and present partial results.

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### 1. INTRODUCTION TO GENERAL RELATIVITY AND BLACK HOLES

It is apparently the Reverend John Michell who was the first to suggest in a paper communicated to the Royal Society of London in November 1783 that if one assumed that light consisted of particles subject to the law of universal gravitation, then the Universe could in principle contain stars that would be completely invisible to any external observer. This conclusion was arrived at independently by Laplace

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in 1796 in his *Exposition du système du monde*, and the idea is credited to him in several classical references in General Relativity, such as *The Large Scale Structure of Space-Time* by Hawking and Ellis [33].<sup>1</sup> The concept of a dark star would have probably remained as a mathematical curiosity had it not been for the advent of the Theory of Relativity and for the important advances in the study of stellar structure which took place in the early decades of the twentieth century, following the development of quantum statistical mechanics. These ideas led a nineteen year old Indian student, while on his way in July of 1930 from Madras, India, to Cambridge, England, where he was to begin his doctoral studies, to discover the extraordinary consequence of General Relativity—that a star which is sufficiently massive will, after having exhausted its nuclear fuel, collapse gravitationally to a point at which the very structure of space-time will become singular. In spite of the severe opposition expressed to this daring work by Sir Arthur Eddington, the leading astronomer of his time, the Indian student's work was completely validated by further analysis and he went on to receive a Nobel Prize many years later, in 1983, for his discovery of this lower mass limit. His name was Subrahmanyan Chandrasekhar, and he in turn became one of the most eminent theoretical astrophysicists of his time.

It took a long period of maturation before a proper understanding was developed of what the space-time geometry would be near such a collapsed object. However, it was already apparent from the early work dating back to 1915 of Karl Schwarzschild on the relativistic space-time geometry near a stationary spherically symmetric star, that the singularity should not be directly visible to an external observer who would be at rest with respect to the star, and that it should be hidden behind a one-way membrane, known today as an *event horizon*. This led the physicist John Wheeler to refer informally to these collapsed stars as *black holes*. This evocative way of referring to these very strange objects soon became the standard terminology.

In a lecture given in memory of Karl Schwarzschild at the Astronomischen Gesellschaft in Hamburg, Germany, in 1986, Chandrasekhar stated the following [12]:

Black holes are macroscopic objects with masses varying from a few solar masses to millions of solar masses. To the extent that they may be considered as stationary and isolated, to that extent, they are all, every single one of them, described *exactly* by the Kerr solution. This is the only instance we have of an exact description of a macroscopic object. Macroscopic objects, as we see them around us, are governed by a variety of forces, derived from a variety of approximations to a variety of physical theories. In contrast, the only elements in the construction of black holes are our basic concepts of space and time. They are, thus, almost by definition, the most perfect macroscopic objects there are in the universe. And since the general theory of relativity provides a single two parameter family of solutions for their description, they are the simplest as well.

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<sup>1</sup>The reader is referred to the excellent survey by Israel [40] on the genesis and evolution of the concept of a dark star, starting from Michell and culminating in the laws of black hole thermodynamics.

The two parameter family of exact solutions referred to by Chandrasekhar is the family of *Kerr solutions* of the Einstein field equations of General Relativity, discovered by Roy Kerr in 1963 [42]. These generalize the Schwarzschild geometry to the case in which the black hole has non-zero angular momentum, and are stationary and axi-symmetric as opposed to stationary and spherically symmetric. In the introduction to an earlier paper, written on the occasion of Einstein's centenary, Chandrasekhar wrote about the Kerr solution [10]:

The special significance of Kerr's solution for astronomy derives from the theorems of Carter (1971) and Robinson (1975) which establish its uniqueness for an exact description of black holes that occur in nature. But Kerr's solution has also surpassing theoretical interest: it has many properties that have the aura of the miraculous about them. These properties are revealed when one considers the problem of the reflection and transmission of waves of different sorts (electromagnetic, gravitational, neutrino, and electron waves) by the Kerr black hole.

Regarding the "miraculous properties", he went on to say in the conclusion of the same paper:

What, may we inquire, are these properties? In many ways, the most striking feature is the separability of all the standard equations of mathematical physics in Kerr geometry.

The goal of this paper is to survey some recent developments in the study of linear wave equations in Kerr geometry, and thereby to give a description of some of the key physical properties of black holes from a rigorous mathematical perspective. We remark that there are also other very interesting and important results in mathematical relativity, such as, for example, black hole uniqueness theorems [37], Hawking radiation [52, 35], the formation of singularities [14], the positive mass theorem [48, 53], and the Penrose inequalities [39, 3, 4]. But this paper has its main focus on the topics that we have worked on: the dynamics and decay properties of various fields in the Kerr geometry. Put differently, linear waves in the Kerr geometry will serve both as vehicle and guide for our voyage to black hole physics. As stated above by Chandrasekhar, the setting for our voyage is provided by the space-time geometry corresponding to an exact solution of the Einstein field equations of gravitation, namely the Kerr solution. We shall see that the presence of a black hole and a space-time singularity manifests itself in remarkable and often unexpected ways through the behavior of waves in the space-time geometry of the black hole's exterior. Among the unexpected aspects in the behavior of waves, we shall explain that waves corresponding to localized initial data for the Dirac and scalar wave equations in Kerr geometry will always decay in any localized region of space as  $t$  tends to infinity, suggesting that the corresponding particles either "fall into the black hole" or "escape to infinity". Some other features are that massive Dirac waves decay in Kerr geometry at a rate *slower* than in Minkowski space, and the fact that spatially localized wave packets can be used to extract energy and angular momentum from the Kerr black hole. We shall also see that even though the geometric properties of the Kerr metric make it possible to solve all the known linear wave equations in Kerr geometry by separation of variables, it also gives rise to specific challenges in the analysis of these wave equations, which require the development of new techniques. These challenges stem from the presence of angular

momentum in the Kerr black hole, which in turn causes the conserved energy for particles and waves to not be everywhere positive. But we are anticipating what is to come, and before talking about black holes, we need to set the stage by recalling some of the essentials of General Relativity, and even before that, of Special Relativity.

Space-time in Special Relativity is given by *Minkowski space-time*, that is  $\mathbb{R}^4$ , endowed with the standard inner product of Lorentzian signature  $(+, -, -, -)$ . Of particular significance are the *light cones*, consisting of the sets of points satisfying an equation of the form

$$(1.1) \quad (t - t_0)^2 - (x - x_0)^2 - (y - y_0)^2 - (z - z_0)^2 = 0.$$

The light cone (1.1) describes the set of points swept out in space-time by a light signal emanating from the point  $(x_0, y_0, z_0)$  at time  $t_0$ . (We always work in units in which the speed of light is equal to one.) The famous Michelson-Morley experiment showed that the speed of light is the same for all observers in uniform motion. In particular, the observers cannot move faster than light, and therefore their space-time trajectories, or world lines, lie everywhere inside the light cone. Such lines are called *time-like*. The generators of the light cone (1.1) correspond to the world lines of light rays and are referred to as *null lines*. The light cones thus introduce a causal structure in space-time, according to which every observer in uniform motion has a future and a past, given by the interior of the upper, respectively lower, sheet of the light cone based at its space-time position. The set of affine transformations of Minkowski space-time preserving the Lorentzian inner product (and in particular the light cones) forms the so-called Poincaré group, the semi-direct product of the Lorentz group  $O(1, 3)$  by the group  $\mathbb{R}^4$  of space-time translations. All of the remarkable physical consequences of Special Relativity, such as length contraction, time dilation, and red shift, correspond to the mathematical properties of the Poincaré group.

While Minkowski space is a model of space-time which is very well suited to the study of electromagnetic phenomena, such as the propagation of light, it is not sufficiently general to give a description of space-time that includes gravitation. Indeed, the structure of Minkowski space-time is anchored around the privileged role played by observers in uniform motion and therefore by linear changes of coordinates. But we already know from Newtonian physics that gravitation manifests itself by causing observers in free fall to be accelerated. General Relativity emerges from Special Relativity by incorporating the Principle of Equivalence, which equates accelerated frames of reference with the presence of a gravitational field and allows in particular for general local coordinate systems, which retain the Minkowskian character of space-time. In this way, General Relativity becomes a geometric theory of gravitation, in which space-time is taken to be a four-dimensional manifold  $M_4$ , endowed with a pseudo-Riemannian metric  $g$  of Lorentzian signature  $(+, -, -, -)$ , implying that the tangent space at each point of space-time is isometric to Minkowski space-time. The world lines of test particles with non-zero rest mass acted upon only by gravity now become *time-like geodesic curves* in  $(M_4, g)$ , i.e., geodesics  $\gamma$  whose tangent vector satisfies

$$0 < g(\dot{\gamma}, \dot{\gamma}) \equiv g_{ij}(\gamma) \dot{\gamma}^i \dot{\gamma}^j,$$

where, according to Einstein's summation convention, we sum over repeated indices. Likewise, light rays propagate along *null geodesics* with  $g(\dot{\gamma}, \dot{\gamma}) = 0$ . These geodesic

curves are the analogue of the time-like and null lines of Minkowski space-time. In a local coordinate system the geodesic equation is given by

$$(1.2) \quad \frac{d^2 x^i}{ds^2} + \Gamma^i_{kj} \frac{dx^k}{ds} \frac{dx^j}{ds} = 0,$$

where the coefficients  $\Gamma^k_{ij}$ , known as the *Christoffel symbols*, are built from the metric and its first derivatives,

$$(1.3) \quad \Gamma^k_{ij} = \frac{1}{2} g^{kr} (\partial_j g_{ir} + \partial_i g_{jr} - \partial_r g_{ij}).$$

The geodesic equation can also be written as

$$(1.4) \quad \nabla_{\dot{\gamma}} \dot{\gamma} = 0,$$

where  $\nabla$  is the Levi-Civita connection, often referred to as a *covariant derivative*, defined for a general vector field  $X$  by

$$\nabla_i X = \left( \partial_i X^j + \Gamma^j_{ik} X^k \right) \frac{\partial}{\partial x^j},$$

where  $\partial/\partial x^k$  denotes the coordinate basis of the tangent space. The Levi-Civita connection is the unique metric, torsion-free connection on the Lorentzian manifold.

In 1915 Albert Einstein discovered the relativistic field equations of gravitation, which account for the presence of the gravitational field through space-time curvature. To formulate these equations, we need to introduce the *Riemann curvature tensor*, which quantifies how a space-time  $(M_4, g)$  deviates from being Minkowskian in a neighborhood of any point. The Riemann curvature tensor  $R^l_{ijk}$  is defined by the relations

$$\nabla_i \nabla_j X - \nabla_j \nabla_i X = R^l_{ijk} X^k \frac{\partial}{\partial x^l},$$

valid for any vector field  $X$ . It measures the degree to which the covariant derivatives fail to commute. One can easily see from the expression of the Christoffel symbols (1.3) that the Riemann tensor is linear in the second derivatives of the components  $g_{ij}$  of the metric and quadratic in the first derivatives. Its expression in local coordinates is quite complicated, and we will not give it here. What we will retain is the interpretation of the Riemann tensor as a measure of the “non-flatness” of a metric, by recalling the classical theorem of Riemannian geometry which says that the condition  $R^l_{kij} \equiv 0$  is equivalent to  $(M_4, g)$  being locally isometric to Minkowski space-time.

Gravitational forces can be understood in General Relativity as tidal forces acting on nearby point particles. Mathematically, this is made precise by the *Jacobi equation*, which shows how curvature affects the behavior of neighboring geodesics. Consider thus a one parameter family of time-like geodesics  $\gamma : (s, \alpha) \mapsto \gamma(s, \alpha)$ , where  $s$  is arclength and  $\alpha$  is a parameter labeling the geodesic curves in our family. The vector field  $U := \partial\gamma/\partial s$  for fixed  $\alpha$  is the unit tangent vector along the corresponding geodesic, while the vector field  $V := \partial\gamma/\partial\alpha$  restricted to a geodesic curve  $\gamma(s, \alpha_0)$  of our family measures the deviation between that curve and the neighboring geodesics. The Jacobi equation expresses the second derivative of  $V$  along the geodesic in terms of the Riemann curvature tensor,

$$\frac{d^2 V^i}{ds^2} = -R^i_{kjl} U^k V^j U^l.$$

The right side of this equation measures the tidal force for an observer moving along the geodesic.

Einstein discovered the field equations of gravitation by the requirement that one recovers Newtonian gravity in the non-relativistic limit (i.e., small perturbations of Minkowski space and particles moving slowly compared to the speed of light) and taking into account the conservation laws for energy and momentum. The field equations read

$$(1.5) \quad R_{ij} - \frac{1}{2} R g_{ij} = 8\pi T_{ij}.$$

The left-hand side of (1.5) is built from the *Ricci tensor*  $R_{ij}$  and *scalar curvature*  $R$ , which are obtained by taking the trace of the Riemann tensor,

$$R_{ij} = R^l{}_{ilj}, \quad R = R^i{}_i.$$

The right-hand side of (1.5) is the *energy-momentum tensor* of the matter fields. The left hand-side of (1.5) is a symmetric tensor, which is divergence-free<sup>2</sup> according to the second Bianchi identities, a set of integrability conditions satisfied by the Riemann tensor. As a consequence, the energy-momentum tensor must also be symmetric and divergence-free; this expresses the local conservation of energy and momentum. The Einstein field equations need to be augmented by a set of field equations for the matter fields. For example, in the case of the interaction of gravity with electromagnetism, one obtains the source-free *Einstein-Maxwell equations*,

$$(1.6) \quad R_{ij} = 8\pi(F_{ik}F_j{}^k - \frac{1}{4}g_{ij}F_{kl}F^{kl}),$$

$$(1.7) \quad \nabla_l F^{kl} = 0, \quad \nabla_i F_{jk} + \nabla_j F_{ki} + \nabla_k F_{ij} = 0,$$

where  $F_{ij}$  denotes the electromagnetic field tensor (and we always use the metric to raise or lower indices).

The Einstein field equations (1.5) are a highly complicated system of non-linear partial differential equations, which are extremely difficult to analyze in full generality. This holds true even in the vacuum case ( $T_{ij} \equiv 0$ ), where they reduce to the vacuum Einstein equations

$$(1.8) \quad R_{ij} = 0.$$

An important analytic result on the Einstein equations was obtained in the work by Christodoulou and Klainerman [16], where the non-linear stability of Minkowski space is proved. This work has inspired intensive research on the analysis of the Einstein equations, with and without matter. Since in this article we cannot enter all aspects of mathematical relativity, we shall restrict our attention to special solutions of the Einstein equation and to the analysis of *linear* wave equations in a given space-time geometry.

The first non-flat exact solution of these equations is a one-parameter family of static and spherically symmetric metrics discovered by Karl Schwarzschild in 1915,

$$(1.9) \quad ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2).$$

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<sup>2</sup>One could of course add any constant multiple of the metric tensor  $g_{ij}$  to the left-hand side of (1.5) and still obtain a tensor with zero divergence. This amounts to adding a cosmological constant to the Einstein equations, but this constant is usually taken to be zero in applications to black holes, where space-time is assumed to be asymptotically flat.

This metric describes a black hole of mass  $M > 0$ . The metric is singular at  $r = 2M$ , but this singular behavior is just an artifact of the coordinate system being used. Indeed, the metric can be extended analytically across the event horizon by introducing the Regge-Wheeler coordinate

$$u = r + 2m \log(r - 2M),$$

and an advanced null coordinate

$$v = t + u,$$

in which the metric is given by

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - 2dvdr - r^2(d\theta^2 + \sin^2\theta d\varphi^2).$$

Now the locus  $r = 2M$  is a null hypersurface which corresponds to the boundary of the black hole and is called the *event horizon*. Working in these coordinates, one can show that future-directed time-like or null curves can only cross the hypersurface  $r = 2M$  from outside to inside, which explains why this hypersurface is referred to as an event horizon. Physically speaking, the event horizon defines the infinite red-shift surface as seen by a distant observer. Unlike the event horizon  $r = 2M$ , the locus  $r = 0$  is a true curvature singularity, which is not removable by any change of coordinates. One can, for example, verify that the square of the Riemann tensor blows up at  $r = 0$ ,  $R_{ijkl}R^{ijkl} \sim M^2r^{-6}$ .

According to Noether's theorem, there is a close connection between symmetries of space-time and conservation laws. In General Relativity, this connection is made precise using the notion of a *Killing field*. A Killing vector field  $X$  is characterized by the so-called Killing equation

$$\nabla_i X_j + \nabla_j X_i = 0.$$

This equation implies that the Lie derivative of the metric vanishes, meaning that the flow of the vector field  $X$  is an isometry of space-time. In order to get the corresponding conservation law, one contracts the energy momentum tensor with the Killing field. The resulting vector field  $Y^i = T^{ij} X_j$  is divergence-free. Namely,

$$\operatorname{div}(Y) = \nabla_i(T^{ij} X_j) = (\nabla_i T^{ij}) X_j + T^{ij} \nabla_i X_j = 0,$$

where we applied the Killing equation and used that the energy-momentum tensor is symmetric and divergence-free as a consequence of the Einstein equations. Applying Gauss' theorem, one concludes that every boundary integral of  $Y$  vanishes,

$$(1.10) \quad \int_{\partial V} Y_i \nu^j d\mu_{\partial V} = 0$$

(where  $V$  is any open subset of space-time with smooth boundary,  $\nu_j$  denotes a unit normal, and  $d\mu_{\partial V}$  is the volume form on  $\partial V$ ). Taking for  $V$  a set whose boundary consists of two space-like hypersurfaces corresponding to two time slices of an observer, (1.10) gives rise to a *conservation law* for a spatial integral.

We remark that the above method is also useful in the situation when space-time has no symmetries. In this case, one can try to find a vector field which is an approximate Killing field in the sense that the expression  $\nabla_i X_j + \nabla_j X_i$  vanishes up to a small error term. Then in (1.10) one also gets error terms, but nevertheless one can hope to obtain useful estimates. This is the so-called *vector field method* which has been used in many papers to study the behavior of waves in the Schwarzschild

geometry (see for example [5, 17, 18]). An extension of the method to conformal symmetries goes back to Morawetz [45]. For a general exposition on the role of vector fields in the analysis of Euler-Lagrange systems of partial differential equations of hyperbolic type, we refer to [15].

## 2. THE KERR METRIC AND THE BLACK HOLE UNIQUENESS THEOREM

It took nearly fifty years until an exact solution of the Einstein equations describing the outer geometry of a *rotating* black hole in equilibrium was found. This is the Kerr metric given by

$$(2.1) \quad ds^2 = \frac{\Delta}{U} (dt - a \sin^2 \vartheta d\varphi)^2 - U \left( \frac{dr^2}{\Delta} + d\vartheta^2 \right) - \frac{\sin^2 \vartheta}{U} (a dt - (r^2 + a^2) d\varphi)^2,$$

where

$$(2.2) \quad U = r^2 + a^2 \cos^2 \vartheta, \quad \Delta = r^2 - 2Mr + a^2,$$

and the coordinates  $(t, r, \vartheta, \varphi)$  are in the range

$$(2.3) \quad -\infty < t < \infty, \quad M + \sqrt{M^2 - a^2} < r < \infty, \quad 0 < \vartheta < \pi, \quad 0 < \varphi < 2\pi.$$

Here the parameters  $M$  and  $a$  describe the mass and the angular momentum per unit mass of the black hole. It is easily verified that in the case  $a = 0$ , one recovers the Schwarzschild metric (1.9).

For this metric to describe a black hole we need to assume that  $M^2 > a^2$ , giving a bound for the angular momentum relative to the mass. In this so-called *non-extreme case*, it can be verified by an argument similar to the one used for the Schwarzschild solution that the null hypersurface

$$(2.4) \quad r = r_1 := M + \sqrt{M^2 - a^2}$$

defines the event horizon, the boundary of the black hole. Here we shall only consider the region  $r > r_1$  outside the event horizon. The coefficients of the Kerr metric are independent of  $t$  and  $\varphi$ , showing that the space-time geometry is stationary and axi-symmetric. One of the key features of the Kerr geometry is the existence of an *ergosphere*, that is a region which lies outside the event horizon, in which the vector field  $\frac{\partial}{\partial t}$  is *space-like*. In order to determine the ergosphere, we consider the norm of the vector field  $\xi = \frac{\partial}{\partial t}$ ,

$$(2.5) \quad g_{ij} \xi^i \xi^j = g_{tt} = \frac{\Delta - a^2 \sin^2 \vartheta}{U} = \frac{r^2 - 2Mr + a^2 \cos^2 \vartheta}{U}.$$

This shows that  $\xi$  is space-like in the open region of space-time where

$$(2.6) \quad r^2 - 2Mr + a^2 \cos^2 \vartheta < 0,$$

which defines the ergosphere. It is a bounded region of space-time outside the event horizon, which intersects the event horizon at the poles  $\vartheta = 0, \pi$ . The implications of the ergosphere for the analysis of wave equations cannot be overstated because the ergosphere corresponds to a region in which the conserved energy associated to a given field will *fail to be positive* (see for example (5.2) and (5.3) below).

The physical significance of the Kerr metric manifests itself through the *black hole uniqueness theorems* of Israel [38], Carter [8] and Robinson [47, 37]. Physically, these theorems indicate that the stationary end state of a rotating black hole should be given precisely by the non-extreme Kerr geometry. From the mathematical point of view, these theorems are uniqueness results for a class of boundary value problems

for the vacuum Einstein equations, and their proof requires additional mathematical assumptions. As boundary conditions one assumes the asymptotic flatness of space-time in the form of weak asymptotic simplicity (see [52, p. 282]), plus the existence of an event horizon with the natural spherical topology. Furthermore, one assumes axi-symmetry and pseudo-stationarity (pseudo-stationarity means that there is a Killing field which is assumed to be time-like only in the asymptotic end). As technical assumptions which are also motivated from physics, one needs to impose a causality condition as well as time orientability, and one needs to fix the topology of space-time to be  $\mathbb{R}^2 \times S^2$ . (The review article by Carter [9] and the books [33, 37] specify and discuss all the conditions in detail.) The theorem is as follows:

**Theorem 2.1.** *Under the above assumptions, every solution of the vacuum Einstein equations admits a global chart  $(t, r, \vartheta, \varphi)$  in which the metric is the Kerr solution (2.1).*

### 3. LINEAR WAVE EQUATIONS IN THE KERR GEOMETRY AND THEIR SEPARATION

In this section we give an overview of the physically relevant linear wave equations and present their separability properties in the Kerr geometry. For simplicity, we begin with the scalar wave equation, which in a general space-time reads

$$(3.1) \quad g^{ij} \nabla_i \nabla_j \Phi = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^i} \left( \sqrt{-g} g^{ij} \frac{\partial}{\partial x^j} \right) \Phi = 0,$$

where  $g$  denotes the determinant of the metric  $g_{ij}$ . For the Kerr metric, this becomes

$$(3.2) \quad \left[ -\frac{\partial}{\partial r} \Delta \frac{\partial}{\partial r} + \frac{1}{\Delta} \left( (r^2 + a^2) \frac{\partial}{\partial t} + a \frac{\partial}{\partial \varphi} \right)^2 - \frac{\partial}{\partial \cos \vartheta} \sin^2 \vartheta \frac{\partial}{\partial \cos \vartheta} - \frac{1}{\sin^2 \vartheta} \left( a \sin^2 \vartheta \frac{\partial}{\partial t} + \frac{\partial}{\partial \varphi} \right)^2 \right] \Phi = 0.$$

We denote the square bracket in this equation by  $\square$ , (although it is actually a scalar function times the wave operator in (3.1)). We now explain the separation of variables as discovered by Carter [9]. Due to the stationarity and axi-symmetry, we can separate the  $t$ - and  $\varphi$ -dependence with the usual plane-wave ansatz

$$(3.3) \quad \Phi(t, r, \vartheta, \varphi) = e^{-i\omega t - ik\varphi} \phi(r, \vartheta),$$

where  $\omega$  is a quantum number which could be real or complex and which corresponds to the “energy”, and  $k \in \mathbb{Z}$  is a quantum number corresponding to the projection of angular momentum onto the axis of symmetry of the black hole. Substituting (3.3) into (3.2), we see that the wave operator splits into the sum of radial and angular parts,

$$(3.4) \quad \square \Phi = (\mathcal{R}_{\omega, k} + \mathcal{A}_{\omega, k}) \Phi,$$

where  $\mathcal{R}_{\omega, k}$  and  $\mathcal{A}_{\omega, k}$  are given by

$$(3.5) \quad \mathcal{R}_{\omega, k} = -\frac{\partial}{\partial r} \Delta \frac{\partial}{\partial r} - \frac{1}{\Delta} ((r^2 + a^2)\omega + ak)^2,$$

$$(3.6) \quad \mathcal{A}_{\omega, k} = -\frac{\partial}{\partial \cos \vartheta} \sin^2 \vartheta \frac{\partial}{\partial \cos \vartheta} + \frac{1}{\sin^2 \vartheta} (a\omega \sin^2 \vartheta + k)^2.$$

We can therefore separate the variables  $r$  and  $\vartheta$  with the multiplicative ansatz

$$(3.7) \quad \phi(r, \vartheta) = R(r) \Theta(\vartheta),$$

to obtain for given  $\omega$  and  $k$  the system of ordinary differential equations

$$(3.8) \quad \mathcal{R}_{\omega,k} R_\lambda = -\lambda R_\lambda, \quad \mathcal{A}_{\omega,k} \Theta_\lambda = \lambda \Theta_\lambda .$$

The separation constant  $\lambda$  is an eigenvalue of the angular operator  $\mathcal{A}_{\omega,k}$  and can thus be thought of as an angular quantum number. In the spherically symmetric case (i.e.,  $a = 0$ ),  $\lambda$  goes over to the usual eigenvalues  $\lambda = l(l+1)$  of the total angular momentum. We point out that the last separation (3.7) is not obvious because it does not correspond to an underlying space-time symmetry.

For the Dirac equation describing a spin  $\frac{1}{2}$  field, the situation is more complicated because it is a system of differential equations and involves a rest mass  $m$ . The separability of the Dirac equation in the Kerr metric was first established by Chandrasekhar in 1976, [11], using an ingenious new method. The Dirac equation for a particle of mass  $m$  reads

$$(3.9) \quad (i\gamma^j \nabla_j - m)\Psi = 0,$$

where the Dirac matrices  $\gamma^j$  are related to the metric by the anti-commutation relations

$$\frac{1}{2} (\gamma^j \gamma^k + \gamma^k \gamma^j) = g^{jk} \mathbf{1} ,$$

and  $\nabla$  is a connection on the spinors. We write the Dirac equation in a Newman-Penrose null frame [11]  $(l, n, m, \bar{m})$ , i.e., in a frame in which the metric takes the form

$$g_{ij} = l_i n_j + n_i l_j - m_i \bar{m}_j - \bar{m}_i m_j .$$

Choosing a suitable frame and after an appropriate linear transformation of the Dirac spinor,

$$\Psi \rightarrow \hat{\Psi} = S \Psi ,$$

where  $S = S(r, \vartheta)$  is a diagonal matrix, the Dirac equation becomes

$$(3.10) \quad (\mathcal{R} + \mathcal{A}) \hat{\Psi} = 0 ,$$

where  $\mathcal{R}$  and  $\mathcal{A}$  are certain first-order matrix differential operators. The separation of variables is achieved by assuming that each component of the transformed spinor  $\hat{\Psi}$  is a product of functions of one variable, according to a specific separation pattern,

$$(3.11) \quad \hat{\Psi}(t, r, \vartheta, \varphi) = e^{-i\omega t} e^{-i(k+\frac{1}{2})\varphi} \begin{pmatrix} X_-(r) Y_-(\vartheta) \\ X_+(r) Y_+(\vartheta) \\ X_+(r) Y_-(\vartheta) \\ X_-(r) Y_+(\vartheta) \end{pmatrix}, \quad k \in \mathbb{Z} .$$

By substituting (3.11) into (3.10), we obtain the separated matrix equations

$$(3.12) \quad \mathcal{R} \hat{\Psi} = \lambda \hat{\Psi}, \quad \mathcal{A} \hat{\Psi} = -\lambda \hat{\Psi},$$

where  $\lambda$  is a separation constant, under which the transformed Dirac equation (3.10) decouples into a system of ODEs.

The remaining field equations of physical interest are those describing electromagnetic and linearized gravitational waves. It is a remarkable and very useful fact

that these waves are all governed by a single second-order equation, the so-called *Teukolsky master equation* [11]. It reads

$$(3.13) \quad \left[ \frac{\partial}{\partial r} \Delta \frac{\partial}{\partial r} - \frac{1}{\Delta} \left( (r^2 + a^2) \frac{\partial}{\partial t} + a \frac{\partial}{\partial \phi} - (r - M)s \right)^2 - 4s(r + ia \cos \theta) \frac{\partial}{\partial t} + \frac{\partial}{\partial \cos \theta} \sin^2 \theta \frac{\partial}{\partial \cos \theta} + \frac{1}{\sin^2 \theta} \left( a \sin^2 \theta \frac{\partial}{\partial t} + \frac{\partial}{\partial \phi} + is \cos \theta \right)^2 \right] \Psi_s = 0.$$

The parameter  $s$  denotes the spin. We explain how this equation comes about, beginning with the case  $s = \pm 1$  of a Maxwell field. The source-free Maxwell's equations are given by

$$(3.14) \quad \nabla_i F^{kl} = 0, \quad \nabla_i F_{jk} + \nabla_j F_{ki} + \nabla_k F_{ij} = 0,$$

where  $F_{ij}$  is the electromagnetic field tensor. Choosing the same Newman-Penrose null frame as in the separation of the Dirac equation, we can combine the components of the field tensor into the three complex functions

$$\Psi_1 = F_{ij} l^i m^j, \quad \Xi_0 = \frac{1}{2\rho} F_{ij} (l^i n^j + \bar{m}^i m^j), \quad \Psi_{-1} = \frac{1}{\rho^2} F_{ij} \bar{m}^i n^j,$$

where  $\rho = -(r - ia \cos \vartheta)^{-1}$ . Then Maxwell's equations (3.14) can be written as

$$\mathcal{D} \begin{pmatrix} \Psi_1 \\ \Xi_0 \\ \Psi_{-1} \end{pmatrix} = 0,$$

where  $\mathcal{D}$  is a first-order matrix differential operator. The key point is that, multiplying this equation from the left by a suitable first-order matrix differential operator, one obtains decoupled second-order equations for  $\Psi_1$  and  $\Psi_{-1}$ , which are precisely the Teukolsky equation for  $s = \pm 1$ . It is important to note that this process of cross-differentiation and elimination makes use of some key commutation identities between certain covariant derivative operators in the Kerr metric. These identities hold as a consequence of the special algebraic structure of the Riemann tensor in the Kerr geometry; namely, it is of type D in the Petrov-Penrose classification, meaning that the Weyl tensor has two repeated principal null directions. If  $\Psi_1$  or  $\Psi_{-1}$  are known, then the remaining components are readily obtained by differentiation using the so-called Teukolsky-Starobinsky identities [11].

Finally, for  $s = \pm 2$ , the Teukolsky equation (3.13) is obtained similarly by cross-differentiation and elimination on the systems of first-order equations obtained by linearizing the Bianchi identities

$$\nabla_i R_{jklm} + \nabla_j R_{kilm} + \nabla_k R_{ijlm} = 0,$$

around the Kerr metric. If we denote by  $\psi_0^{(1)}$  and  $\psi_4^{(1)}$  the first-order perturbations of the Newman-Penrose curvature components  $\psi_0$  and  $\psi_4$  defined by

$$\psi_0 = C_{ijkl} l^i m^j l^k m^l, \quad \psi_4 = C_{ijkl} n^i \bar{m}^j n^k \bar{m}^l,$$

where  $C_{ijkl}$  denotes the Weyl conformal curvature tensor, then the unknowns in the Teukolsky equations with  $s = \pm 2$  are given by

$$\Psi_2 = \psi_0^{(1)}, \quad \Psi_{-2} = \rho^{-4} \psi_4^{(1)}.$$

We remark that for  $s = 0$ , the Teukolsky equation (3.13) reduces to the scalar wave equation (3.1). The case  $s = 1/2$  corresponds to the massless Dirac equation.

We also point out that the reduction to a second-order scalar equation does not work for the massive Dirac equation. Just like the scalar wave equation, the Teukolsky equation can be reduced to ordinary differential equations by the separation ansatz

$$\Psi_s = e^{-i\omega t - ik\varphi} R_s(r) \Theta_s(\vartheta).$$

These are the separability properties that were referred to by Chandrasekhar [10] in the excerpt quoted in the Introduction as “having the aura of the miraculous”.

Clearly, the above linear wave equations can also be analyzed in the special case  $a = 0$ , where the Kerr geometry simplifies to the spherically symmetric and static Schwarzschild geometry (1.9). Since in this paper we shall concentrate on rotating black holes, we merely mention a few important papers on the analysis of waves in the Schwarzschild geometry. In the fundamental paper [41] it is proven that the solutions of the scalar wave equation are pointwise bounded, uniformly in time. The corresponding rates of decay are derived in [19] and [43, 44]. In [5] the boundedness of solutions to Maxwell’s equations in the Schwarzschild geometry is proved. Finally, [20] considers the scalar wave equations in the Kerr metric in the case  $a \ll M$  of small angular momentum, where the metric is close to Schwarzschild.

#### 4. DYNAMICS OF DIRAC WAVES

To get a more detailed picture of the properties of a rotating black hole, it is very helpful to consider dynamical situations where physical objects are moving in the Kerr geometry. The simplest situation is to consider point particles, whose motion is described by the geodesic equation (1.2). As is worked out in detail in [11], there are stable orbits where a point particle “rotates around” the black hole. Apart from the circular orbits, there are elliptic orbits which are not quite closed due to the perihelion shift. Furthermore, there are orbits where the particle falls into the black hole or escapes to infinity. Considering quantum mechanical particles makes the situation more interesting, because due to the Heisenberg Uncertainty Principle, these phenomena could happen simultaneously with certain probabilities. To clarify the picture, in this section we consider the dynamics of Dirac waves in the Kerr geometry. We shall first see how the Cauchy problem for the Dirac equation in Kerr geometry can be solved by means of an integral representation for the propagator, realized as a superposition of “modes” arising from the separation of variables. We then analyze this integral representation to show that in the presence of a black hole, a quantum mechanical Dirac particle cannot remain in a localized region of space for arbitrarily large times, in other words that the quantum mechanical particle corresponding to the Dirac wave function will either eventually “fall into the black hole” or “escape to infinity”. We also discuss the decay rates. Our analysis is of course limited by the fact that we look at a test Dirac field in Kerr geometry as opposed the fully coupled axi-symmetric Einstein-Dirac equations, but the non-existence results proved in [22] for black hole solutions of the spherically symmetric Einstein-Dirac-Maxwell equations suggest that the same conclusions should hold in the fully coupled axi-symmetric case.<sup>3</sup>

The first step is to bring the Dirac equation (3.10) into Hamiltonian form in a way that is compatible with the separation of variables. This is not a trivial step

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<sup>3</sup>A major obstacle in the analysis of the fully coupled axi-symmetric Einstein-Dirac-Maxwell equations by similar methods is that a complete separation of variables into ordinary differential equations seems most unlikely, and one has to deal with the full system of PDEs.

because the coefficients of the  $\partial_t$ -term depend on both  $r$  and  $\vartheta$ . The Dirac equation becomes

$$(4.1) \quad i \frac{\partial}{\partial t} \Psi = H \Psi,$$

where  $H$  is a first-order matrix differential operator. It is important for the analysis of this evolution equation to introduce a positive scalar product  $\langle \cdot | \cdot \rangle$ , with respect to which  $H$  is symmetric. To this end, we consider the Dirac current  $\bar{\Psi} \gamma^j \Psi$ , which, as a consequence of the Dirac equation, it is divergence-free. Furthermore, it is time-like and future-directed, so that its inner product  $\bar{\Psi} \gamma^0 \Psi$  with the unit normal to the space-like hypersurface  $t = \text{const}$  is positive, even inside the ergosphere. We introduce  $\langle \Psi | \Phi \rangle$  by integrating the corresponding bilinear form  $\bar{\Psi} \gamma^0 \Phi$  over the hypersurface  $t = \text{const}$ . Then the Gauss divergence theorem yields that  $\langle \Psi | \Phi \rangle$  is time independent. Hence, using (4.1),

$$0 = \frac{d}{dt} \langle \Psi(t) | \Phi(t) \rangle = i (\langle H \Psi(t) | \Phi(t) \rangle - \langle \Psi(t) | H \Phi(t) \rangle),$$

and thus  $H$  is indeed a symmetric operator on the corresponding Hilbert space  $\mathcal{H}$ .

After constructing a self-adjoint extension, we can use the functional calculus and the spectral theorem in Hilbert spaces to obtain

$$\Psi(t) = e^{-itH} \Psi_0 = \int_{\sigma(H)} e^{-i\omega t} dE_\omega.$$

Using the separation of variables, we can analyze the spectrum and the spectral measure. More precisely, as a technical tool for analyzing the spectral measure, we first set up the problem in a finite radial box, with Dirichlet boundary conditions on the walls of the box that make the Dirac operator essentially self-adjoint. The Hamiltonian for this problem has a purely discrete spectrum and one thus obtains a spectral representation of the propagator as an infinite sum of discrete eigenstates, expressible as products of eigenfunctions of the radial and angular operators arising from the separation of variables. The next step is to take limits as the walls of the box tend to the event horizon and to infinity. In these limits, the sum over the discrete eigenvalues goes over to an integral representation for the solution of the Cauchy problem. We thus obtain the following integral representation.

**Theorem 4.1.** *Consider the Cauchy problem for the Dirac equation in the Kerr geometry,*

$$(4.2) \quad (i\gamma^j \nabla_j - m)\Psi(t, x) = 0, \quad \Psi(0, x) = \Psi_0(x),$$

for initial data  $\Psi_0 \in C_0^\infty((r_1, \infty) \times S^2)^4$ . Then

$$(4.3) \quad \Psi(t, x) = \frac{1}{\pi} \sum_{k, n \in \mathbb{Z}} \int_{-\infty}^{+\infty} e^{-i\omega t} \sum_{a, b=1}^2 t_{ab}^{k\omega n} \Psi_a^{k\omega n}(x) \langle \Psi_b^{k\omega n} | \Psi_0 \rangle d\omega,$$

where the sums and the integrals converge in the  $\mathcal{H}$ -norm.

In the above integral representation, the integer  $k$  is the quantum number corresponding to the projection of the angular momentum on the axis of symmetry of the black hole, and  $n$  is the generalized total angular momentum quantum number arising from the separation of variables, corresponding to the discrete spectrum of the angular part of the Dirac operator in the Kerr geometry. The  $\Psi_a^{k\omega n}$  are solutions of the Dirac equation which arise from the separation of variables and satisfy

asymptotic boundary conditions at infinity and near the event horizon. More precisely, for  $|\omega| > m$ , the  $\Psi_{k\omega n}^k$  behave near the event horizon like incoming spherical waves for  $a = 1$  and outgoing spherical waves for  $a = 2$ , whereas in the case  $|\omega| < m$ , the  $\Psi_a^{k\omega n}$  is a linear combination of both incoming and outgoing spherical waves near the event horizon, such that the solution has exponential decay at infinity. The coefficients  $t_{ab}^{k\omega n}$  are given explicitly in terms of the reflection and transmission coefficients.

Applying the Riemann-Lebesgue lemma to the above integral representation gives the following local decay result.

**Theorem 4.2.** *Consider the Cauchy problem (4.2) for the Dirac equation in the Kerr geometry, where the initial data  $\Psi_0$  is in  $L^2((r_1, \infty) \times S^2)^4$  and in  $L_{loc}^\infty$  near the horizon, i.e.,  $|\Psi_0(x)| < c$  for  $x \in (r_1, r_1 + \epsilon) \times S^2$ . Let  $\delta > 0$  be given, let  $R > r_1 + \delta$ , and consider the compact space-like hyper-surface  $K_{\delta,R}$  of  $\mathcal{E}$  given by*

$$(4.4) \quad K_{\delta,R} := \{(t, r, \theta, \phi) \mid r_1 + \delta \leq r \leq R, t = \text{const}\}.$$

*Then the probability for the Dirac particle to be inside  $K_{\delta,R}$  tends to zero as  $t \rightarrow \infty$ , that is*

$$(4.5) \quad \lim_{t \rightarrow \infty} \int_{K_{\delta,R}} (\bar{\Psi} \gamma^0 \Psi)(t, x) \, d\mu = 0,$$

where  $d\mu$  denotes the induced volume element on  $K_{\delta,R}$ .

Theorem 4.2 implies that the Dirac spinor  $\Psi$  decays to zero in  $L_{loc}^\infty$ , or equivalently that the Dirac particle must eventually either disappear into the black hole, or escape to infinity. In order to get a more detailed physical picture, one would like to determine the probability of these outcomes in terms of the Cauchy data. Likewise, one would like to determine the rates of decay of the Dirac spinor in  $L_{loc}^\infty$ , as  $t$  tends to infinity. Both of these questions were addressed in [26], under the additional assumption that in the integral representation (4.3) only a finite number of angular momentum modes are present,

$$(4.6) \quad |k| \leq k_0, \quad |l| \leq l_0.$$

We now state the main results, beginning with the decay rates.

**Theorem 4.3.** *Consider the Cauchy problem as in Theorem 4.2, with initial data normalized by  $\langle \Psi_0 \mid \Psi_0 \rangle = 1$ . Suppose that (4.6) holds. Then we have:*

*i) If for any  $k$  and  $n$ ,*

$$(4.7) \quad \limsup_{\omega \searrow m} |\langle \Psi_2^{k\omega n} \mid \Psi_0 \rangle| \neq 0, \quad \text{or} \quad \liminf_{\omega \nearrow -m} |\langle \Psi_2^{k\omega n} \mid \Psi_0 \rangle| \neq 0,$$

*then, as  $t \rightarrow \infty$ ,*

$$(4.8) \quad |\Psi(x, t)| = ct^{-5/6} + O(t^{-5/6-\epsilon}),$$

where  $c = c(x) \neq 0$ , and  $\epsilon < 1/30$ .

*ii) If for all  $k, n$  and  $a = 1, 2$ ,  $\langle \Psi_a^{k\omega n} \mid \Psi_0 \rangle = 0$  for all  $\omega$  is a neighborhood of  $\pm m$ , then for any fixed  $x$ ,  $\Psi(x, t)$  decays rapidly in  $t$ .*

Note that the decay rate obtained in Theorem 4.3 is *slower* than the rate of decay of  $t^{-3/2}$  one obtains for the solutions of the Dirac equation in Minkowski space [26]. At first glance, this result seems to be somewhat counterintuitive. Indeed, because the Kerr metric is asymptotically flat, the Dirac spinor should behave near infinity like a solution of the Dirac equation in Minkowski space, where the rate of decay

is in  $t^{-3/2}$ . On the other hand, near the event horizon, the Dirac particle should behave like a massless particle, and its wave function should have rapid decay. One would therefore expect the rate of decay in the Kerr metric to be *at least* as fast as  $t^{-3/2}$ . This naive picture is in fact incorrect. Indeed, as shown in [26], the slower rate of decay can be understood by the fact that in the Kerr geometry, the energy spectrum of the initial data for the Dirac equation oscillates more and more as  $\omega$  approaches the rest mass  $m$  of the Dirac particle (essentially as  $\sin((m - \omega)^{-1/2})$  as opposed to  $(\omega - m)^{1/2}$  in Minkowski space). Upon taking the Fourier transform in time, these oscillatory contributions lead to a slower decay rate. Another way of understanding this effect is that the backscattering of the outgoing wave near infinity slows down the decay.

We now turn to the probability estimates. The probability for the particle to escape to infinity is given by

$$(4.9) \quad p = \lim_{t \rightarrow \infty} \int_{r > R} \bar{\Psi} \gamma^j \Psi(t, x) \nu_j d\mu,$$

where  $R > r_1$ . In [26] it is shown that the probability  $p$  is independent of  $R$  and can be expressed in terms of the quantities appearing in the integral representation (4.3) by

$$p = \frac{1}{\pi} \sum_{|k| \leq k_0} \sum_{|l| \leq l_0} \int_{\mathbb{R}[-m, m]} \left( \frac{1}{2} - 2|t_{12}^{k\omega n}|^2 \right) |\langle \Psi_2^{k\omega n} | \Psi_0 \rangle|^2 d\omega.$$

By analyzing this expression, one gets the following result.

**Theorem 4.4.** *Consider the Cauchy problem as in Theorem 4.2, with initial data normalized by  $\langle \Psi_0 | \Psi_0 \rangle = 1$ . Then the following hold:*

- (i) *If the outgoing initial energy distribution satisfies  $\langle \Psi_2^{k\omega n} | \Psi_0 \rangle \neq 0$  for some  $\omega$  such that  $|\omega| > m$ , then  $p > 0$ .*
- (ii) *If the initial energy distribution satisfies for  $a = 1$  or  $a = 2$ ,  $\langle \Psi_a^{k\omega n} | \Psi_0 \rangle \neq 0$  for some  $\omega$  such that  $|\omega| > m$ , then  $p < 1$ .*
- (iii) *If the initial energy distribution is supported in the interval  $[-m, m]$ , then  $p = 0$ .*
- (iv) *If for any  $k$  and  $n$ ,*

$$(4.10) \quad \limsup_{\omega \searrow m} |\langle \Psi_2^{k\omega n} | \Psi_0 \rangle| \neq 0 \quad \text{or} \quad \liminf_{\omega \nearrow -m} |\langle \Psi_2^{k\omega n} | \Psi_0 \rangle| \neq 0,$$

*then  $0 < p < 1$ .*

- (v) *We have  $p = 1$  if and only if for all  $k$  and  $n$  the following conditions hold:*

$$(4.11) \quad \langle \Psi_1^{k\omega n} | \Psi_0 \rangle = 0, \quad \text{if } |\omega| \leq m,$$

$$(4.12) \quad \langle \Psi_1^{k\omega n} | \Psi_0 \rangle = -2t_{12}^{k\omega n} \langle \Psi_2^{k\omega n} | \Psi_0 \rangle \quad \text{if } |\omega| > m.$$

We point out that in the above theorems, we always considered solutions supported outside the event horizon. One can also analyze the Dirac equation across the event horizon in the weak sense, and show that there are no time-periodic weak solutions which are in  $L^2$  away from the event horizon [23, 24].

We finally mention a few other rigorous results on the dynamics of quantum mechanical waves in the Kerr geometry which we cannot describe in this article. First, the scattering theory for Dirac particle has been developed in [36, 21, 2].

A remarkable result on the Klein-Gordon equation was obtained in [34]. The paper [35] is devoted to the Hawking effect, which leads to the creation of fermions by a rotating black hole.

5. AN INTEGRAL REPRESENTATION AND DECAY FOR SCALAR WAVES

For the scalar wave equation, we derive an integral representation which is similar to that obtained for the Dirac operator in Theorem 4.1, but the proof is significantly more difficult and requires new techniques. The integral representation will imply decay as for the Dirac equation, but it will, in addition, lead to the phenomenon of superradiance, which will be explained in Section 6. In contrast to the Dirac operator, the scalar wave operator in Kerr geometry does *not* admit a conserved quantity which is positive everywhere outside the event horizon, making it impossible to apply the spectral theory of self-adjoint operators in Hilbert space. First of all, the energy  $E[\Phi]$  is not positive. Namely, using the invariance of the scalar wave Lagrangian

$$(5.1) \quad \mathcal{L}[\Phi] = |\nabla\Phi|^2$$

under time translations, we obtain by Noether’s theorem the expression

$$(5.2) \quad E[\Phi] = \int_{r_1}^\infty dr \int_{-1}^1 d(\cos\vartheta) \int_0^{2\pi} \frac{d\varphi}{2\pi} \mathcal{E},$$

where  $\mathcal{E}$  is the energy density

$$(5.3) \quad \begin{aligned} \mathcal{E} = & \left( \frac{(r^2 + a^2)^2}{\Delta} - a^2 \sin^2\vartheta \right) |\partial_t\Phi|^2 + \Delta |\partial_r\Phi|^2 \\ & + \sin^2\vartheta |\partial_{\cos\vartheta}\Phi|^2 + \left( \frac{1}{\sin^2\vartheta} - \frac{a^2}{\Delta} \right) |\partial_\varphi\Phi|^2. \end{aligned}$$

One sees that all the terms in the energy density are positive, except for the coefficient of  $|\partial_\varphi\Phi|^2$ , which is positive if and only if  $r^2 - 2Mr + a^2 \cos^2\vartheta > 0$ , that is, precisely outside the ergosphere. The difficulty is that the conservation of  $E[\Phi]$  does not give a Sobolev estimate for  $\Phi$ . In particular, energy conservation does not rule out the situation where  $\Phi$  blows up in time, in such a way that the energy density tends to minus infinity inside the ergosphere and to plus infinity outside the ergosphere. Moreover, in contrast to the Dirac equation, the conserved charge is non-positive. In [27] it is shown that no other first-order or higher-order positive conserved energy exists for the scalar wave equation in Kerr geometry.

We begin by stating a theorem which gives an integral spectral representation for the solution of the Cauchy problem for the scalar wave equation. This theorem can be thought of as the analogue of Theorem 4.1 for the scalar wave equation.

**Theorem 5.1.** *Given initial data  $\Psi_0 \in C_0^\infty(\mathbb{R} \times S^2)^2$ , the solution of the Cauchy problem for the scalar wave equation (3.2) can be represented as*

$$(5.4) \quad \Psi(t, r, \vartheta, \varphi) = \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} e^{-ik\varphi} \sum_{n \in \mathbb{N}} \int_{-\infty}^\infty \frac{d\omega}{\omega\Omega} e^{-i\omega t} \sum_{a,b=1}^2 t_{ab}^{k\omega n} \Psi_{k\omega n}^a(r, \vartheta) \langle \Psi_{k\omega n}^b, \Psi_0 \rangle,$$

where

$$(5.5) \quad \Omega(\omega) = \omega - \omega_0 \quad \text{and} \quad \omega_0 = -\frac{ak}{r_1^2 + a^2}.$$

Here the sums and the integrals converge in  $L^2_{\text{loc}}$ .

We now describe the various terms that appear in the above integral representation in some detail, since they will be used in our discussion of superradiance in Section 6. Recall from the discussion of the separation of variables for the scalar wave equation that with the separation ansatz (3.3), the scalar wave operator splits into the sum of a radial operator (3.5) and an angular operator (3.6), with separated ordinary differential equations (3.8). The angular operator  $\mathcal{A}_{\omega,k}$  has a purely discrete spectrum of non-degenerate eigenvalues  $0 \leq \lambda_1 < \lambda_2, \dots$ . The corresponding eigenfunctions  $\Theta_n^{\omega,k}$  are the spheroidal wave functions. In order to bring the radial equation into a convenient form, we introduce a new radial function  $\phi(r)$  by

$$(5.6) \quad \phi(r) = \sqrt{r^2 + a^2} R(r),$$

and define the Regge-Wheeler variable  $u \in \mathbb{R}$  by

$$(5.7) \quad \frac{du}{dr} = \frac{r^2 + a^2}{\Delta},$$

mapping the event horizon to  $u = -\infty$ . The radial equation then takes the form of a Schrödinger equation,

$$(5.8) \quad \left( -\frac{d^2}{du^2} + V(u) \right) \phi(u) = 0.$$

For fixed separation constants  $k > 0$ ,  $\omega$  and  $\lambda_n$ , the potential  $V(u)$  has the asymptotics

$$\lim_{u \rightarrow -\infty} V(u) = -\Omega^2, \quad \lim_{u \rightarrow \infty} V(u) = -\omega^2.$$

Thus there are fundamental solutions  $\acute{\phi}$  and  $\grave{\phi}$  of (5.8) which behave asymptotically like plane waves at the event horizon and near infinity, respectively,

$$\acute{\phi}(u) \sim e^{i\Omega u} \text{ as } u \rightarrow -\infty, \quad \grave{\phi}(u) \sim e^{-i\omega u} \text{ as } u \rightarrow \infty$$

(these functions are the so-called Jost solutions, see [28] for details). As the corresponding time-dependent solutions behave in time like the plane wave  $e^{-i\omega t}$ , these solutions have a physical interpretation as incoming and outgoing waves in the asymptotic regions. The functions  $\acute{\phi}$  and  $\overline{\acute{\phi}}$  form a fundamental system, and thus we can represent  $\grave{\phi}$  as

$$(5.9) \quad \overline{\acute{\phi}(u)} = A \acute{\phi}(u) + B \overline{\acute{\phi}(u)}.$$

The coefficients  $A$  and  $B$  are the reflection and transmission coefficients. The quantities  $t_{ab}$  in the integral representation (5.4) are explicit functions of  $A$  and  $B$ . Finally, the functions  $\Psi_{k\omega n}^a(r, \vartheta)$ ,  $a = 1, 2$ , are the solutions of the wave equation (3.2), with fixed quantum numbers  $k, \omega, n$ , corresponding to the real-valued fundamental solutions of the radial equation given by

$$(5.10) \quad \phi^1 = \text{Re } \acute{\phi}, \quad \phi^2 = \text{Im } \acute{\phi}.$$

Our next result is a decay theorem for scalar waves in Kerr geometry analogous to Theorem 4.2 for the Dirac operator.

**Theorem 5.2.** *Consider the Cauchy problem for the wave equation in the Kerr geometry for smooth initial data which is compactly supported outside the event*

horizon and has fixed angular momentum in the direction of the rotation axis of the black hole, i.e., for some  $k \in \mathbb{Z}$ ,

$$(\Phi_0, \partial_t \Phi_0) = e^{-ik\varphi} (\Phi_0, \partial_t \Phi_0)(r, \vartheta) \in C_0^\infty((r_1, \infty) \times S^2)^2.$$

Then the solution decays in  $L_{\text{loc}}^\infty((r_1, \infty) \times S^2)^2$  as  $t \rightarrow \infty$ .

Theorem 5.2 can be interpreted as a linear stability result for the Kerr black hole under perturbations by massless scalar fields for a finite number of azimuthal angular momentum modes. It would be desirable to analyze the convergence of the sum over all  $k$ -modes. For this, one must obtain estimates which are uniform in  $k$ . Such estimates have not worked out, except in the case  $a \ll m$ , a perturbation of the Schwarzschild space-time (see [20], where the authors obtain a boundedness result using the vector field method). No theorems are known on rates of decay or probability estimates for scalar fields that would be analogous to Theorems 4.3 and 4.4 for Dirac fields. Proving such theorems in Kerr geometry appears to be a significant challenge.

We conclude this section by sketching the proof of Theorem 5.1. As for the Dirac equation, we reformulate the wave equation in Hamiltonian form. Letting

$$(5.11) \quad \Psi = \begin{pmatrix} \Phi \\ i\partial_t \Phi \end{pmatrix},$$

the wave equation (3.2) takes the form

$$(5.12) \quad i\partial_t \Psi = H\Psi,$$

where  $H$  is the Hamiltonian

$$(5.13) \quad H = \begin{pmatrix} 0 & 1 \\ \alpha & \beta \end{pmatrix},$$

and where  $\alpha$  and  $\beta$  are certain differential operators. The general strategy for constructing an integral representation for the propagator is the same as in the case of the Dirac operator, and requires as a first step that the problem be set up in a finite radial box with Dirichlet boundary conditions on the walls of the box. Unfortunately, the lack of a positive conserved energy makes it impossible to associate a positive definite invariant inner product to the time evolution of the Hamiltonian, and one has to consider the Hamiltonian  $H$  as acting on a function space endowed with an inner product of *indefinite* signature. In the case of a finite box, the inner product space is a *Pontrjagin space* [6], that is, a complex vector space  $\mathcal{K}$  endowed with a non-degenerate inner product  $\langle \cdot, \cdot \rangle$  and an orthogonal direct sum decomposition  $\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_-$ , such that  $(\mathcal{K}_+, \langle \cdot, \cdot \rangle)$  and  $(\mathcal{K}_-, -\langle \cdot, \cdot \rangle)$  are both Hilbert spaces, with  $\mathcal{K}_-$  being finite-dimensional. Classical results of Pontrjagin [6] imply that any self-adjoint operator  $A$  on a Pontrjagin space has a spectral decomposition, which is similar to the one given by the spectral theorem in Hilbert spaces, except that there is, in general, an additional finite point spectrum in the complex plane (which is symmetric about the real axis). We now consider the vector space  $\mathcal{P}_{r_L, r_R} = (H^{1,2} \oplus L^2)([r_L, r_R] \times S^2)$  with Dirichlet boundary conditions  $\Psi_1(r_L) = 0 = \Psi_1(r_R)$ . We endow this vector space with the inner product associated to the energy (5.2). It can be shown [27] that for every  $r_R > r_1$  there is a countable set  $E \subset (r_1, r_R)$  such that for all  $r_L \in (r_1, r_R) \setminus E$ , the inner product space  $\mathcal{P}_{r_L, r_R}$  is a Pontrjagin space with the same topology as  $(H^{1,2} \oplus L^2)([r_L, r_R] \times S^2)$ , and that the Hamiltonian on the Pontrjagin space  $\mathcal{P}_{r_L, r_R}$

with domain  $\mathcal{D} = C^\infty([r_L, r_R] \times S^2)^2 \subset \mathcal{P}_{r_L, r_R}$  is essentially self-adjoint. If the size of the box increases, the number of complex conjugate pairs becomes larger, and in the infinite volume limit all these spectral points move onto the real axis. This is made precise using contour methods and estimates for the resolvent. Whiting’s mode stability result [54], which guarantees the absence of unstable exponentially growing modes for the separated wave equation, is crucial to our proof.

6. A RIGOROUS TREATMENT OF SUPERRADIANCE FOR SCALAR WAVES

One of the most fascinating aspects of the classical physics of black holes is given by the so-called *Penrose process* [46], which shows that one can extract energy and angular momentum from the Kerr black hole so as to lower the angular momentum to zero by suitably exploiting the effect of the ergosphere on the dynamics of point particles. The basic idea of the Penrose process is as follows (see [52] for more detail and [51] for more realistic scenarios involving collisions of charged particles). First recall that the conserved energy of a point particle of momentum  $p$  is given by  $\langle p, \partial_t \rangle$ , and that this inner product clearly need not be positive inside the ergosphere, where the vector field  $\partial_t$  is space-like. Now consider a rocket which flies into the ergosphere, where it splits into two objects whose energies have opposite signs. By a suitable choice of the energy and momenta, one can arrange that the object of negative energy crosses the event horizon and reduces the energy and angular momentum of the black hole, whereas the other object escapes to infinity, carrying (due to energy conservation) more energy than the original rocket. In this way, one can extract energy from the black hole, at the expense of reducing its angular momentum. Christodoulou [13] showed that the infinitesimal changes of mass  $\delta M$  and angular momentum  $\delta(aM)$  of the black hole satisfy the inequalities

$$(6.1) \quad \delta(aM) \leq \frac{r_1^2 + a^2}{a} \delta M < 0,$$

and as a consequence he showed that it is not possible to reduce the mass of the black hole via the Penrose process below the so-called *irreducible mass*

$$(6.2) \quad M_{\text{irr}}^2 := \frac{1}{2} \left( M^2 + \sqrt{M^4 - (aM)^2} \right).$$

Superradiance is the wave analogue of the Penrose process. One considers a wave entering the black hole. One part of the wave enters the event horizon, whereas the other part is scattered at the black hole and gives rise to an outgoing wave (see Figure 1). By arranging the energy of the infalling wave to be negative, one can again extract energy from the black hole. This effect was first studied by Zel’dovich [55] and Starobinsky [49] on the level of modes, i.e., by considering the transmission and reflection coefficients in (5.9) associated to the Schrödinger equation (5.8). In this analysis, the quantities  $\omega^2|A|^2$  and  $\omega^2|B|^2$  have the interpretations as the energy flux of the incoming and outgoing waves, respectively. Thus the relative energy gain  $\mathfrak{R}$  is given by

$$(6.3) \quad \mathfrak{R} = \frac{|B|^2}{|A|^2}.$$

Computing the Wronskians of  $\phi$  and  $\bar{\phi}$  near the event horizon and near infinity gives the relation

$$(6.4) \quad |A|^2 - |B|^2 = \frac{\Omega}{\omega}.$$

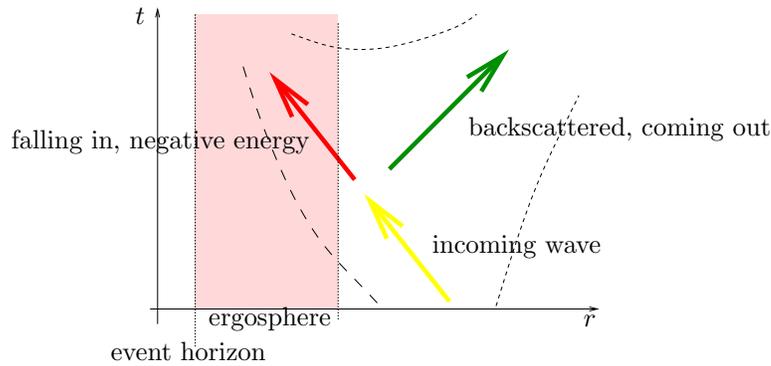


FIGURE 1. Superradiant scattering

If the right side of (6.4) is positive, the outgoing flux is smaller than the incoming flux, and this corresponds to ordinary scattering. However, if the right side of (6.4) is negative, then the outgoing flux is larger than the incoming flux, and according to (6.3) we gain energy. This is termed *superradiant scattering*. Using (5.5), superradiant scattering appears precisely when  $\omega$  is in the range

$$(6.5) \quad 0 < |\omega| < |\omega_0|.$$

Starobinsky [49] computed  $\mathfrak{R}$  and found a relative gain of energy of about 5% for  $k = 1$  and less than 1% for  $k \geq 2$ . Teukolsky and Press [50] made a similar mode analysis for higher spin and found numerically an energy gain of at most 4.4% for Maxwell ( $s = 1$ ) and up to 138% for gravitational waves ( $s = 2$ ). For more recent developments on the mode analysis of superradiance and related physical effects we refer to [7].

Unfortunately, the mode analysis does not give information on the dynamics. Thus for a rigorous treatment of energy extraction one needs to consider the time-dependent situation. This was done numerically in [1] for wave packet initial data. The main result of [29] treats the time-dependent situation rigorously for the Cauchy problem. As initial data we take wave packets of the form

$$(6.6) \quad \Psi_0 = \Theta_{\tilde{n}, \tilde{\omega}}(\vartheta) e^{-ik\varphi} \frac{\eta_L(u)}{\sqrt{r^2 + a^2}} \left[ c_{in} e^{-i\tilde{\omega}u} \begin{pmatrix} 1 \\ \tilde{\omega} \end{pmatrix} + c_{out} e^{i\tilde{\omega}u} \begin{pmatrix} 1 \\ -\tilde{\omega} \end{pmatrix} \right],$$

where  $L$  is a large parameter, and  $\eta_L$  is a smooth cutoff function of the form

$$\eta_L(u) = \frac{1}{\sqrt{L}} \eta\left(\frac{u - L^2}{L}\right),$$

with  $\eta \in C_0^\infty(\mathbb{R}_+)$ . Here  $\Theta_{\tilde{n}, \tilde{\omega}}(\vartheta)$  is an eigenfunction of the angular operator  $\mathcal{A}$ . The energy radiated to infinity is defined by

$$(6.7) \quad E_{out} = \lim_{t \rightarrow \infty} \langle \Psi(t), \chi_{(2r_1, \infty)}(r) \Psi(t) \rangle,$$

where  $\chi$  is the characteristic function.

**Theorem 6.1.** *For any  $R > r_1$  and  $\delta > 0$  there is initial data*

$$\Psi_0 \in C_0^\infty((R, \infty) \times S^2)^2$$

of the form (6.6) such that the limit in (6.7) exists and

$$\left| \frac{E_{\text{out}}}{\langle \Psi_0, \Psi_0 \rangle} - \mathfrak{R} \right| \leq \delta$$

with  $\mathfrak{R}$  as in (6.3).

For the proof we consider the integral representation of Theorem 5.1 for the wave packet initial data (6.6). The crucial analytical ingredient in the proof is the time-independent energy estimate for the outgoing wave as derived in [31].

It should be stressed that we allow the initial data to be supported arbitrarily far away from the event horizon. This is important in order to avoid artificial initial data which would not correspond to an energy extraction mechanism. For example, if one allows the support of the initial data to intersect the ergosphere, one could take initial data with zero total energy, in which case the quotient  $E_{\text{out}}/\langle \Psi_0, \Psi_0 \rangle$  could be made arbitrarily large.

### 7. THE STABILITY PROBLEM FOR KERR BLACK HOLES

The remaining challenge is to prove the *linear stability* of the Kerr black hole under electromagnetic and gravitational perturbations. As Frolov and Novikov put it [32, S. 143], this is “one of the few truly outstanding problems that remain in the field of black hole perturbations”. Since the linear perturbations of the Kerr metric are described by the solutions of the Teukolsky equation (3.13), proving linear stability amounts to showing that the solutions of the Cauchy problem for the Teukolsky equation for compactly supported Cauchy data decay in  $L_{\text{loc}}^\infty$  as  $t$  tends to infinity. In other words, the task is to prove the analogue of Theorem 5.2 for the solutions of the Teukolsky equation (3.13) in the cases  $s = 1$  and  $s = 2$ . The analysis of the Teukolsky equation for higher spin appears to be considerably more difficult than that of the scalar wave equation. This is because, in contrast to the scalar wave equation (5.1), the Teukolsky equation has no variational formulation, and thus there is no simple method for deriving the conserved quantities. Indeed, the expressions for the physical energy and charge are very complicated and involve higher derivatives of the field  $\Psi_s$ . The only result so far on the Teukolsky equation in Kerr is the deep and important mode stability theorem proved by Whiting in [54].

More recently, a rigorous stability result was proved for the Teukolsky equation in the simpler setting of the Schwarzschild geometry [30] for electromagnetic and gravitational wave perturbations, which we now describe. The evolution of a massless wave of spin  $s$  in the Schwarzschild geometry is described by the Teukolsky equation (set  $a = 0$  in (3.2))

$$(7.1) \quad \left[ \partial_r \Delta \partial_r - \frac{1}{\Delta} (r^2 \partial_t - (r - M)s)^2 - 4sr \partial_t + \partial_{\cos \vartheta} \sin^2 \vartheta \partial_{\cos \vartheta} + \frac{1}{\sin^2 \vartheta} (\partial_\varphi + is \cos \vartheta)^2 \right] \Phi(t, r, \vartheta, \varphi) = 0,$$

where  $\Delta = r^2 - 2Mr$ . We consider (7.1) with  $C_0^\infty$  initial data

$$(7.2) \quad \Phi|_{t=0} = \Phi_0, \quad \partial_t \Phi|_{t=0} = \Phi_1.$$

Then the following theorem holds (see [30]).

**Theorem 7.1.** *For spin  $s = 1$  or  $s = 2$ , the solution of the Cauchy problem (7.1), (7.2) for  $(\Phi_0, \Phi_1) \in C_0^\infty((2M, \infty) \times S^2)^2$  decays in  $L_{\text{loc}}^\infty((2M, \infty) \times S^2)$  as  $t \rightarrow \infty$ .*

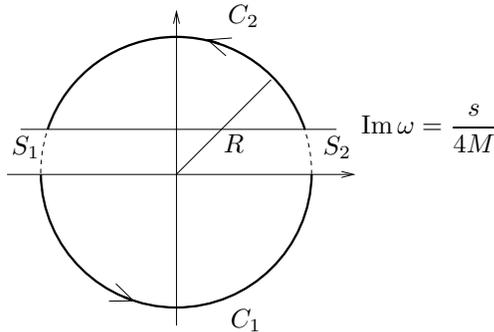


FIGURE 2. The contour  $C_R$

The proof has some novel features, and we shall discuss some of them.

After separating the time and angular dependence similar to (3.3), (3.8) (with  $k \in \mathbb{Z}$ ,  $\lambda \in \mathbb{R}$  and  $\omega \in \mathbb{C}$ ) and replacing  $r$  by the Regge-Wheeler variable  $u$ , the Teukolsky equation can be reduced to a one-dimensional Schrödinger equation

$$-\frac{d^2}{du^2} \phi(u) + V(u) \phi(u) = 0$$

with potential  $V$  given by

$$V(u) = -\omega^2 + is\omega \left[ \frac{2(r-M)}{r^2} - \frac{4\Delta}{r^3} \right] + \frac{(r-M)^2 s^2}{r^4} + \frac{\partial_u^2 r}{r} + \lambda \frac{\Delta}{r^4},$$

where  $\lambda$  is an eigenvalue of the angular operator. Due to the presence of the  $\partial_t$ -term in (7.1),  $V$  is complex even for real  $\omega$ . Thus most standard techniques are unavailable.

First one easily checks that

$$\lim_{u \rightarrow -\infty} V(u) = -\left( \omega - \frac{is}{4M} \right)^2.$$

Then writing the time-dependent equations in Hamiltonian form,

$$i\partial_t \Psi = H \Psi, \quad \Psi = (\Phi, \partial_t \Phi),$$

with  $H$  a matrix differential operator, we can show that if  $\omega$  is outside the strip  $0 \leq \text{Im} \omega \leq \frac{s}{4M}$ , then  $\omega$  lies in the resolvent set of  $H$ , and the resolvent  $R_\omega$  is holomorphic there. By means of suitable resolvent estimates, we show that any  $\Psi \in C_0^\infty$  has the integral representation

$$\Psi(u) = -\frac{1}{2\pi i} \lim_{R \rightarrow \infty} \int_{C_R} (R_\omega \Psi)(u) d\omega,$$

where  $C_R$  is the contour shown in Figure 2, which has two connected components  $C_1$  and  $C_2$ . Using contour deformation techniques, we can prove the following non-standard integral representation theorem for the solution.

**Theorem 7.2.** *For spin  $s = 1$  or  $s = 2$ , the solution of the Cauchy problem for the Teukolsky equation (7.1) with initial data  $\Psi_0 = (\Phi, \partial_t \Phi)|_{t=0} \in C_0^\infty(\mathbb{R})^2$  has the*

representation

$$\begin{aligned} \Psi(t, u) = & -\frac{1}{2\pi i} \int_{\mathbb{R}} e^{i\omega t} \left( (\mathfrak{R}_\omega \Psi_0)(u) + \frac{\Psi_0(u)}{\omega + i} \right) d\omega \\ & + \frac{1}{2\pi i} \int_{\mathbb{R} + \frac{is}{2M}} e^{i\omega t} \left( (R_\omega \Psi_0)(u) + \frac{\Psi_0(u)}{\omega + i} \right) d\omega, \end{aligned}$$

where  $\mathfrak{R}_\omega$  is the limit of the integral kernel of the resolvent from the lower half plane,

$$\mathfrak{R}_{\omega_0}(u, v) := \lim_{\omega \rightarrow \omega_0, \operatorname{Im} \omega < 0} R_\omega(u, v).$$

Both integrands in the above integral representation are in  $L^1$ .

Decay is an immediate consequence of this theorem: for  $t \rightarrow \infty$ , the first integral tends to zero by the Riemann-Lebesgue lemma, while the second integral tends exponentially to zero.

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