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The birth of model theory: Löwenheim's theory in the frame of the theory of relatives, by Calixto Badesa, Princeton University Press, Princeton, NJ, 2004, xiv+240 pp., ISBN 978-0-691-05853-5, hardcover, US$64.00

What do the theorems of Gödel–Deligne, Chevalley–Tarski, Ax–Grothendieck, Tarski–Seidenberg, and Weil–Hrushovski have in common? And what do they have to do with the book under review? Each of these theorems was proven by techniques in a particular mathematical area and by model theoretic methods. In fact, these model theoretic methods often show a pattern that extends across these areas.

What are model theoretic methods? Model theory is the activity of a "self-conscious" mathematician. This mathematician distinguishes an object language (syntax) and a class of structures for this language and "definable" subsets of those structures (semantics). Semantics provides an interpretation of inscriptions in the formal language in the appropriate structures. At its most basic level this allows the recognition that syntactic transformations can clarify the description of the same set of numbers. Thus, $x^2 - 3x < -6$ is rewritten as $x < -2$ or $x > 3$; both formulas define the same set of points if they are interpreted in the real numbers.

After clarifying these fundamental notions, we give an anachronistic survey of three themes of twentieth century model theory: the study of a) properties of first order logic, b) specific first order theories, and c) classification of first order theories. In this survey we will highlight the increasing interactions between "pure" model theory and the analysis of topics in core mathematics. Then we return to the book at hand and see how Löwenheim's 1915 paper [Löw67] set the stage for these developments.

On the syntactic side, first order logic contains several *logical symbols*: equality; the quantifiers $\forall, \exists$; a sequence of variables, v_i; and the logical connectives $\wedge, \neg, \vee$. A vocabulary τ for first order logic consists of a collection of relation and function symbols that is appropriate for the area of mathematics being formalized. Terms of the language based on τ are built up inductively from constant and variables using the function and relation symbols of the language. An atomic formula has the form $R(t_1, \dots t_n)$ where R is a relation symbol with n-arguments and the t_i are terms. The first order language associated with τ is the least set of formulas containing the atomic τ-formulas and closed under the Boolean operations and quantification over individuals. Formulas in which each variable is bound by a quantifier are called τ-sentences.

On the semantical side, a structure for this vocabulary consists of a domain (also denoted A) and a relation on A^n for each relation symbol with n arguments (a function from A^n into A for each function symbol with n arguments) and with equality interpreted as identity. In a structure A, each variable-free term denotes an element of A. Truth in a structure is also defined inductively: if there are no quantifiers, a formula is true in the structure A if the interpretation of the terms lies in the relation which is the interpretation of the formula. The truth of Boolean combination and or quantified formulas is defined in the natural way; e.g.,

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$A \models (\exists v)\phi(v)$ if for some $a \in A$, $A \models \phi(a)$. By a first order theory T we mean a set of sentences for a vocabulary τ. The theory T is complete if every τ-sentence or its negation is in T. The cardinality ($|T|$) of T is the number of nonlogical symbols in the vocabulary τ. An L-structure A is called a model of an L-sentence (or a theory T) if the sentence (each sentence in T) is true in A.

"First order" means that the quantification is only over elements of the structure. In contrast, second order logic allows quantification over subsets and relations. Thus in second order logic one can specify the natural numbers up to isomorphism with a full second order induction axiom: every subset that is closed under successor and contains 0 is the entire universe. The Löwenheim theorem, which is the focus of the book under review, asserts that every first order sentence which has a model (of any size) has a model whose domain is countable. The more modern version of this theorem, called the Löwenheim–Skolem theorem, asserts that any set of first order sentences with an infinite model has models of every cardinality. The statement of the theorem is due to Malcev in 1936 (see [Mal72]). Our formulation hides a fundamental contribution of Gödel [Göd29]. Consistency is a syntactic condition: a set of sentences Σ is consistent if (with respect to some well-behaved notion of proof) it is not possible to deduce a contradiction from Σ. Via Gödel's completeness theorem, it is equivalent to say that Σ has a model. The finitary nature of proof implies the compactness theorem: a set of sentences is consistent if and only if each finite subset is. In the 1970s, categorical formulations [MR77] showed the equivalence of the Gödel completeness theorem with results of Deligne on the existence of enough points on a coherent topos. This observation led to the sobriquet "Gödel–Deligne".

The study of first order logic produces, in addition to the completeness and compactness theorems, a number of applications of compactness to show syntactical characterizations of semantic properties. For example, a class of structures is closed under substructure if and only if it can be axiomatized by sentences whose only quantifiers are initial occurrences of $\forall$. Similarly, a class is closed under unions of chains if and only if it can be axiomatized by $\forall\exists$-sentences—the prefix is a sequence of universal quantifiers followed by a sequence of existential quantifiers.

The second theme, the study of particular theories, had already begun in the 1920s and early 1930s. A theory T admits quantifier elimination if every formula is equivalent over T to a formula with no quantifiers. Tarski and Pressburger began such studies in the 1920s with such results as quantifier elimination for the theory of the natural numbers under addition (with predicates for divisibility by n) [Pre30]. Already in 1931, Tarski proved quantifier elimination for the first order theory of the ordered real field [Tar31]. He noted in a footnote to [Tar51] that similar arguments (elimination theory) show quantifier elimination for algebraically closed fields. Robinson independently obtained the result by ideal theoretic methods [Rob54]. Chevalley phrased it as "the projection of a constructible set is constructible". Joyal [Joy75] popularised the name Chevalley–Tarski theorem.

Abraham Robinson introduced the important notion of model completeness—every formula is equivalent to one with only initial existential quantifiers—e.g., the real field without order in the vocabulary. This concept allowed Robinson to provide a context, differentially closed fields [Rob59], for the Ritt–Kolchin theory of differential algebra. Shelah's later development of stability theory led to proving the uniqueness of differential closure (see [Blu68, Sac72]) in characteristic 0. But this closure is not minimal [She73]. Indeed, Shelah's analysis provided a structural

condition (eni-DOP) which resulted in the proof twenty years later by Hrushovski and Sokolovic that there are $2^{\aleph_0}$ nonisomorphic countable differentially closed fields (see [Mar07]).

Tarski and Vaught introduced the crucial notion of elementary submodel [TV56]: a structure A is an elementary submodel of a structure B if every sentence (with parameters from A!) has the same truth value in both A and B. This allows the description of the correct category for first order model theory: the collection of models of a complete first order theory with elementary embeddings as morphisms.

The Ax–Grothendieck theorem [Ax68, Gro66] asserts an injective polynomial map on an affine algebraic variety over $\mathbb{C}$ is surjective. The model theoretic proof [Ax68] (see also [Tao]) observes the condition is axiomatized by a family of "for all–there exist" first order sentences ϕ_i (one for each pair of a map and a variety). Such sentences are preserved under direct limit and the ϕ_i are trivially true on all finite fields. So they hold for the algebraic closure of F_p for each p (as it is a direct limit of finite fields). Note that $T = \mathrm{Th}(\mathbb{C})$, the theory of algebraically closed fields of characteristic 0, is axiomatized by a schema Σ asserting each polynomial has a root and stating for each p that the characteristic is not p. Since each ϕ_i is consistent with every finite subset of Σ, it is consistent with Σ and so proved by Σ, since the consequences T of Σ form a complete theory.

Work of many model theorists led to the understanding that first order theories admitting elimination of quantifiers provided the most fruitful field of study. Elimination of quantifiers can arise in two radically different ways. Morley [Mor65] noticed that there is an extension by explicit definition of any complete first order theory to one with elimination of quantifiers. Most studies in pure model theory adopt the convention that this has taken place. But this extension requires a large price; the vocabulary is no longer tied to the basic concepts of the area of mathematics. Thus it is a major enterprise to work with specific algebraic structures and add a few intelligible definitions to obtain quantifier elimination (or the weaker model completeness). But there is a clear understanding in either context that it is desirable to have a limited number (of alternations) of quantifiers available so that definable sets can be analyzed. Further applications of quantifier elimination include the Ax–Kochen–Ershov [AK65a, AK65b, AK66, Ers65] work on valued fields solving Artin's conjecture, Macintyre's proof of quantifier elimination for p-adic fields [Mac76], and Denef's proof of the rationality of the Poincaré series [Den84].

The notion of categoricity was introduced by Huntington at the beginning of the twentieth century: a sentence or theory Σ is categorical if it has exactly one model (up to isomorphism). The Löwenheim theorem (as generalized) tells us that no first order theory with an infinite model is categorical. Model theorists eventually discovered that the most useful form of this concept was categoricity in power (all models of Σ with the same cardinality are isomorphic). Morley [Mor65] proved that a first order theory in a countable vocabulary is categorical in one uncountable cardinality if and only if it is categorical in every uncountable cardinality. More importantly, there is a structure theory for models of each such theory. There is a formula whose solutions admit a dimension theory similar to that for vector spaces. And every model is determined by the solution set in it of that formula [BL71].

Around 1970 Shelah burst on the scene with a revolutionary program, classifying theories. The 25-page abstract introducing [She70] outlines the new paradigm for model theory. The Stone space $S(A)$ of a set $A \subset M \models T$ is the space of ultrafilters on the Boolean algebra of formulas (with one free variable) with parameters from

A (up to T-equivalence). Shelah defined for each countable theory T its stability function: $f_T(\kappa) = \sup\{|S(A)| : |A| \leq \kappa\}$. His remarkable theorem asserts that f_T is one of four functions, the identity (ω-stable), the identity above $2^{\aleph_0}$ (superstable), κ^ω (stable), and 2^κ (unstable). More importantly, if the function is one of the first three (T is stable), then the models of T admit a "dimension theory". By categorizing first order theories into a small finite number of classes, he provided a tool that continues to dominate the area today. The prescient ideas of [She75] are still being mined or rediscovered thirty years later.

Stability theory, summarized in [She91], begins the classification of complete first order theories by translating the four possibilities for stability function $f_T(\kappa)$ into certain "syntactic conditions". For example, slightly roughly, a theory is unstable if it interprets a linear ordering on n-tuples for some n. If a theory is not stable it has the maximal number of models in every uncountable cardinality. If it is stable, there is a notion of independence, generalizing that of van der Waerden's. This notion of independence allows one to ascribe dimensions to certain subsets of the universe and eventually (modulo some more technical conditions) to decompose each model into a tree of countable submodels.

This classification leads to two major kinds of theories (the main gap): classifiable and creative/chaotic. If a theory T is classifiable, then all models of any cardinality are controlled by countable submodels via a mechanism (decomposition into a tree of submodels) which is the same for all such theories. In particular, this implies that the number of models in cardinality $\aleph_\alpha$ is bounded by $\beth_\beta(\alpha)$ (for some $\beta < |T|^+$).[1] In contrast, the number of models in $\aleph_\alpha$ of a chaotic theory is $2^{\aleph_\alpha}$; essentially new methods of creating models are always needed. The general idea of a structure theory is to isolate "definable" subsets of models of a theory that admit a dimension theory analogous to that in vector spaces. And then to show that all models are controlled by a family of such dimensions. Theories that are categorical in power are the simplest case. There is a single dimension and the control is very direct.

The fine structure of the independent sets was investigated by the methods of geometric stability theory in the 1980s. The spark was the proof by Zilber [Zil84] and Cherlin, Harrington, and Lachlan [CHL85] that no complete theory categorical in every infinite cardinality could be axiomatized by a single first order sentence. Although this problem is phrased as a logical one, the solution reveals deep structural properties of each model of such a theory (compare [Pil94]). The [CHL85] proof relied on the classification of finite simple groups. Zilber's proof avoided this by providing a new proof of the classification of finite two-transitive groups that is fully worked out in [Eva86]. Hrushovski took a decisive step by interpreting groups and fields in structures based on technical model theoretic properties of the structure and then using algebraic properties of the interpreted structure to solve purely model theoretic problems (e.g., [Hru90]). Pillay's book [Pil96] summarises this synthesis of the local geometric analysis with Shelah's techniques for global analysis: orthogonality, canonical bases, regular types, etc. Hrushovski combined these methods with a deep understanding of Diophantine geometry to provide fundamental advances related to the Manin–Mumford conjecture [Bou99, Hru96].

[1] The beth function is defined by recursion: $\beth_{\beta+1}(\alpha) = 2^{\beth_\beta(\alpha)}$ where $\beth_0(\alpha) = \aleph_\alpha$ and sups are taken at limit ordinals.

At the base of Shelah's hierarchy are the so-called ω-stable theories. A large team of model theorists [BN94, ABC08] has been investigating groups of finite Morley rank (a particularly strong form of ω-stability). They have developed a strong analogy to the analysis of the finite simple groups aimed at the conjecture: a simple group of finite Morley rank is an algebraic group over an algebraically closed field. This study builds on and extends the finite simple group machinery. The Weil–Hrushovski theorem, "every constructible group is definably isomorphic to an algebraic group" (Theorem 4.13 of [Poi87]), arose in this analysis. In the appendix to [Hru02], Hrushovski discusses the integration of sheaf theoretic methods with this viewpoint.

There are however weaker notions of "well-behaved theory" available, which allow the investigation of the definable subsets of chaotic theories; these include the study of theories that are simple, o-minimal, or do not have the independence property (are dependent). Although the notion of simple theory was introduced by Shelah in [She80], its significance only became clear with the further development of both applications to difference fields (e.g., [CH99]) and a definitive grounding of the model theoretic notions as a weakening of the notion of independence in [KP97]. The notion of o-minimality returns to our first example. An ordered structure is o-minimal if every definable set is a Boolean combination of intervals [PC86]. Tarski's quantifier elimination theorem [Tar31] (also known as the Tarski–Seidenberg theorem) shows the real field is o-minimal. More dramatically, Wilkie showed the real field with exponentiation is o-minimal [Wil96]. This spurred a still-continuing study of o-minimal structures which has many connections with real algebraic geometry [dD99] and led, for example, to a solution of a problem of Hardy [vdDMM97].

A later generation of model theorists takes quantifier elimination a step further and seeks "elimination of imaginaries". Shelah introduced the notion of imaginary elements—a name for each equivalence class of each definable equivalence relation. Poizat [Poi83] provided the tools for exploiting this concept in algebraic contexts by noting that many important theories admit elimination of imaginaries. This concept is exploited in a fusion of the study of theories without the independence property with that of valued fields [HHM07]. The role of definability as a tool for mathematical investigation is further highlighted by the model theoretic explanations of motivic integration (e.g., [CL08, DL02, HKxx]).

There are important mathematical structures, e.g., complex exponentiation, which exhibit the Gödel undecidability phenomena and so cannot be analyzed by these techniques *in first order logic*. However, Zilber has provided a means for such analysis in the logic $L_{\omega_1,\omega}(Q)$ [Zil04, Zil05]. (Now formulas may contain countable disjunctions and conjunctions; the quantifier Q, means, "there exist uncountably many".) This study also draws on Shelah's notion of excellent classes of sentences in $L_{\omega_1,\omega}$ [She83a, She83b]. Zilber's analysis led to a number of striking conjectures in algebraic number theory; Shelah provides a more general theory with profound connections to axiomatic set theory [Bal, She00]. As in the first order case, he provides conditions on countable structures that determine the behavior of models of all cardinalities. In this case, Zilber identifies algebraic conditions [Zil06, Zil03] (studied by such as Serre and Bashmakov), which are the special cases of excellence for covers of Abelian varieties.

The mention of $L_{\omega_1,\omega}$ brings us back to the book at hand. Badesa analyzes in detail one of the seminal papers in model theory. The division of logical languages into

first and second order, finitary and infinitary had not been made when Löwenheim wrote. Coming from the algebraic tradition of Pierce and Shroeder, Löwenheim was dealing implicitly with infinitary first order logic—now formalized as $L_{\infty,\omega}$. Lówenheim's actual paper had limited direct influence; the work of Skolem quickly superceded it. Van Heijenoort's epic sourcebook of mathematical logic [VH67] reprinted the Löwenheim paper along with an introduction identifying an alleged flaw in the proof. Badesa describes two possible interpretations of Löwenheim's theorem: a) if a first order formula has a model, then it has a countable model; b) if a first order formula ϕ has a model M, then M has a countable substructure M_0 in which ϕ is also true. Badesa coherently argues that Van Heijenoort misinterpreted Löwenheim's argument as giving a flawed argument for a) when in fact Löwenheim gave a more correct argument for b). As Avigad [Avi06] points out in a review that focuses more directly on Badesa's book, the source of the confusion is that the distinction between syntax and semantics that is fundamental for the model theoretic advances described in this review were not available to Löwenheim but rather arose in the context of his work. Strikingly, crucial problems in the modern study of infinitary logic stem precisely from the failure of the upward Löwenheim–Skolem theorem.

The essence of model theory is its "metamathematical" orientation. The common properties (originally syntactic but increasingly structural) of various mathematical theories or families of theories are isolated and general arguments are provided that enable generalizations and improvements. This review has only sampled the developments in this field in the almost full century since Löwenheim's seminal result.

Added after posting

After the posting of this review, it was pointed out to the author that there was an error in the footnote on p. 180. The corrected version of the footnote follows.

> [1] The beth function is defined by recursion: $\beth_{\beta+1}(\kappa) = 2^{\beth_\beta(\kappa)}$ and $\beth_0(\kappa) = \kappa$ and sups are taken at limit ordinals.

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John T. Baldwin
University of Illinois at Chicago