BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 47, Number 2, April 2010, Pages 187–279
S 0273-0979(10)01292-9
Article electronically published on February 2, 2010

# GEOMETRIC CYCLES, ARITHMETIC GROUPS AND THEIR COHOMOLOGY

JOACHIM SCHWERMER

Abstract. It is the aim of this article to give a reasonably detailed account of a specific bundle of geometric investigations and results pertaining to arithmetic groups, the geometry of the corresponding locally symmetric space $X/\Gamma$ attached to a given arithmetic subgroup $\Gamma \subset G$ of a reductive algebraic group $G$ and its cohomology groups $H^*(X/\Gamma, \mathbb{C})$. We focus on constructing totally geodesic cycles in $X/\Gamma$ which originate with reductive subgroups $H \subset G$. In many cases, it can be shown that these cycles, to be called geometric cycles, yield non-vanishing (co)homology classes. Since the cohomology of an arithmetic group $\Gamma$ is strongly related to the automorphic spectrum of $\Gamma$, this geometric construction of non-vanishing classes leads to results concerning, for example, the existence of specific automorphic forms.

## Contents



Received by the editors September 12, 2008, and, in revised form, June 8, 2009.
2000 *Mathematics Subject Classification.* Primary 11F75, 22E40; Secondary 11F70, 57R95.
*Key words and phrases.* Arithmetic groups, geometric cycles, cohomology, automorphic forms.
This work was supported in part by FWF Austrian Science Fund, grant number P 16762-N04.

## Introduction

**Prelude: a historical remark.** An *arithmetic group*, roughly speaking, is a group of matrices with integral entries defined by polynomial equations. For example, a subgroup of finite index in the general linear group of $n \times n$ matrices with entries in the ring of integers $\mathcal{O}_k$ of an algebraic number field $k$ or, more generally, in an order of a finite-dimensional division algebra over $k$, is an arithmetic group. This notion includes the "familiar cases" of the special linear group $SL_n(\mathbb{Z})$ or the group $Sp_n(\mathbb{Z})$ of all symplectic transformations on the symplectic space $\mathbb{Z}^{2n}$ with its standard alternating form, and their subgroups of finite index. Among them are the principal congruence subgroups $\Gamma(q) = \{\gamma \in SL_n(\mathbb{Z}) \mid \gamma \equiv 1 \bmod q\}$, $q$ a positive integer. The theory of arithmetic groups has its origins in number theory, in particular, in the arithmetic of quadratic forms. The concept of reduction, as developed by Gauss, Dirichlet, and Minkowski, among others, provided a powerful way to select, from the infinitely many forms which are integrally equivalent to a given form, one which is characterized in an intrinsic way by suitable conditions for the entries. The suggestion made by Gauss in 1831 to interpret an integral quadratic form as a lattice in space ultimately proved decisive in Minkowski's approach to the theory of quadratic forms. Minkowski unfolded a new form of reduction theory by working with lattices as geometric objects.[1] His investigations, in particular his geometric point of view, served as substantial stimuli for Siegel's studies of quadratic forms and discontinuous groups in the context of classical groups.

**Suite.** The theory of arithmetic groups, as subsequently developed within the realm of the theory of algebraic groups by Borel, Harder, Serre, and Raghunathan, among others, still has a very distinctive geometric flavor. Each arithmetic group acts on a homogeneous space which is defined by the ambient Lie group. Number theory, group theory and geometry are interwoven in a most fruitful way. Moreover, there are close connections with the theory of automorphic forms. It is the aim of this article to give a reasonably detailed account of a specific bundle of geometric investigations and results pertaining to arithmetic groups, the geometry of the corresponding locally symmetric space $X/\Gamma$ attached to a given arithmetic subgroup

[1] The historically inclined reader might find some more details shedding light on this development in [130] or [132].

$\Gamma \subset G$ of a reductive algebraic group $G$ and its cohomology groups $H^*(X/\Gamma, \mathbb{C})$.[2] We focus on constructing totally geodesic cycles in $X/\Gamma$ which originate with reductive subgroups $H \subset G$. In many cases, it can be shown that these cycles, to be called geometric cycles, yield non-vanishing (co)homology classes. Since the cohomology of an arithmetic group $\Gamma$ is strongly related to the theory of automorphic forms with respect to $\Gamma$ this geometric construction of non-vanishing classes leads to results concerning, for example, the existence of specific automorphic forms.

More precisely, let $G$ be a connected semi-simple algebraic $\mathbb{Q}$-group,[3] and $X = X_G$ the corresponding symmetric space of maximal compact subgroups of the group $G(\mathbb{R})$ of real points of $G$. A torsion-free arithmetic subgroup $\Gamma \subset G(\mathbb{Q})$ acts properly and freely on $X$, and the quotient space $X/\Gamma$ is a complete Riemannian manifold of finite volume. Let $H$ be a reductive $\mathbb{Q}$-subgroup of $G$, let $K_H$ be a maximal compact subgroup of the real Lie group $H(\mathbb{R})$, and let $X_H = K_H \backslash H(\mathbb{R})$. If $x_0 \in X$ is fixed under the natural action of $K_H \subset G(\mathbb{R})$ on $X$, then the assignment $h \mapsto x_0 h$ defines a closed embedding

$$X_H = K_H \backslash H(\mathbb{R}) \longrightarrow X;$$

that is, the orbit map identifies $X_H$ with a totally geodesic submanifold of $X$. We will focus on the following points:

- First, by replacing $\Gamma$ with a subgroup of finite index, it is possible to guarantee that the embedding $X_H \longrightarrow X$ passes to an embedding (see Section 6.5)
$$j_{H|\Gamma} : X_H/\Gamma_H \longrightarrow X/\Gamma$$
[with $\Gamma_H = \Gamma \cap H(\mathbb{Q})$] whose image $j_{H|\Gamma}(X_H/\Gamma_H)$ is a totally geodesic submanifold, to be called a **geometric cycle** in $X/\Gamma$.
- Second, up to a subgroup of finite index in $\Gamma$, we may suppose that the manifolds $X/\Gamma$ and (every connected component of) $X_H/\Gamma_H$ are orientable.
- Third, we are interested in cases where
  - a geometric cycle $Y$ is orientable and
  - its fundamental class is not homologous to zero in $X/\Gamma$, in singular homology or homology with closed supports, as necessary.

  The usual way to go about the second question is to construct an orientable submanifold $Y'$ of complementary dimension such that the intersection product (so defined) of its fundamental class with that of $Y$ is non-zero. In doing so, if $X/\Gamma$ is non-compact, we have to assume that at least one of the cycles $Y, Y'$ is compact, while the other one need not be. In order to find a non-zero intersection product, if at all possible, it is often necessary to replace the arithmetic group $\Gamma$ by a suitable subgroup of finite index.

[2]These arithmetic quotients arise in many different areas of mathematics (topology, algebraic geometry, differential geometry, arithmetic). Some arithmetic quotients are algebraic varieties; some are not. Many of them are moduli spaces for familiar structures such as quadratic forms or abelian varieties. Understanding the cohomology of these spaces is an essential first step in understanding the spaces themselves. But the cohomology of arithmetic quotients is very subtle and it is notoriously difficult to compute. In many cases the cohomology vanishes; in other cases the rank of the cohomology may be very large. In this paper a wide range of techniques for understanding the cohomology of arithmetic quotients is provided, along with many specific successful computations.

[3]This terminology refers to an algebraic group, such as $SL_n$, that can be defined, as a subset of the groups of all $n \times n$ matrices, as the zeroes of a collection of polynomials with coefficients in the algebraic number field $\mathbb{Q}$. For $SL_n$ it is the single degree $n$ polynomial, $\det(A) - 1$.

**The exposition: overview and principal results.** The current exposition breaks fairly naturally into several parts which increase in their level of sophistication and technical understanding.

*A menagerie of examples.* The first three sections comprise Part I and introduce this circle of questions and ideas. Groups of units of algebraic number fields serve as our first example. We view these groups as discrete subgroups $\Gamma$ in an ambient Lie group $G_{\mathbb{R}}$ and we will explain how Dirichlet's unit theorem implies that the quotient $G_{\mathbb{R}}/\Gamma$ is compact. Then we proceed with a detailed study of various families of arithmetically defined hyperbolic $n$-manifolds, that is, arithmetic quotients attached to hyperbolic $n$-space. We start off with a treatment of arithmetically defined Kleinian groups, that is, the case $n = 3$. In Theorem A [Section 2.4] we indicate how the theory of geometric cycles has been used to prove a fundamental conjecture of Waldhausen for arithmetically defined hyperbolic three-manifolds in a specific case. In Theorem B [Section 3.3] and Theorem C [Section 3.4] we describe results regarding the far-reaching generalization of Waldhausen's conjecture to the case of $n$ dimensions: every arithmetically defined hyperbolic $n$-manifold $M$ has a finite covering $N$ such that every Betti number of $N$ is non-zero.

Mathematical prerequisites in the theory of algebraic and arithmetic groups have been kept to a minimum in these sections, but the examples involve a certain amount of algebraic number theory. These sections are not a prerequisite for later parts of the paper, and they may be skipped by the reader who wishes to proceed directly to the general case. However the examples in the *menagerie* involve the fundamental construction of the *group of units of a quadratic form*, which may help in gaining familiarity with the concepts pertaining to arithmetic groups and their ambient algebraic groups.

*Arithmetic groups and their cohomology.* Part II gives a brief account of the underlying general notion of an arithmetic subgroup of an algebraic group $G$ defined over some algebraic number field $k$, discusses congruence groups and the concept of neatness, and describes the arithmetic quotient $X/\Gamma$ associated to an arithmetic group. We present the fundamental result of Borel and Harish Chandra, to the effect that $X/\Gamma$ has finite volume if and only if $G$ has no non-trivial rational character, and it is compact if and only if , in addition, every rational unipotent element belongs to the radical of $G$. In Section 5.2, we review this compactness criterion. If $X/\Gamma$ is of finite volume but not compact, the adjunction of corners provides a compact manifold $\overline{X}/\Gamma$ so that the inclusion $X/\Gamma \longrightarrow \overline{X}/\Gamma$ is a homotopy equivalence. Next we turn to various realizations of the cohomology of these arithmetic quotients.

Section 5.5 provides the link between topology and representation theory. It culminates in the assertion that the deRham cohomology groups of $X/\Gamma$ are related in a natural way to relative Lie algebra cohomology groups. It is because of this result that representation-theoretic methods can be used to describe the cohomology of $X/\Gamma$. Later on this line of thought is extended in Section 13 where, following a fundamental result of Franke, we interpret these cohomology groups in terms of the theory of automorphic forms.

*Geometric cycles.* In Part III, Sections 6 and 9 are fundamental to the eventual construction of non-bounding geometric cycles. We work through the three steps in the general scheme outlined above. More precisely, let $G$ denote a connected semi-simple algebraic group defined over an algebraic number field $k$, $\Gamma \subset G(k)$ an

arithmetic subgroup. A given reductive $k$-subgroup $H$ of $G$ gives rise to a natural map $j_{H|\Gamma} : X_H/\Gamma_H \longrightarrow X/\Gamma$, where $\Gamma_H = \Gamma \cap H(k)$. The basic results, Theorems D, E and F in Section 6, guarantee that this map (by passing to a finite covering if necessary) is a proper, injective, closed embedding, and so that each connected component of the image is an orientable, totally geodesic submanifold of $X/\Gamma$, to be called a geometric cycle in $X/\Gamma$.

Several techniques may be used to prove that such a geometric cycle represents a non-trivial homology class. One method is by showing that the cycle intersects a second geometric cycle, of complementary dimension, in a single point with multiplicity $\pm 1$. Unfortunately, geometric cycles of complementary dimension usually intersect in a more complicated set, possibly of dimension greater than zero. To handle this situation, the theory of "excess intersections" is developed in Section 9; in particular, we discuss a useful formula for the intersection number of a pair of suitable geometric cycles. Under suitable conditions the intersection number of two such cycles can be expressed as the sum of the Euler numbers of the excess bundles corresponding to the connected components of the intersection, Theorem I [Section 9.2]. The resulting criterion for non-triviality appears as Theorem J [Section 9.3].

However, prior to that, we study the space $I_G^*$ of differential forms on $X$ which are invariant under the corresponding real Lie group. In Section 7, Theorem G, we give the interpretation of the space $I_G^*$ as the cohomology of the compact dual $X_u$ of the symmetric space $X$. The space $I_G^*$ consists of closed (even harmonic) forms; thus we get a homomorphism $\beta_\Gamma^* : H^*(X_u, \mathbb{R}) \longrightarrow H^*(X/\Gamma, \mathbb{R})$. In the case of a compact arithmetic quotient, $\beta_\Gamma^*$ is injective. The compact dual symmetric space is a relatively simple object, so this gives a completely understandable subspace of $H^*(X/\Gamma, \mathbb{R})$. As an example, we describe the case of real rank one symmetric spaces.

In specific cases, the approach via the cohomology of the compact dual symmetric space can be used to detect non-vanishing (co)homology classes carried by geometric cycles. Indeed, as explained in Section 8, given a geometric cycle $X_H/\Gamma_H$ in a compact quotient $X/\Gamma$, where $X$ and $X_H$ are both of non-compact type, one can use the homomorphism induced in cohomology by the embedding $X_{H,u} \longrightarrow X_u$ of the compact dual symmetric spaces to derive a criterion which ensures that the fundamental class $o_{X_H/\Gamma_H}$ of the geometric cycle $X_H/\Gamma_H$ is a non-trivial class in $H_*(X/\Gamma, \mathbb{R})$. As an application, in Theorem H [Section 8.3], we discuss the case of arithmetic quotients attached to quaternionic hyperbolic $n$-space. However, this approach does not lead to a construction of non-trivial classes for arithmetic quotients of hyperbolic $n$-space.

Following the first approach, the general analysis of intersection numbers of geometric cycles in Section 9 allows us to construct non-vanishing (co)homology classes for the arithmetic quotients in question. In Section 10, we exhibit some results regarding various families of compact arithmetic quotients attached to classical groups, whereas Section 11 is devoted to results concerning certain families of arithmetically defined groups in exceptional groups over number fields. In the former case, arithmetic quotients attached to arithmetically defined subgroups of special orthogonal groups serve as examples, Theorem L [Section 10.2]. In the latter case, we describe the construction of groups of type $G_2$ and $F_4$ as automorphism groups of Cayley (or octonian) algebras and of Albert algebras, respectively. Then

we discuss the geometric construction of non-vanishing homology classes for arithmetic quotients associated to these groups. The main result is Theorem M [Section 11.1]. In all cases, by passing to a finite covering of the arithmetic quotient if necessary, the geometric cycles represent "new" classes; that is, they do not come from the compact dual symmetric space via the map $\beta_\Gamma^*$. This is a consequence of the general result, given in Theorem K [Section 10.1], due to Millson and Raghunathan.

In Section 12, concluding Part III, our focus is on non-compact arithmetic quotients (but of finite volume) attached to a reductive algebraic $\mathbb{Q}$-group $G$ which is isotropic over $\mathbb{Q}$. We discuss geometric cycles which originate with a Levi subgroup $L$ of a proper parabolic $\mathbb{Q}$-subgroup $P$ of $G$. By passing over to a suitable subgroup of finite index the fundamental class of such a non-compact cycle detects a non-trivial class in homology with closed supports, and, by duality, a non-trivial cohomology class in $H^*(X/\Gamma, \mathbb{R})$. The method of proof for this result, Theorem N [Section 12.3], relies on analyzing the intersection of this cycle with a compact cycle of complementary dimension attached to the unipotent radical $N$ of $P$. These non-compact cycles, usually called *modular symbols*, are natural generalizations of the "classical" modular symbols. In general, the adjunction of corners $\overline{X}/\Gamma$ of the non-compact locally symmetric space $X/\Gamma$ provides a suitable framework to understand the geometric significance of both the modular symbols and the related compact cycles. Thus, our treatment includes a review of the construction of the compactification $\overline{X}/\Gamma$ of $X/\Gamma$.

Which portion of the cohomology $H^*(X/\Gamma, \mathbb{R})$ is generated by modular symbols or what their arithmetic meaning is, in particular, with regard to special values of $L$-functions [54], are still open questions.

*Towards automorphic forms.* The main objective of Part IV is to demonstrate in which way the geometric construction of (co)homology classes for arithmetically defined subgroups $\Gamma$ of a reductive $k$-group, $k$ an algebraic number field, has some important consequences for the existence of automorphic forms with specific properties. These will come up in the guise of the related automorphic representation generated by all translates of a given automorphic form by elements of the group $G(\mathbb{R})$ of real points of $G$. In Section 15, we substantiate this close relation between geometry and automorphic theory by discussing some families of examples, in particular, arithmetic quotients attached to symmetric spaces of real rank one and Hermitian symmetric domains of type IV, respectively. The interpretation of the cohomology groups of $\Gamma$ in terms of relative Lie algebra cohomology, discussed in Section 13, serves as the technical launching pad which makes this close connection possible. Moreover, as a prerequisite, we need the classification of irreducible unitary representations of real reductive Lie groups with non-zero cohomology. In Section 14, we briefly review this classification result [154] and derive some general vanishing results for certain analytically defined subspaces in the cohomology of arithmetic groups. We make this classification explicit in cases where the Lie group is the exceptional split group of type $G_2$ or a special orthogonal group of real rank one.

**Interlude: a historical comment.** There are various origins for the general scheme just outlined which differ in nature and methodological approach.

In 1970, Harder [52] studied the cohomology of arithmetic subgroups of the special linear group $SL_2/k$ defined over some number field $k$ and its relation with

the theory of automorphic forms, in particular, the theory of Eisenstein series. One of the methodological tools he introduced in his analysis [52, Section 3] was the action of the Galois group $\mathrm{Gal}(k/\mathbb{Q})$ on the arithmetic quotients of interest. Rohlfs [118] took up this idea in the general case and studied the corresponding set of fixed points. Its connected components are geometric cycles in the sense above. Results on the associated Lefschetz number of this action captured the cohomological contribution of these geometric cycles. This approach also worked in other cases of rational automorphisms of finite order on the underlying $k$-group $G$ ([120], [121], [84], [123]).

Another origin lies in the work of Millson [99] and Millson-Raghunathan [102]. Millson's result on the non-vanishing of the first Betti number of certain arithmetically defined compact hyperbolic $n$-manifolds hinges upon an explicit construction of an oriented, totally geodesic submanifold of codimension one. In the second paper, the authors construct non-bounding special cycles for the case of groups of units of certain quadratic or Hermitian forms which do not represent zero over their field of definition. This amounts to a geometric construction of cohomology classes for certain compact arithmetic quotients. Their approach differs from the scheme in so far as their object of concern is a classical real Lie group, say $SO(p,q), SU(p,q)$ or $Sp(p,q)$, without specifying an underlying rational structure. Then, in order to match the various conditions as, for example, orientability, they do a case-by-case construction of suitable discrete subgroups with compact quotients. This work is now superseded by the general results in [122] where, as one aspect in the study of geometric cycles and corresponding intersection numbers, the systematic use of non-abelian Galois cohomology serves as a suitable general framework to analyze the connected components of the intersection of special cycles and the questions of orientability involved.

**Beyond this exposition.** In view of the richness of the interplay between geometry and automorphic theory and the various results which are beyond the scope of this survey, the account we give can only be very selective in its choices and can only touch upon the most salient features of this interaction. In particular, the global approach in the theory of automorphic forms via representations of the group $G$ over the adèles of $k$, the related interpretation of the arithmetic quotients and the arithmetic involved will not be discussed. This manuscript will focus on the geometric rather than the representation-theoretical aspects of the construction of non-vanishing cohomology classes for arithmetic groups. However, we refer to [37], [90], [91], [131] for some recent results regarding the cohomology of arithmetic groups and their interpretation in terms of automorphic forms. These involve Eisenstein series, residues of such as well as cuspidal automorphic representations.

**Appendices.** As additional aids for various classes of readers we include a series of appendices, filling in background material. Among them one finds short expository accounts of central simple algebras over algebraic number fields, Weil's restriction of scalars, groups of units of quadratic forms, Lie algebra cohomology, and cohomology. The latter one includes products in cohomology and cohomology of manifolds, in particular, fundamental classes.

I thank James Cogdell, Fritz Grunewald and the referee for their careful reading of this manuscript. They made numerous insightful criticisms which, as I believe, enabled me to improve the exposition.

## Notation

(1) Let $F$ be an arbitrary finite extension of the field $\mathbb{Q}$, and denote by $\mathcal{O}_F$ its ring of integers. The set of places will be denoted by $V$, while $V_\infty$ (resp. $V_f$) will refer to the set of Archimedean (resp. non-Archimedean) places of $F$. The completion of $F$ at a place $v \in V$ is denoted by $F_v$, and its ring of integers by $\mathcal{O}_v (v \in V_f)$. For a given place $v \in V$ the normalized absolute value $|\cdot|_v$ on $F_v$ is defined as usual: if $v \in V_\infty$ is a real place, we let $|\cdot|_v$ be the absolute value; if $v \in V_\infty$ is a complex place we put $|x_v|_v = x_v \cdot \bar{x}_v$; and if $v \in V_f$ is a finite place we put $|x_v|_v = N_v^{-ord_v(x_v)}$, where $N_v$ denotes the cardinality of the residue field at the place $v$.

Suppose the extension $F/\mathbb{Q}$ has degree $d = [F : \mathbb{Q}]$. Let $S$ be the set of distinct embeddings $\sigma_i : F \to \mathbb{C}$, $1 \leq i \leq d$. Among these embeddings some factor through $F \to \mathbb{R}$. Let $\sigma_1, ..., \sigma_s$ denote these real embeddings $F \to \mathbb{R}$. Given one of the remaining embeddings $\sigma : F \to \mathbb{C}, \sigma(k) \not\subset \mathbb{R}$, to be called imaginary, there is the conjugate one $\bar{\sigma} : F \to \mathbb{C}$, defined by $x \mapsto \overline{\sigma(x)}$, where $\bar{z}$ denotes the usual complex conjugation of the complex number $z$. Then the number of imaginary embeddings is an even number, which we denote by $2t$. We number the $d = s + 2t$ embeddings $\sigma_i : F \to \mathbb{C}$ , $i = 1, ..., d$ in such a way that, as above, $\sigma_i$ is real for $1 \leq i \leq s$, and $\overline{\sigma}_{s+i} = \sigma_{s+i+t}$ for $1 \leq i \leq t$.

The set $V_\infty$ of Archimedean places of $F$ is naturally identified with the set of embeddings $\{\sigma_i\}_{1 \leq i \leq s+t}$; that is, we take all real embeddings and one representative for each pair of conjugate imaginary embeddings.

(2) The algebraic groups we consider will be linear groups; i.e., such a group $G$ defined over a field $k$ is affine viewed as an algebraic variety. It comes with an embedding $\rho : G \longrightarrow GL_n$ (defined over $k$) of $G$ into some general linear group. The additive group $(k, +)$ and the multiplicative group $(k^*, \cdot)$ of $k$, considered as algebraic groups, will be denoted by $G_a$ and $G_m$, respectively; they are both linear groups. If $G$ is any algebraic group, the connected component $G^0$ of the identity element of $G$ in the Zariski topology is a normal subgroup of finite index.

As usual, the radical (resp. unipotent radical) of a connected $k$-group is denoted by $RG$ (resp. $R_uG$). We recall that $RG$ (resp. $R_uG$) is the unique maximal connected solvable (resp. unipotent) normal subgroup in $G$. These subgroups are $k$-closed. In particular, if $k$ is a perfect field, the radical and the unipotent radical of every $k$-group are defined over $k$. By definition, if $G$ is not connected, its radical (resp. unipotent radical) is that of $G^0$, also to be denoted by $RG$ (resp. $R_uG$).

If $H$ is an algebraic group defined over a field $k$, and $k'$ is a commutative $k$-algebra containing $k$, we denote by $H(k')$ the group of $k'$-valued points of $H$. When $k'$ is a field containing $k$ we denote by $H/k'$ the $k'$ algebraic group $H \times_k k'$ obtained from $H$ by extending the ground field from $k$ to $k'$.

Let $k$ be a perfect field, e.g., a field of characteristic zero. An algebraic $k$-group is said to be reductive (resp. semi-simple) if its unipotent radical $R_uG$ (resp. its radical $RG$) is the trivial group $\{e\}$.

A $k$-torus $T$ is an algebraic group defined over $k$ which over the algebraic closure $\overline{k}$ of $k$ becomes a torus, i.e., is isomorphic to a product of copies of the multiplicative group $G_m(\overline{k}) = \overline{k}^*$ of $\overline{k}$. A $k$-torus $T$ is said to be $k$-split if it is isomorphic to such a product over $k$. The number of copies in such a product is the dimension of $T$, to be denoted $\dim T$. We say that a semi-simple algebraic group defined over $k$ splits over $k$ if it has a maximal $k$-split torus.

Let $G$ be an algebraic group defined over a field $k$. If $G$ is reductive or if the field $k$ is perfect, then the maximal $k$-split tori of $G$ are conjugate under $G(k)$. Thus they all have the same dimension. We denote by $\mathrm{rank}_k G = \mathrm{rk}_k G$ this common dimension and call it the $k$-rank of $G$. If $\mathrm{rank}_k G > 0$, then $G$ is said to be isotropic over $k$; if $\mathrm{rank}_k G = 0$, $G$ is said to be anisotropic over $k$.

(3) Let $k$ be an algebraic number field. If $G$ is an algebraic group defined over $k$, then a subgroup $\Gamma$ of $G(k)$ is arithmetic or arithmetically defined if, given an embedding $\rho : G \longrightarrow GL_n$ over $k$, the group $\rho(\Gamma)$ is commensurable with $\rho(G) \cap GL_n(\mathcal{O}_k) =: G_{\mathcal{O}_k}$; that is, the intersection $\rho(\Gamma) \cap G_{\mathcal{O}_k}$ has finite index both in $\rho(\Gamma)$ and $G_{\mathcal{O}_k}$. This notion is independent of the choice of a faithful representation $\rho : G \longrightarrow GL_n$.

(4) Let $\Gamma$ be a group, and let $R = \mathbb{Z}[\Gamma]$ be its group algebra. Given an $R$-module $M$, the Eilenberg-MacLane cohomology $H^*(\Gamma, M)$ of $\Gamma$ with coefficients in $M$ is defined. The cohomological dimension of $\Gamma$, to be denoted $\mathrm{cd}(\Gamma)$, is the upper bound (finite or infinite) of integers $m$ such that an $R$-module $M$ exists with $H^m(\Gamma, M) \neq 0$. Let $\Gamma'$ be a subgroup of finite index of $\Gamma$. Then, by [137], $\mathrm{cd}(\Gamma)$ is equal to $\mathrm{cd}(\Gamma')$ or $\infty$ (resp. $\mathrm{cd}(\Gamma')$ if $\Gamma$ is without torsion).

Suppose $\Gamma$ has torsion-free subgroups of finite index. Choosing one of these, say $\Gamma'$, we define the virtual cohomological dimension $\mathrm{vcd}(\Gamma)$ of $\Gamma$ to be $\mathrm{cd}(\Gamma')$.

# Part I
## A menagerie of examples

### 1. Groups of units

Even though we start the theory with semi-simple groups, in its development we need many other types of algebraic groups including reductive and unipotent groups. The simplest reductive but not semi-simple group is the multiplicative group $G_m$, and the simplest unipotent group is the additive group $G_a$. Therefore it is natural, despite the fact that these groups are abelian, to illustrate key notions with at least one of them, namely groups of units in algebraic number fields. We focus on Dirichlet's unit theorem and place it in a geometric frame. Finally we briefly discuss the highly non-trivial (and non-abelian) case of the multiplicative group $SL_1(D)$ of a division algebra, a natural generalization of the former example.

1.1. **Groups of units in number fields.** Let $k$ be an algebraic number field of degree $d = [k : \mathbb{Q}]$. An element $a \in k$ is said to be integral (or an algebraic integer) if there is a monic polynomial $f \in \mathbb{Z}[X]$ such that $f(a) = 0$. The set $\mathcal{O}_k$ of algebraic integers in $k$ forms an integral domain; in fact, it is a Dedekind domain. By definition, the group $\mathcal{O}_k^*$ of units of $k$ is the multiplicative group of invertible elements in $\mathcal{O}_k$. As an abstract group, due to Dirichlet's unit theorem, $\mathcal{O}_k^*$ is a finitely generated $\mathbb{Z}$-module, and

$$\mathcal{O}_k^* \cong \mu_k \times \mathbb{Z}^{s+t-1}$$

is a direct product of a finite cyclic group $\mu_k$, consisting entirely of the roots of unity in $k$, and a free module of rank $r := s + t - 1$, where $s$, and $2t$, denote the number of real, and imaginary, embeddings $\sigma : k \longrightarrow \mathbb{C}$.

The most familiar approach to this fact relies on some geometric techniques, introduced into number theory by Minkowski at the end of the 19th century, notably, his lattice point theorem. This method has proved to be of fundamental importance not only in arithmetic but also in other branches of mathematics such as the theory of quadratic forms.

In the rest of this section we explain how the group of units $\mathcal{O}_k^*$ may be naturally viewed as a discrete arithmetic subgroup $\Gamma$ of a real Lie group $G_{\mathbb{R}}$ whose quotient $G_{\mathbb{R}}/\Gamma$ is a compact space.

We number the $d = s + 2t$ embeddings $\sigma_i : k \to \mathbb{C}$ , $i = 1, ..., d$, in such a way that $\sigma_i$ is real for $1 \leq i \leq s$, and $\overline{\sigma}_{s+i} = \sigma_{s+i+t}$ for $1 \leq i \leq t$. The $d$-dimensional $\mathbb{R}$-algebra $k_{\mathbb{R}} = k \otimes_{\mathbb{Q}} \mathbb{R}$ decomposes as a product of $\mathbb{R}$-algebras with an embedding

$$\phi : k \longrightarrow k_{\mathbb{R}} = k \otimes_{\mathbb{Q}} \mathbb{R} = \mathbb{R}^s \times \mathbb{C}^t,$$
$$x \mapsto x \otimes 1 = (x^1, ..., x^s, x^{s+1}, ..., x^{s+t}),$$

where $x^i := \sigma_i(x)$, $i = 1, ..., s+t$. The coordinate maps $k_{\mathbb{R}} \to \mathbb{C}, y = x \otimes \nu \mapsto \sigma_i(x)\nu$, $i = 1, ..., s+t$, are $\mathbb{R}$-linear forms on $k_{\mathbb{R}}$, also denoted by $x^i$. By defining $x^i = x_i$ for $i = 1, ..., s$ and $x^i = x_i + x_{i+t}\sqrt{-1}$ for $i = s+1, ..., s+t$, that is, by separating real and imaginary parts for $i > s$, we obtain $d$ real coordinates $x_i$, $i = 1, ..., s + 2t$, on $k_{\mathbb{R}}$. These coordinates define a Euclidean metric on $k_{\mathbb{R}}$. The norm $n_{k/\mathbb{Q}} : k \longrightarrow \mathbb{Q}$, defined by

$$n_{k/\mathbb{Q}}(x) = \prod_{1,..,d} x^i = \prod_{i=1,..,s} x_i \prod_{i=s+1,..,s+t} (x_i^2 + x_{i+t}^2),$$

can be extended to a norm on $k_{\mathbb{R}}$, given by the right-hand side, to be denoted $n : k_{\mathbb{R}} \longrightarrow \mathbb{R}$. For a given $x \in k_{\mathbb{R}}$ the absolute value $|n(x)|$ is the product over the normalized absolute values of its coefficients. Then the image $\phi(\mathcal{O}_k)$ in $k_{\mathbb{R}}$ under the embedding $\phi$ is a (full) lattice in $k \otimes_{\mathbb{Q}} \mathbb{R}$. The volume of the fundamental parallelepiped of $\phi(\mathcal{O}_k)$ with respect to the coordinate system established above is given by $\mathrm{vol}(k_{\mathbb{R}}/\phi(\mathcal{O}_k)) = 2^{-t}\sqrt{|d_k|}$, where $d_k$ denotes the discriminant of $k$.

Within this geometric framework, the set $\{x \in k_{\mathbb{R}}^* |\ |n(x)| = 1\}$ carries the structure of a locally compact topological group, in fact, a Lie group, to be denoted $G_{\mathbb{R}}$, and $\phi(\mathcal{O}_k^*) =: \Gamma$ is a discrete subgroup. The assignment

$$x \mapsto (\log|x^1|, ..., \log|x^s|, 2\log|x^{s+1}|, ..., 2\log|x^{s+t}|)$$

defines a continuous homomorphism

$$\psi : G_{\mathbb{R}} \longrightarrow \mathbb{R}^{s+t}.$$

The kernel of the restriction of $\psi$ to $\phi(\mathcal{O}_k^*)$ is the finite group $\phi(\mu_k)$, and its image $\psi(\phi(\mathcal{O}_k^*))$ is a full lattice $L$ in the hyperplane $H \subset \mathbb{R}^{s+t}$ defined by the equation $\sum_{i=1,..,s} x_i + 2\sum_{i=s+1,..,s+t} x_i = 0$. Then $\psi$ passes to a surjective mapping $\psi : G_{\mathbb{R}}/\Gamma \longrightarrow H/L \cong (S^1)^{s+t-1}$ with finite fibers. In this way, Dirichlet's unit theorem implies that the quotient $G_{\mathbb{R}}/\Gamma$ is compact. Thus, in this new framework, properties of the quotient space encode the structural description of the $\mathbb{Z}$-module $\mathcal{O}_k^*$ (as given by Dirichlet's unit theorem) and vice versa.

1.2. **Division algebras over number fields.** Passing from a number field $k$ to a finite-dimensional division algebra $D$ over $\mathbb{Q}$, or even over $k$, we encounter new arithmetic groups. By definition, an order $\Lambda$ in $D$ is a subring of $D$ containing the unit element $1_D$ which is a finitely generated $\mathcal{O}_k$-module with $k\Lambda = D$. The latter condition characterizes a full $\mathcal{O}_k$-lattice in $D$. If $G = SL_1(D)$ denotes the algebraic

$k$-group determined by the elements in $D$ of reduced norm one (see Appendix A), then any order $\Lambda$ in $D$ gives rise to an arithmetic subgroup $\Gamma_\Lambda$ of $G$.

Which form a suitable unit theorem in this case might take is an open problem [69]. However, the following is true: Let $D$ be a division algebra defined over $\mathbb{Q}$, and suppose that $D \otimes_{\mathbb{Q}} \mathbb{R}$ is isomorphic to a matrix algebra over $\mathbb{R}$. Then $\Gamma_\Lambda$ presents itself acting on a homogeneous space $X$ which is defined by the ambient Lie group $G_{\mathbb{R}}$, and the quotient space $X/\Gamma_\Lambda$ is compact [163].

## 2. Arithmetically defined Kleinian groups

In this section, we review the basic definitions and aspects of arithmetically defined hyperbolic 3-manifolds and Kleinian groups. Firstly, there is the class of Bianchi groups, that is, subgroups of finite index in some $PGL_2(\mathcal{O}_d)$, where $\mathcal{O}_d$ denotes the ring of integers in an imaginary quadratic number field $\mathbb{Q}(\sqrt{d})$, $d < 0$. These groups give rise to non-compact hyperbolic 3-manifolds but of finite volume. Secondly, compact arithmetically defined hyperbolic 3-manifolds are determined by groups which originate with an order in a division quaternion algebra $D$ over an algebraic number field $k$ subject to certain conditions. This is explained in Section 2.3.[4]

We then discuss the existence of totally geodesic surfaces in arithmetically defined hyperbolic 3-manifolds. In the case of Bianchi groups, one finds an abundance of families of these. They are instrumental in constructing non-bounding cycles. In the case of compact arithmetically defined hyperbolic 3-manifolds the very existence of non-vanishing (co)homology classes and their nature is still a challenging issue.[5]

2.1. **Hyperbolic 3-manifolds.** Every orientable hyperbolic 3-manifold is isometric to the quotient $H^3/\Gamma$ of hyperbolic 3-space $H^3$ by a discrete torsion-free subgroup $\Gamma$ of the group $Iso(H^3)^0$ of orientation-preserving isometries of $H^3$. The latter group is isomorphic to the (connected) group $PGL_2(\mathbb{C})$, the real Lie group $SL_2(\mathbb{C})$ modulo its center $\{\pm \mathrm{Id}\}$. The space $H^3$ is diffeomorphic to $\mathbb{R}^3$ and hence is contractible.

Hyperbolic 3-space can be realized in various models. It is characterized as the unique 3-dimensional connected, simply connected Riemannian manifold with constant sectional curvature $-1$. In the framework of this article, $H^3$ is best described as the symmetric space attached to the real Lie group $G = SL_2(\mathbb{C})$, that is, $H^3 = K\backslash G$, where $K$ denotes the maximal compact subgroup $SU(2)$ of unitary matrices in $SL_2(\mathbb{C})$. Since two maximal compact subgroups of $G$ are conjugate by an inner automorphism, the space $H^3$ may also be viewed as the space of maximal compact subgroups of $G$. In general, we refer to Section 3.1 in which hyperbolic $n$-space is described.

By definition, a Kleinian group $\Gamma$ is a discrete subgroup of the group $Iso(H^3)^0 = PGL_2(\mathbb{C})$ of orientation-preserving isometries of $H^3$. The group $\Gamma$ is said to have finite covolume if $H^3/\Gamma$ has finite volume, and is said to be cocompact if $H^3/\Gamma$ is compact. If the Kleinian group $\Gamma$ has torsion, then $H^3/\Gamma$ is an orbifold (that is, it

[4]For the convenience of the reader, we include in Appendix A the necessary background material on central simple algebras, in particular, quaternion algebras over an algebraic number field.

[5]The exposition given here relies on the author's article [129] where more details can be found.

locally looks like the quotient of a Euclidean space by a finite group); otherwise it is a manifold.

Among hyperbolic 3-manifolds, the ones originating with arithmetically defined Kleinian groups form a class of special interest. These arithmetic Kleinian groups fall naturally into two classes, according to whether $H^3/\Gamma$ is compact or not. However, this quotient always has finite volume with respect to the hyperbolic metric. In the following we discuss these two classes in more detail.

### 2.2. **Bianchi groups.**

*The groups.* Let $k$ be an imaginary quadratic number field; that is, $k$ is of the form $\mathbb{Q}(\sqrt{d})$, $d < 0$, $d$ a square-free integer. The ring of algebraic integers in $k$ is denoted by $\mathcal{O}_d$; it is a Dedekind domain. This ring forms a $\mathbb{Z}$-lattice in $\mathbb{C}$ with basis 1, $\omega_d$, where $\omega_d = \sqrt{d}$ when $d \equiv 2, 3 \bmod 4$ and $\omega_d = (1/2)(1 + \sqrt{d})$ when $d \equiv 1 \bmod 4$. In the former case, the discriminant $d_k$ of $k$ is $4\,d$ whereas $d_k = d$ in the other case. We consider the projective general linear group $G = PGL_2/k$ viewed as an algebraic group defined over $k$. The group $G(k) = PGL_2(k)$ of $k$-points of $G$ is the quotient of $GL_2(k)$ modulo its center. A subgroup $\Gamma$ of the group $G(k)$ is arithmetically defined (or an arithmetic group) if it is commensurable with the group $\Gamma_d := PGL_2(\mathcal{O}_d)$, that is, if $\Gamma \cap \Gamma_d$ has finite index both in $\Gamma$ and $\Gamma_d$. These groups may be viewed as discrete subgroups of the group $PGL_2(\mathbb{C}) = PGL_2(k \otimes \mathbb{R})$ of real points of $G$.

As early as 1892, L. Bianchi studied this class of groups, today named after him. These groups and all their subgroups of finite index have finite covolume. Let $\mu(\Gamma)$ denote the volume of $H^3/\Gamma$ with respect to the hyperbolic metric. Following G. Humbert, the value $\mu(\Gamma_d)$ can be expressed in terms of invariants only depending on the underlying field $k$ (see, e.g., [46] or [18, Thm. 7.3.]). More precisely, one has

$$\mu(\Gamma_d) = (|d_k|^{3/2}/\pi^2)\zeta_k(2),$$

where $d_k$ denotes the discriminant of $k$ and $\zeta_k$ is the $\zeta$-function attached to $k$. Following the work of Bianchi and Humbert [13], Swan [146] exhibited fundamental domains for the action of $\Gamma_d$ on $H^3$ for small values of $d$. A range of geometric or group-theoretical results such as, for example, presentations for $\Gamma_d$ is based on this approach.

*Totally geodesic surfaces in $H^3/\Gamma$.* Let $\sigma \in \mathrm{Gal}(k/\mathbb{Q}) \cong \mathbb{Z}/(2)$ be the non-trivial Galois automorphism of the field extension $k/\mathbb{Q}$. It acts on $H^3/\Gamma$ provided $\Gamma \subset \Gamma_d$ has finite index and is preserved under the involution $\sigma$. The set $\mathrm{Fix}(\langle\sigma\rangle, H^3/\Gamma)$ of fixed points consists of finitely many connected components, each of which is a totally geodesic submanifold of $H^3/\Gamma$, of some dimension (0, 1, or 2). The connected components are parametrized in a beautiful way, by the non-abelian Galois cohomology set $H^1(\langle\sigma\rangle, \Gamma)$. We explain this construction as well as its algebraic description in a general framework in Section 6. In many cases it is possible to count the number of components of each dimension, and even the number of 2-dimensional components with non-vanishing Euler characteristic [119].

### 2.3. **Kleinian groups originating with orders in quaternion algebras.** *Appendix A contains a brief account of some facts and results on quaternion algebras (and central simple algebras) needed in this subsection.*

*Arithmetically defined Kleinian groups.* Let $\Gamma$ be a discrete subgroup of $PGL_2(\mathbb{C})$. Then $\Gamma$ is said to be arithmetically defined if there exist an algebraic number field $k/\mathbb{Q}$ with exactly one complex place $w$ (that is, $t = 1$ in the usual enumeration of the places of an algebraic number field as in 1.1), an arbitrary (but possibly empty) set $T$ of real places, a $k$-form $G$ of $PGL_2/k$ such that $G(k_v)$ is compact for $v \in T$ and an isomorphism

$$PGL_2(\mathbb{C}) \xrightarrow{\sim} G(k_w), \ w \text{ the complex place},$$

which maps $\Gamma$ onto an arithmetic subgroup of $G(k)$ naturally embedded into $G(k_w)$. Naturally, the case of Bianchi groups dealt with in Section 2.2 can be subsumed under this construction in terms of quaternion algebras as well. Here the quaternion algebra is $Q = M_2(k)$, $k$ an imaginary quadratic extension of $\mathbb{Q}$, and $G$ is the split form $PGL_2/k$ itself; that is, $T$ is the empty set and the choice of $Q$ is equivalent to the specification that the ramification set $\mathrm{Ram}(Q) = \emptyset$ (see Appendix A.4 for this notion).

*Groups originating with orders in division algebras.* Given an algebraic number field $k$ with exactly one complex place and an arbitrary non-empty set $T$ of real places we consider a $k$-form $G$ of $PGL_2/k$ which is of the form $SL_1(D)$, where $D$ is a division quaternion algebra over $k$ which ramifies (at least) at all real places $v \in T$. Then an arithmetically defined subgroup $\Gamma$ originates with an order $\Lambda$ in $D$. By definition, an order $\Lambda$ in $D$ is a subring of $D$ containing the unit element $1_D$ which is a finitely generated $\mathcal{O}_k$-module with $k\Lambda = D$. The latter condition characterizes a full $\mathcal{O}_k$-lattice in $D$. Then any subgroup $\Gamma$ of $G(k)$ which is commensurable with $G_\Lambda$ gives rise to a compact hyperbolic 3-manifold $H^3/\Gamma$.

This construction exhausts all possible types of arithmetically defined subgroups of $PGL_2(\mathbb{C})$ that give rise to a compact hyperbolic 3-manifold $H^3/\Gamma$.

*Examples.* We discuss some families of examples. Suppose that the defining field $k$ (which has exactly one complex place) contains a subfield $k'$ such that the degree $[k : k']$ of the extension $k/k'$ is 2. Due to the assumption on $k$, $k'$ is a totally real extension field of $\mathbb{Q}$. Let $\mathrm{Gal}(k/k') = \{\mathrm{Id}_k, c\}$ denote its Galois group.

Let $D$ be a quaternion division algebra over $k$ underlying a given inner form $G'/k$ of $G/k = PGL_2/k$ so that the finite set $S$ of places $v \in V$ where $G'(k_v)$ is not isomorphic to $G(k_v)$ contains $T$. As a quaternion division algebra, $D$ is isomorphic to its opposite algebra, and the class of $D$ is of order 2 in the Brauer group $Br(k)$ of $k$. In our situation at hand, as explained in Appendix A.4, given a central simple $k$-algebra $A$ of degree $\deg(A)$ there is the associated central simple $k'$-algebra $N_{k/k'}(A)$ of degree $\deg(A)^2$, to be called the norm of the $k$-algebra $A$. This construction induces a group homomorphism

$$N_{k/k'} : Br(k) \longrightarrow Br(k'), \quad [A] \mapsto [N_{k/k'}(A)],$$

of the respective Brauer groups [71, 3.13].

In our context we have to distinguish the two cases:

(I) The class $[N_{k/k'}(D)]$ has order 1 in $Br(k')$,
(II) The class $[N_{k/k'}(D)]$ has order 2 in $Br(k')$.

In case (I), the class of the $k'$-algebra $N_{k/k'}(D)$ of degree 4 is the unit element in $Br(k')$. As a consequence, $N_{k/k'}(D)$ is isomorphic to the matrix algebra $M_4(k')$; that is, the algebra splits over $k'$. In such a case, by using results of Albert, the

quaternion algebra $D$ posseses an involution $\tau$ of the second kind[6] of a particular type. There exists a unique quaternion $k'$-subalgebra $B \subset D$ such that $D = B \otimes_{k'} k$ and $\tau$ is of the form $\tau = \gamma_0 \otimes c$, where $\gamma_0$ is the quaternionic conjugation.

In case (II), the $k'$-algebra $N_{k/k'}(D)$ of degree 4 is (up to isomorphism) of the form $M_2(Q)$, where $Q$ is a quaternion division algebra over $k'$.

Now suppose that $\Gamma \subset SL_1(D)$ is a torsion-free arithmetic subgroup given by an order $\Lambda$ in $D$, $D$ a quaternion division algebra over $k$. In case (I), via the group $SL_1(B)$ and a suitable order $\Lambda_B$ in $B$, the $k'$-subalgebra $B \subset D$ gives rise to a totally geodesic hypersurface of the form $H^2/\Gamma_B$ in the compact hyperbolic 3-manifold $H^3/\Gamma$. In case (II), there is evidence that the corresponding hyperbolic 3-manifolds $H^3/\Gamma$ do not admit totally geodesic hypersurfaces. However, $H^3/\Gamma$ may very well admit submanifolds of codimension one which are not totally geodesic.

2.4. **Construction of (co)homology classes.** In this subsection we discuss various approaches to construct non-trivial classes in (co)homology of an arithmetically defined hyperbolic 3-manifold.

*Bianchi groups.* Suppose that the arithmetically defined hyperbolic manifold $H^3/\Gamma$ is non-compact; that is, the arithmetic group $\Gamma$ is a torsion-free subgroup of finite index in some $\Gamma_d = PGL_2(\mathcal{O}_d)$, $\mathcal{O}_d$ the ring of integers in the imaginary quadratic number field $\mathbb{Q}(\sqrt{d})$. By use of reduction theory, one may view $M_\Gamma = H^3/\Gamma$ as the interior of a compact manifold $\overline{M}_\Gamma$ with boundary $\partial(\overline{M}_\Gamma)$ so that the inclusion $M_\Gamma \longrightarrow \overline{M}_\Gamma$ is a homotopy equivalence [109]. The boundary is a disjoint union of finitely many tori corresponding to the $\Gamma$-conjugacy classes of Borel subgroups of $PGL_2(k)$. The long exact cohomology sequence of the pair $(\overline{M}_\Gamma, \partial(\overline{M}_\Gamma))$ contains the segment

$$H^1(\overline{M}_\Gamma, \mathbb{Q}) \xrightarrow{r^1} H^1(\partial(\overline{M}_\Gamma), \mathbb{Q}) \longrightarrow H^2_c(\overline{M}_\Gamma, \mathbb{Q}).$$

The image of the restriction map $r^1$ has dimension equal to $(1/2)$ dim $H^1(\partial(\overline{M}_\Gamma))$, that is, equal to the number of components in $\partial(\overline{M}_\Gamma)$. This leads to a non-vanishing result for the first cohomology group of $H^3/\Gamma$ in this case.

The existence of cohomology classes in $H^1(H^3/\Gamma, \mathbb{Q}) = H^1(\overline{M}_\Gamma, \mathbb{Q})$ which restrict trivially under $r^1$ to the cohomology of the boundary components is a more challenging issue. It is related to the very existence of cuspidal automorphic forms for the arithmetic groups $\Gamma \subset \Gamma_d$ that we deal with ([49]). Quite deep results in the theory of automorphic forms have been used to obtain non-vanishing results for the cohomology of $H^3/\Gamma$. In reverse, a non-vanishing class in ker $r^1$ gives rise to a cuspidal automorphic form.

However, from the geometric point of view, the arithmetically defined non-compact hyperbolic 3-manifolds of Bianchi type admit totally geodesic submanifolds. In particular, totally geodesic hypersurfaces arise as 2-dimensional components $F(\gamma)$ of the set of fixed points under the involution induced by the non-trivial Galois automorphism of the underlying imaginary quadratic extension $k/\mathbb{Q}$. Their existence made possible the construction of non-bounding cycles and eventually lead to non-vanishing results for the cohomology of Bianchi groups. In more group-theoretical terms, a Bianchi group $\Gamma_d$ has a subgroup of finite index which admits a homomorphism onto a non-abelian free group. This was first proved in [48] by use of an explicit geometric construction of certain strips within a fundamental domain

[6] i.e., $\tau$ fixes $k'$ elementwise and is of order 2 on the center of $D$.

for $\Gamma_d$, together with some arithmetic results. Nowadays the analogous result of Lubotzky [92] (see Section 3.3 below) pertaining to oriented hyperbolic $n$-manifolds of finite volume which contain a totally geodesic submanifold of codimension one sheds new light on this assertion. His approach relies on methods of geometric group theory.

*Betti numbers in the compact case.* We now consider a compact arithmetically defined hyperbolic 3-manifold as constructed in Section 2.3. Within Thurston's geometrization program for 3-manifolds, the class of hyperbolic 3-manifolds plays a fundamental role. However, due to its underlying connections with number theory, the arithmetically defined hyperbolic 3-manifolds seem to be in many ways more tractable.

A fundamental conjecture in 3-manifold theory, stated by Waldhausen in 1968, says: Given an irreducible 3-manifold $M$ with infinite fundamental group there exists a finite cover $M'$ of $M$ which is Haken; that is, it is irreducible and contains an embedded incompressible surface. One knows that 3-manifolds which are virtually Haken are geometrizable. This so-called virtual Haken conjecture is the source for the (even stronger) virtual positive Betti number conjecture which states within the class of hyperbolic 3-manifolds $M = H^3/\Gamma$ that there exists a finite cover $M'$ with non-vanishing first Betti number $b_1(M')$. The following result confirms this conjecture in a specific case.

**Theorem A.** *Let $H^3/\Gamma = M$ be a compact arithmetically defined hyperbolic 3-manifold. Suppose that the defining field $k$ contains a subfield $k'$ so that the field extension $k/k'$ has degree two. Then there exists a finite covering $N$ of $M$ with non-vanishing first Betti number $b_1(N)$.*

Various approaches which are substantially different in nature lead to a proof of this result.

Firstly, suppose that $\Gamma \subset SL_1(D)$ is an arithmetic subgroup originating with an order in a quaternion division algebra $D$ over $k$ which belongs to case (I) in Section 2.3. Then, as seen, $H^3/\Gamma$ admits a totally geodesic hypersurface. Originally initiated by Millson [99], later on strengthened by Lubotzky [92], one can use this totally geodesic submanifold of codimension one to construct non-bounding cycles. This geometric approach applies to this case but can easily be extended to higher-dimensional orientable $n$-hyperbolic manifolds which contain a totally geodesic submanifold of codimension one.

Secondly, in case (II), due to the lack of totally geodesic hypersurfaces in $H^3/\Gamma$ one has to pursue a different idea. This class of cocompact arithmetically defined Kleinian groups can be interpreted within the theory of unit groups of skew-Hermitian forms in quaternionic vector spaces. By a thorough analysis of a quaternionic theta series and an associated period integral, Li and Millson [87] obtained a non-vanishing result in this case as well.

However, within the realm of the theory of automorphic forms, there is a unified approach to the non-vanishing result in cases (I) and (II). By use of the Jacquet–Langlands correspondence, the construction of cuspidal automorphic forms for Bianchi groups and corresponding non-vanishing cohomology classes in these cases leads to the existence of cohomological automorphic forms for the quaternionic division algebra $D$ and the order therein [80, Section 6]. It is worth noting

that the approach works as well in other cases of interest, for example, in case the defining field $k$ is a cubic non-normal extension of $\mathbb{Q}$ [129, Theorem 4.9].

## 3. Arithmetically defined hyperbolic $n$-manifolds

Within the classification of Riemannian globally symmetric spaces, hyperbolic $n$-space is one of the four classes of rank one symmetric spaces of negative curvature. It can be described by the corresponding Riemannian symmetric pair $(SO_0(n,1), SO(n))$ of non-compact type. In this section we exhibit various families of arithmetically defined locally symmetric spaces which arise as quotients of hyperbolic $n$-space. Of particular interest is the family of hyperbolic manifolds $H^n/\Gamma(f)$, where $H^n$ denotes hyperbolic $n$-space and $\Gamma(f)$ is the "unit group associated to a quadratic form $f$", that is, the group of integral matrices in $SO_0(n,1)$ that preserve $f$ (Section 3.2). For $n$ odd, there are other types of arithmetically defined hyperbolic $n$-manifolds.

In the case of a unit group of a quadratic form, geometric ideas can be used to prove a non-vanishing result for the first Betti number of the corresponding manifold up to a finite covering, thereby confirming a fundamental conjecture in real hyperbolic geometry (see Section 3.3). In the sequel we discuss similar results for other arithmetically defined hyperbolic $n$-manifolds as well as generalizations pertaining to arbitrary Betti numbers for cocompact arithmetic groups, a result due to Millson-Raghunathan.

### 3.1. Hyperbolic $n$-space $H^n$.

*The hyperboloid model.* The real vector space $\mathbb{R}^{n+1}$ of dimension $n+1$, endowed with the non-degenerate indefinite bilinear form $b(x,y) = \sum_{1\le i\le n} x_i\, y_i - x_{n+1}\, y_{n+1}$ is a pseudo-Euclidean space. Let $q(x) = \sum_{1\le i\le n} x_i^2 - x_{n+1}^2$ be the corresponding quadratic form on $\mathbb{R}^{n+1}$ determined by $b$. Since $b$ is indefinite, the number $q(x)$ can be positive, negative, or zero. The orthogonal complement of a vector $x \in \mathbb{R}^{n+1}$ with respect to $q$ is, by definition, the subspace $\langle x\rangle^\perp = \{y \in \mathbb{R}^{n+1} \mid b(y,x) = 0\}$ in $\mathbb{R}^{n+1}$. If $x \in \mathbb{R}^{n+1}$ is, what is usually called a time-like vector, that is, $q(x) < 0$, then the restriction of $q$ to the complement $\langle x\rangle^\perp$ is positive definite. The hyperboloid model of real hyperbolic $n$-space $H^n$ is the upper sheet of the hyperboloid $H = \{u \in \mathbb{R}^{n+1} \mid q(u) = -1\}$,

$$H^n = \{x \in \mathbb{R}^{n+1} \mid q(x) = -1,\ x_{n+1} > 0\}.$$

Note that $b(x,y) \le -1$ for all $x, y \in H^n$, with equality if and only if $x = y$. The hyperbolic distance between two points $x, y \in H^n$ is defined to be the real number

$$d_H(x,y) = \eta(x,y),$$

where $\eta(x,y)$ denotes the Lorentzian time-like angle between $x$ and $y$. It is uniquely determined by the fact that $\cosh d_H(x,y) = -b(x,y)$. This distance function $d_H$ is a metric on $H^n$.

*The Riemannian manifold $H^n$ and its isometry group.* We consider the group $O(n,1) = \{g \in GL(\mathbb{R}^{n+1}) \mid b(g(u), g(v)) = b(u,v),\ u, v \in \mathbb{R}^{n+1}\}$ of Lorentz transformations of $(\mathbb{R}^{n+1}, b)$. This group consists of those matrices $A \in M_{n+1}(\mathbb{R})$ such that $A^t J A = J$, where $J$ is the diagonal matrix $diag(1, ..., 1, -1)$. We denote by $O(n,1)_+$ the subgroup of index two in $O(n,1)$ that preserves $H^n$. This group $O(n,1)_+$ acts by isometries on $H^n$. This action is transitive, and the group of

isometries $Iso(H^n, d_H)$ coincides with $O(n,1)_+$. The stabilizer $K$ of the point $(0,...,0,1)$ is naturally isomorphic to the group $O(n) \times O(1)$. The group $O(n,1)_+$ has two connected components corresponding to matrices of determinant $+1$ and $-1$. We denote by $SO_0(n,1)$ the connected component of the identity.

The pseudo-Euclidean metric $ds^2 = \sum_{1\le i\le n} dx_i^2 - dx_{n+1}^2$ on $\mathbb{R}^{n+1}$ induces a Riemannian metric on the smooth submanifold $H^n$ by restricting the given bilinear form on the tangent space $T_x(\mathbb{R}^{n+1})$ to $T_x(H^n)$ for each $x \in H^n$. Recall that the latter tangent space is naturally isomorphic to $\langle x\rangle^\perp$, the orthogonal complement of $\langle x\rangle$, and the restriction of $b$ to $\langle x\rangle^\perp$ is positive definite for all $x \in H^n$. The group $Iso(H^n, d_H) = O(n,1)_+$ preserves the Riemannian metric on $H^n$. We may identify $H^n$ with the quotient of $O(n,1)_+$ by the stabilizer $K$ of $(0,...,0,1) \in H^n$, that is, with $K\backslash O(n,1)_+$. The stabilizer $K$ is a maximal compact subgroup in $O(n,1)_+$. As well, we also may identify hyperbolic $n$-space with the quotient of $SO_0(n,1)$ by $SO(n)$.

### 3.2. **Groups of units acting on $H^n$.**

*Standard arithmetic unit groups.* Let $k$ be a totally real algebraic number field of degree $d = [k : \mathbb{Q}]$, $\mathcal{O}$ its ring of integers, and let $\sigma_1, ..., \sigma_d : k \to \mathbb{R}$ be the distinct embeddings of $k$ into $\mathbb{R}$. We suppose that $\sigma_1 = \mathrm{Id}$. Let

$$f(x) = \sum_{1\le j\le n+1} \alpha_j x_j^2$$

be a non-degenerate quadratic form on $k^{n+1}$ of signature $(n,1)$, all of whose non-trivial conjugates are positive definite; that is, the form

$$f^{\sigma_i} = \sigma_i(\sum_{1\le j\le n+1} \alpha_j x_j^2) = \sum_{1\le j\le n+1} \sigma_i(\alpha_j) x_j^2$$

is positive definite for $i = 2, ..., d$. Let $O(f)$ be the group of isometries of $f$; that is,

$$O(f) = \{\phi \in GL(k^{n+1}) \mid f(\phi(x)) = f(x) \text{ for all } x \in k^{n+1}\}.$$

The *group of units of $f$* consists of those isometries with integral entries; that is,

$$\Gamma(f) = \{\gamma \in GL(\mathcal{O}^{n+1}) \mid f(\gamma(x)) = f(x) \text{ for all } x \in k^{n+1}\}.$$

A standard arithmetic unit group of the quadratic form $f$ is a subgroup of finite index,

$$\Gamma \subset \Gamma(f) \cap SO(f),$$

where $SO(f)$ denotes the kernel of the determinant map $\det : O(f) \longrightarrow k^*$ into the multiplicative group $k^*$ over $k$. As an algebraic $k$-group, the group $SO(f)$ is connected; it is called the special orthogonal group of $f$.

We refer to Appendix B where we discuss the general case of orthogonal groups, that is, groups of isometries of non-degenerate quadratic $k$-vector spaces $(E, f)$ of finite dimension. In particular we describe the explicit construction of arithmetically defined groups that arise from unit groups of quadratic forms.

A torsion-free standard arithmetic unit group $\Gamma$ may be viewed as a discrete subgroup of the group $SO_0(n,1)$. It acts freely on hyperbolic $n$-space via isometries.

We consider two cases. If $k \neq \mathbb{Q}$, then $H^n/\Gamma$ is compact. If $k = \mathbb{Q}$, and if $n+1 \ge 5$, then $H^n/\Gamma$ has finite volume but it is not compact. In the first case, $f(x) = 0$ has no non-zero solutions $x \in k^{n+1}$. Consequently, the group $SO(f)$ consists of semi-simple elements and it has no non-trivial characters defined over $k$,

so the compactness criterion (of [19], described in Section 5.2 of this paper) applies. In the second case, every non-degenerate quadratic form on $\mathbb{Q}^{n+1}$ which is indefinite over $\mathbb{R}$ has a rational zero, by a theorem of A. Meyer [98]. Hence, the manifold $H^n/\Gamma$ is not compact but it has finite volume. For $n = 2, 3$, this construction gives rise to compact as well as to non-compact hyperbolic $n$-manifolds $H^n/\Gamma$.

Here is a more explicit family of examples whose corresponding quotients $H^n/\Gamma$ are compact. Let $k \neq \mathbb{Q}$ be a totally real number field of degree $d$, let $\epsilon_1 = 1$, resp. $\epsilon_i = -1$, for $i = 2, ..., d$. Then there exists $\alpha \in k$ with $sign(\sigma_i(\alpha)) = \epsilon_i$ , $i = 1, ..., d$. Then the quadratic form $f$ on $k^{n+1}$, defined by

$$f(x) = \sum_{1 \leq j \leq n} x_j^2 - \alpha x_{n+1}^2,$$

has signature $(n, 1)$ and all of its conjugates are positive definite. A subgroup $\Gamma$ of $\Gamma(f) \cap SO(f)$ of finite index serves our purpose.

In general, for $n$ odd, $SO(n, 1)$ has another type of cocompact arithmetically defined subgroups, commensurable to groups of units of skew Hermitian forms in quaternionic vector spaces. If $n = 7$, one should add arithmetic subgroups of the triality form of $SO_8$ to those [147].

*Remark.* Hyperbolic 3-space can be realized in various models. One of these, the Lobachevski model discussed above, provides an interpretation of its group of isometries as an orthogonal group of a real quadratic space of dimension 4. More precisely, the group $Iso(H^3)^0$ of orientation-preserving isometries of $H^3$ can be viewed as the identity component of the real Lie group $SO(3, 1)$. This identification provides the exceptional isomorphism of real Lie groups

$$PGL_2(\mathbb{C}) \xrightarrow{\sim} Iso(H^3)^0 \xleftarrow{\sim} SO_0(3, 1).$$

In the case of hyperbolic 3-manifolds, all possible types of cocompact arithmetically defined subgroups in $PGL_2(\mathbb{C})$ are described in Section 2.3. The construction of standard arithmetic unit groups of quadratic forms as exhibited fits into this description as follows: Let $k$ be a totally real number field, $k \neq \mathbb{Q}$, $f$ a non-degenerate quadratic form on $k^4$ such that for all $v$ not equal to a given $v_0$ in $V_\infty$ the form $f_v$ over $k_v \cong \mathbb{R}$ is positive definite while $f_{v_0}$ over $k_{v_0}$ has signature $(3, 1)$, and let $\Gamma$ be an arithmetic subgroup in $SO(f)$. This class of cocompact Kleinian groups corresponds to the class of arithmetically defined subgroups in $SL_1(D)$, $D$ a quaternion division algebra of the form $B \otimes_k k'$, where $B$ is a quaternion algebra over the totally real field $k$ which ramifies at all places $v \neq v_0$ and $k'$ is a quadratic externsion of $k$ with exactly one complex place (that is, $k' = k(\sqrt{a})$, $a < 0$, $a \in k$, $a$ square free) (see, e.g., [93, Thm. 10.2.1]). This is case (I) in the discussion in Section 2.3.

### 3.3. **The virtual positive Betti number conjecture.**

*Totally geodesic hypersurfaces.* A fundamental conjecture in the theory of real hyperbolic geometry is:

**Conjecture.** *Let $M$ be a compact hyperbolic manifold. Then there exists a finite covering $N$ of $M$ with non-vanishing first Betti number $b_1(N)$.*

This conjecture is a natural outgrowth of the same assertion in the case of hyperbolic 3-manifolds, namely the virtual positive Betti number conjecture (see Section

2.4). In this general form, the conjecture has been verified (in many cases) by using the geometric techniques described in this paper. The very existence of totally geodesic hypersurfaces proved to be decisive in these results.

**Theorem B.** *Let $M$ be an oriented $n$-dimensional hyperbolic manifold of finite volume. Suppose that $M$ contains a totally geodesic submanifold $F$ of codimension one. Then $\Gamma := \pi_1(M)$ has a subgroup of finite index which admits a homomorphism onto a virtually non-abelian free group. In particular, there exists a finite covering $N$ of $M$ with non-vanishing first Betti number $b_1(N)$.*

This was first proven by Millson in the case of standard arithmetic unit groups [99]. His method is the following: First, by passing to a suitable arithmetic subgroup $\Gamma$ of finite index in $\pi_1(M)$, he constructed an oriented totally geodesic submanifold $F$ of codimension one in $H^n/\Gamma$ which does not separate $H^n/\Gamma$, that is, such that $(H^n/\Gamma) \setminus F$ is connected. Second, given such a submanifold $F$, it separates $H^n/\Gamma$ locally; thus, if $p \in F$, we may choose points $x, y$ in a small neighborhood $U$ of $p$ so that $x$ is contained in one of the connected components of $U \setminus (U \cap F)$ and $y$ in the other. We may join $x$ to $y$ by an arc $\alpha$ intersecting $F$ in exactly one point. Since $F$ does not separate $H^n/\Gamma$ we may join $y$ to $x$ by an arc $\beta$ neither meeting $F$ nor $\alpha$ outside $x$ and $y$. Then, by combining $\alpha$ and $\beta$, we obtain a simple closed curve in $H^n/\Gamma$ intersecting $F$ in exactly one point. Thus, there is a compact 1-cycle which has a non-vanishing intersection number with the class of $F$. As a consequence, the first Betti number $b_1(N)$ with $N = H^n/\Gamma$ does not vanish.

The general case is due to Lubotzky [92] who strengthens this geometric approach by taking in a result in geometric group theory. This method applies first to the case just considered and second to the non-arithmetic lattices in $SO(n,1)$ constructed in [45].

We sketch the idea of his proof. Consider the case where $F$ separates $M$; i.e., $M \setminus F$ consists of two parts $M_i, i = 1, 2$, with common boundary $F$. Let $A_i$ be the fundamental group of $M_i, i = 1, 2$, and let $C$ be $\pi_1(F)$. Then the fundamental group $\Gamma$ of $M$ is the free product of $A_1$ and $A_2$ with amalgamated subgroup $C$. The latter group $C$ is of infinite index in $A_1$ and $A_2$. Using Borel's density theorem, one constructs a homomorphism $\pi : \Gamma \longrightarrow S$ onto a finite group $S$ such that $\pi(C) \neq \pi(A_i), i = 1, 2$. Thus, $\Gamma$ is mapped under $\pi$ onto the free product of $\pi(A_1)$ and $\pi(A_2)$ with amalgamated subgroup $\pi(C)$. As an amalgam of finite groups it is virtually free. A further group-theoretical analysis of the situation implies the result.

The case where $F$ does not separate $M$ is analogous, with the free product replaced by an $HNN$ construction.

*Arithmetic groups which are not standard unit groups of quadratic forms.* The result of Millson alluded to above settled the conjecture for arithmetically defined groups $\Gamma \subset Iso(H^n)^0$ which are commensurable to standard arithmetic unit groups. In particular, by the classification of rational structures on the real Lie group $SO_0(n,1) = Iso(H^n)^0$, the problem is solved for the class of non-cocompact arithmetically defined subgroups of $SO_0(n,1)$. Thus we may restrict our attention to the case of cocompact arithmetically defined subgroups of $SO_0(n,1)$. If $n \neq 3, 7$, there is exactly one other family of arithmetically defined subgroups, the one given (up to commensurability) by the group of units of suitable skew-Hermitian forms over a quaternion algebra. This family only exists if $n$ is odd. In the case $n > 5$,

non-vanishing results for the first Betti number of such arithmetic groups were obtained by J.-S. Li [86] by an automorphic approach. In [87], Millson and Li gave a new proof for this class of arithmetic groups, including the case $n = 5$ as well. It is based on a study of the period of a suitable theta series (a closed 1-form on the arithmetic quotient $M$ in question) over a carefully chosen closed geodesic in $M$. For the case $n = 3$ we refer to the previous section. If $n = 7$, in the case of arithmetically defined subgroups of the triality form of $SO_0(7,1)$, the conjecture is still open. For the case of arithmetically defined hyperbolic 3-manifolds we refer to the discussion in Section 2.4, where it is indicated that the conjecture is still open in many cases.

3.4. **A geometric construction of (co)homology classes.** In [102], Millson, jointly with Raghunathan, suggested a new geometric construction of non-vanishing (co)homology classes for some specific families of cocompact arithmetically defined subgroups of classical real Lie groups. They deduced the existence of these cohomology classes by constructing non-bounding cycles in the arithmetic quotient under consideration. In the case at hand, namely the class of standard arithmetic unit groups $\Gamma \subset SO(f)$, these cycles are projections under the natural map $H^n \longrightarrow H^n/\Gamma$ of subsymmetric spaces of $H^n$ which are sets of fixed points of suitable rational involutions on $H^n$. These cycles come in pairs so that one obtains two complementary dimensional closed cycles. By passing over to a subgroup of finite index in $\Gamma$ one can arrange that the two cycles so obtained intersect transversally with all multiplicities giving a positive contribution to the intersection number. Thus these geometric cycles cannot be bounded. We will discuss this approach in a more general context in part III. However, here is the result of Millson-Raghunathan in the case of standard arithmetic unit groups acting on hyperbolic $n$-space. [We refer to Section 10 for a more thorough treatment.]

**Theorem C.** *Let $k \neq \mathbb{Q}$ be a totally real algebraic number field, $f$ a non-degenerate quadratic form on $k^{n+1}$ and $G = SO(f)$ the special orthogonal group of $f$. Suppose that $f$ has signature $(n,1)$ over $k$ and all its conjugates $f^{\sigma_i}$, $i = 2, ..., d$ are positive definite. Let $\Gamma$ be an arithmetic subgroup of $SO(f)$. Then there exists a torsion-free arithmetic subgroup $\Gamma'$ of finite index in $\Gamma$ such that all Betti numbers $b_j(H^n/\Gamma'), j = 1, ..., n-1$ of the oriented compact hyperbolic $n$-manifold $H^n/\Gamma'$ are non-zero.*

# Part II
# Arithmetic groups and their cohomology

## 4. Arithmetic groups

Let $G$ be an algebraic subgroup of some $GL_n$ defined over $\mathbb{Q}$. Thus there exists a finite set of polynomial equations over $\mathbb{Q}$ whose set of solutions in $\mathbb{Q}$ (or any extension field $F$ of $\mathbb{Q}$) is a subgroup $G(F)$ of $GL_n(F)$. Roughly speaking, a subgroup $\Gamma$ of the group $G(\mathbb{Q})$ of $\mathbb{Q}$-rational points in $G$ is arithmetically defined if it is commensurable with $G(\mathbb{Q}) \cap GL_n(\mathbb{Z}) =: G_{\mathbb{Z}^n}$, that is, if $\Gamma \cap G_{\mathbb{Z}^n}$ has finite index in both $\Gamma$ and $G_{\mathbb{Z}^n}$. In this section we review the definition of an arithmetically defined group in the context of algebraic groups defined over an algebraic number field and introduce the concept of congruence subgroups. The reader may wish to consult Appendix B on groups of units of quadratic forms, for a specific example.

The result of Minkowski, proved in 1887, that the principal congruence subgroups $\Gamma(q)$ in $GL_n(\mathbb{Z})$, $n \geq 2$, with $q \geq 3$ are torsion-free is fundamental for the theory. Finally, we discuss the notion of neatness for arithmetic groups. The defining condition is stronger than torsion free and preserved under morphisms of algebraic groups, a property torsion-free arithmetic groups lack. The principal congruence subgroups $\Gamma(q)$ with $q \geq 3$ are neat.

4.1. **Arithmetically defined subgroups of $G$.** Let $V$ be a finite-dimensional vector space over $\mathbb{C}$ endowed with a $\mathbb{Q}$-structure; that is, there is a $\mathbb{Q}$-subspace $V_{\mathbb{Q}}$ of $V$ with $\dim_{\mathbb{Q}} V_{\mathbb{Q}} = \dim_{\mathbb{C}} V$ such that $V_{\mathbb{Q}}$ generates $V$ as a vector space over $\mathbb{C}$. Let $G$ be an algebraic subgroup of $GL(V)$ defined over $\mathbb{Q}$. If $\Lambda$ is a lattice of $V_{\mathbb{Q}}$, that is, a free $\mathbb{Z}$-submodule of $V_{\mathbb{Q}}$ of maximal rank, we define *the group of $\Lambda$-units of $G$* to be

$$G_\Lambda = \{g \in G(\mathbb{Q}) \mid g(\Lambda) = \Lambda\}.$$

By definition, a subgroup $\Gamma$ of the group $G(\mathbb{Q})$ of $\mathbb{Q}$-rational points in $G$ is an arithmetic (or arithmetically defined) subgroup of $G$ if there exists a lattice $\Lambda$ of $V_{\mathbb{Q}}$ such that $\Gamma$ is commensurable with $G_\Lambda$, that is, if the intersection $\Gamma \cap G_\Lambda$ has finite index both in $\Gamma$ and $G_\Lambda$. The notion of an arithmetic subgroup of $G$ only depends on the structure of $G$ as an algebraic $\mathbb{Q}$-group; that is, it is independent of the choice of a faithful representation $\rho : G \longrightarrow GL(V)$ of $G$ and a lattice. Moreover, given such a representation $\rho : G \longrightarrow GL(V)$ and an arithmetic subgroup of $G$, every lattice $\Lambda$ of $V_{\mathbb{Q}}$ is contained in a lattice that is stable under $\Gamma$ [15, Chap. 7].

One can replace $\mathbb{Q}$ in the definition by an arbitrary algebraic number field $k$ of degree $d = [k : \mathbb{Q}]$, and $\mathbb{Z}$ by the ring $\mathcal{O}_k$ of integers in $k$. Given an algebraic subgroup $G$ of $GL_n(\mathbb{C})$ defined over $k$, we define a subgroup $\Gamma$ to be arithmetic if it is commensurable with $G_{\mathcal{O}_k}$. Here we use the following notation: Given an algebraic group $H$, defined over some algebraic number field $l$, and given a subring $A$ of $l$, we denote by $H_A$ the subgroup of elements of $H$ whose coefficients are in $A$ and whose determinant is a unit in $A$.

However, one does not obtain any new groups. Indeed, the latter case can be reduced to the case where $k = \mathbb{Q}$ by use of the so-called (Weil) restriction of scalars; see Appendix C.

4.2. **Congruence subgroups.** Let $G$ be a connected algebraic $k$-group, and let $G_{\mathcal{O}_k}$ be the group of integral points of $G$ relative to some given embedding $\rho : G \to GL_n$. A subgroup $\Gamma$ of $G_{\mathcal{O}_k}$ is called a congruence subgroup if there exists a non-trivial ideal $\mathfrak{q}$ in $\mathcal{O}_k$ such that $\Gamma$ contains the subgroup

$$\Gamma(\mathfrak{q}) := \{\gamma \in G_{\mathcal{O}_k} \mid \gamma \equiv 1 \bmod \mathfrak{q}\}.$$

The latter group is usually known as the principal congruence subgroup of level $\mathfrak{q}$. A congruence subgroup is of finite index in $G_{\mathcal{O}_k}$.

As early as 1887, Minkowski [104], [105] proved that the principal congruence subgroups $\Gamma(q)$ in $GL_n(\mathbb{Z})$, $n \geq 2$, with $q \geq 3$ are torsion free. Indeed, suppose that $g \in \Gamma(q), g \neq \mathrm{Id}$, is an element of order $m$. Then $g$ has the form $I + qH'$ with $H' \in M_n(\mathbb{Z})$. We may suppose that $H' \neq 0$. If $d$ denotes the greatest common divisor of the entries of $H'$, we may write $I + q\ d\ H$ where the entries of $H$ have $\gcd = 1$. Since $I$ and $H$ commute, by use of the binomial theorem, we obtain

$$g^m = I + mqdH + \tbinom{m}{2} q^2 d^2 H^2 + \cdots + \tbinom{m}{m} q^m d^m H^m. \tag{4.1}$$

This implies the congruence $mqdh_{ij} \equiv 0 \mod q^2d^2$ and hence $qd$ divides $mh_{ij}$ for all $i$ and $j$. Since the matrix entries $h_{ij}$ have greatest common divisor equal to 1, $qd$ divides $m$. In the case that $m$ is a prime $p$, it follows that $q = m = p$ and $d = 1$. If $p > 2$, we obtain that $q$ divides the binomial coefficient $\binom{m}{2}$ and, by the equation for $g^m$, $q^2H \equiv 0 \mod q^3$. Then $q$ divides all entries of $H$, and so we have a contradiction. As a consequence, if $q \geq 3$, there is no element in $\Gamma(q)$ whose order is a prime $p \neq 1$. If $m > 1$ there is a prime factor $p$ of $m$ so that $g^{\frac{m}{p}}$ is an element of order $p$ in $\Gamma(q)$. Again we have a contradiction.

As a consequence of this result we obtain that every arithmetic subgroup $\Gamma$ of $G$ contains a torsion-free subgroup of finite index.

4.3. **Neatness.** Let $F$ be a field of characteristic zero, and let $\overline{F}$ be an algebraic closure of $F$. Given an element $g \in GL_n(F)$, we denote by $E(g)$ the multiplicative group in $\overline{F}^*$ generated by the eigenvalues of $g$. We say that $g \in GL_n(F)$ is neat if the group $E(g)$ is torsion free. A subgroup $\Gamma$ of $GL_n(F)$ is neat by definition if each element in $\Gamma$ is neat. Notice that a neat subgroup is also torsion free but, in general, a torsion-free subgroup is not necessarily neat. The crucial aspect of this notion is its hereditary nature. More precisely, if $f : G \subset GL_n \to GL_m$ is a morphism of algebraic groups, $g \in G$ neat, then also $f(g)$ is neat. Torsion-free arithmetic groups lack this property. There is the following result.

**Proposition.** *Let $G$ be a connected algebraic group defined over $\mathbb{Q}$ and let $\Gamma$ be an arithmetic subgroup of $G$. Then $\Gamma$ contains a subgroup of finite index which is neat.*

*The principal congruence subgroups $\Gamma(q)$ in $GL_n(\mathbb{Z})$, $n \geq 2$, are neat if $q \geq 3$.*

Relative to some given embedding $\rho : G \to GL_n$ the group $\Gamma$ is commensurable to $G_{\mathbb{Z}} \subset GL_n(\mathbb{Z})$. Thus, the first assertion follows from the fact that the principal congruence subgroups $\Gamma(q)$ in $GL_n(\mathbb{Z})$ are neat if $q \geq 3$.

For the sake of completeness we indicate the argument. As above, we write $g \in \Gamma(q)$ in the form $g = I + qH$, $H \in M_n(\mathbb{Z})$, $H \neq 0$. The eigenvalues of $g$ may be written as $1 + q\eta_i, i = 1, ..., n$, where $\eta_i$ denotes the eigenvalues of $H$. In fact, as roots of a monic polynomial with integral coefficients, the $\eta_i$ are algebraic integers. Now we suppose that there are integers $e_i, i = 1, ..., n$, not all zero, so that $\prod_i (1 + q\eta_i)^{e_i} = \zeta$ with $\zeta \neq 1$ a root of unity. By renumbering the eigenvalues, we may suppose that there exists an index $i_0$, $1 \leq i_0 \leq n$, so that $e_i \geq 0$ if $i \leq i_0$ and $e_i \leq 0$ if $i > i_0$. Then we have the equation

$$\prod_{i \leq i_0} (1 + q\eta_i)^{e_i} \quad = \quad \zeta \prod_{i > i_0} (1 + q\eta_i)^{-e_i}, \tag{4.2}$$

where all exponents are non-negative integers. This give rise to an identity of the form

$$1 + q\alpha \quad = \quad \zeta(1 + q\beta) \tag{4.3}$$

with algebraic integers $\alpha, \beta$. Thus, $1 - \zeta = q(\beta\zeta - \alpha)$; that is, $1 - \zeta$ is a non-trivial multiple of an algebraic integer. If $\zeta = -1$, then $2 = q(-\beta - \alpha)$. For $q > 2$, this is impossible. If $\zeta \neq -1$, this can only be the case if $\zeta = 1$. Otherwise it contradicts the fact that the ring of algebraic integers of the cyclotomic field $\mathbb{Q}(\zeta)$ is the ring $\mathbb{Z}[\zeta]$.

## 5. Arithmetic quotients

Let $\Gamma$ be an arithmetically defined subgroup of a connected reductive algebraic $\mathbb{Q}$-group $G$. Then it is a discrete subgroup of the real Lie group $G(\mathbb{R})$. Let $K$ be a maximal compact subgroup of $G(\mathbb{R})$. In this section we define the "generalized symmetric space" $X = K\backslash G(\mathbb{R})$ on which $G(\mathbb{R})$, and thus $\Gamma$, acts properly.

We then describe the arithmetically defined quotient space $X/\Gamma$ and state the result of Borel and Harish Chandra, to the effect that $X/\Gamma$ has finite volume if and only if $G$ has no non-trivial rational character, and it is compact if and only if, in addition, every rational unipotent element belongs to the radical of $G$. If $X/\Gamma$ is non-compact, it may be interpreted as the interior of a compact manifold with corners. We briefly review its construction due to Borel and Serre and its role within cohomological investigations. We will take up this topic again in Section 12 when we discuss modular symbols. We conclude this section by interpreting the deRham cohomology of $X/\Gamma$ in terms of relative Lie algebra cohomology groups.

5.1. **Generalities.** Let $G$ be an algebraic group defined over an algebraic number field $k$. We choose an embedding $\rho : G \to GL_N$ and write as before $G_{\mathcal{O}_k} = G(k) \cap GL_N(\mathcal{O}_k)$ for the group of integral points with respect to $\rho$. For simplicity, we suppose that $G$ is reductive and connected.

For every Archimedean place $v \in V_\infty$ corresponding to the embedding $\sigma_v : k \to \overline{k}$ there are given a local field $k_v = \mathbb{R}$ or $\mathbb{C}$ and a real Lie group $G_v = G^{\sigma_v}(k_v)$. The group

$$G_\infty = \prod_{v \in V_\infty} G_v,$$

viewed as the topological product of the groups $G_v$, $v \in V_\infty$, is isomorphic to the group of real points $G'(\mathbb{R})$ of the algebraic $\mathbb{Q}$-group $G' = \mathrm{Res}_{k/\mathbb{Q}} G$ obtained from $G$ by restriction of scalars; see Appendix C. In $G_\infty$, we identify $G(k)$, resp. $G_{\mathcal{O}_k}$, with the set of elements $(g^{\sigma_v})_{v \in V_\infty}$ with $g \in G(k)$, resp. $g \in G_{\mathcal{O}_k}$. If $\Gamma$ is an arithmetic subgroup of $G$, then $\Gamma$ is a discrete subgroup in $G_\infty$.

Each of the groups $G_v$ has finitely many connected components. The factor $G_v$ has maximal compact subgroups, and any two of these are conjugate by an inner automorphism. Thus, if $K_v$ is one of them, the homogeneous space $K_v\backslash G_v = X_v$ may be viewed as the space of maximal compact subgroups of $G_v$. Since $X_v$ is diffeomorphic to $\mathbb{R}^{\mathrm{d}(G_v)}$, where $\mathrm{d}(G_v) = \dim G_v - \dim K_v$, the space $X_v$ is contractible. Notice that, if $G$ is semi-simple, the space $X_v$ is the symmetric space associated to $G_v$. We let

$$X = \prod_{v \in V_\infty} X_v$$

(or we write $X_G$ emphasizing the underlying reductive $k$-group $G$), resp.

$$\mathrm{d}(G) = \sum_{v \in V_\infty} \mathrm{d}(G_v).$$

An arithmetic subgroup $\Gamma$ of $G$ acts properly discontinuously on $X$; that is, given a compact subset $C$ in $X$, there are only finitely many translates $C\gamma, \gamma \in \Gamma$, that meet $C$. In particular, the stabilizers of points are finite subgroups. If $\Gamma$ is a torsion-free arithmetic subgroup, the action of $\Gamma$ on X is free, and the quotient $X/\Gamma$ is a smooth manifold of dimension $\mathrm{d}(G)$. Since $X$ is a contractible space, $X/\Gamma$ is an Eilenberg-MacLane space $K(\Gamma, 1)$. Its cohomology (or homology) is isomorphic

to the Eilenberg-MacLane cohomology of $\Gamma$. More precisely, if $E$ is a $\Gamma$-module, we denote the corresponding local system on $X/\Gamma$ by $\tilde{E}$. Then there are canonical isomorphisms

$$H_q(\Gamma, E) = H_q(X/\Gamma, \tilde{E})$$

and

$$H^q(\Gamma, E) = H^q(X/\Gamma, \tilde{E})$$

for any degree $q$. As a consequence, the cohomological dimension $\mathrm{cd}(\Gamma)$ is at most $\mathrm{d}(G)$. If $X/\Gamma$ is compact, we have $\mathrm{cd}(\Gamma) = \mathrm{d}(G)$; otherwise, $\mathrm{cd}(\Gamma) < \mathrm{d}(G)$.

5.2. **A compactness criterion.** Due to a result of Borel and Harish Chandra [19, Th. 12.3], one has criteria for the quotient $X/\Gamma$ to have finite volume, or to be compact. These only depend on the $k$-structure of the algebraic $k$-group $G$. More precisely, the quotient $G_\infty/G_{\mathcal{O}_k}$ has finite Haar measure (and thus $X/\Gamma$ for any arithmetic subgroup $\Gamma$ of $G$ ) if and only if $G$ has no non-trivial rational character defined over $k$. It is compact if and only if $X_k(G) = \{1\}$, and every unipotent element in $G(k)$ belongs to the radical of $G$.

This result supersedes previously known cases as, for example, the case of unit groups of non-degenerate quadratic spaces over $k$ [140]. This criterion had been conjectured by R. Godement. An important consequence of this criterion is the following assertion: Suppose that there is at least one place $v \in V_\infty$ such that $G_v$ is compact. Then the quotient $G_\infty/G_{\mathcal{O}_k}$ is compact if and only if $X_k(G) = \{1\}$.

The general criterion remains true without the assumption that $G$ is connected if one replaces the condition $X_k(G) = \{1\}$ by the analogous one for the connected component $G^0$ of the identity in $G$.

5.3. **Adjunction of corners.** The arithmetic quotient $X/\Gamma$ is non-compact if and only if $\mathrm{rk}_k G > 0$. In this case it has a natural compactification $\overline{X/\Gamma}$, the so-called "adjunction of corners", in its generality due to Borel and Serre [22]. Assume $\Gamma$ is torsion free. Then $\overline{X/\Gamma}$ is a compact manifold with finitely many corners, having one "corner" for each $\Gamma$-conjugacy class of proper parabolic $k$-subgroups of $G$.[7] This compactification is obtained as the quotient under $\Gamma$ of a $G(k)$-equivariant partial compactification $\overline{X}$, that is, $\overline{X/\Gamma} = \overline{X}/\Gamma$. Since $\overline{X}$ is contractible, the cohomology of $\overline{X}/\Gamma$ may be identified with that of $\Gamma$, that is,

$$H^q(\Gamma, E) = H^q(X/\Gamma, \tilde{E}) = H^q(\overline{X}/\Gamma, \tilde{E}),$$

where $H^q(\Gamma, E)$ denotes the (Eilenberg-MacLane) algebraic cohomology of $\Gamma$ with coefficients in the module $E$. It follows, for example, that this cohomology is finitely generated; see also [22, Section 11] for further cohomological consequences. Of particular interest in our discussion is the fact that the virtual cohomological dimension of an arithmetic subgroup $\Gamma$ in an algebraic $k$-group $G$ is actually

$$\mathrm{vcd}(\Gamma) = \mathrm{d}(G) - \mathrm{rk}_k(G^0/RG),$$

where $\mathrm{rk}_k(G^0/RG)$ denotes the $k$-rank of the semi-simple group $G^0/RG$.

Since the inclusion $X/\Gamma \longrightarrow \overline{X}/\Gamma$ is a homotopy equivalence, the long exact cohomology sequence attached to the pair $(\overline{X}/\Gamma, \partial(\overline{X}/\Gamma))$

$$\to H^*(\overline{X}/\Gamma, \partial(\overline{X}/\Gamma), E) \longrightarrow H^*(\overline{X}/\Gamma, E) = H^*(X/\Gamma, E) \longrightarrow H^*(\partial(\overline{X}/\Gamma), E) \to$$

[7] Hence topologically, it is a compact manifold with boundary, homotopically equivalent to its interior $X/\Gamma$. If $\Gamma$ is not torsion free, then $\overline{X/\Gamma}$ is a compact orbifold with orbifold corners.

is a helpful tool in the analysis of the cohomology of $X/\Gamma$. In particular, this sequence splits the investigation into two parts: the interior cohomology, defined as the kernel of the restriction map $r^* : H^*(\overline{X}/\Gamma, E) \longrightarrow H^*(\partial(\overline{X}/\Gamma), E)$ and to be denoted by $H^*_!(X/\Gamma, E)$, and the so-called "cohomology at infinity", a possible complement to the interior cohomology . The cohomology at infinity is charged to encode all those phenomena in the cohomology that are due to the non-compactness of the quotient $X/\Gamma$.

5.4. **DeRham cohomology.** Let $(\nu, E)$ be a finite-dimensional irreducible representation of the real Lie group $G_\infty$ on a real or complex vector space $E$. Let $\Omega^q(X, E)$ denote the space of smooth $E$-valued differential forms on the space $X$ of degree $q \geq 0$. Endowed with the exterior differentiation $d : \Omega^q(X, E) \longrightarrow \Omega^{q+1}(X, E)$, defined by

$$\begin{aligned}(d\omega)(Y_0, ..., Y_q) : \quad &= \sum_{0\leq i\leq q} (-1)^i Y_i \cdot \omega(Y_0, ..., \hat{Y}_i, ..., Y_q) \\ &+ \sum_{i<j} (-1)^{i+j} \omega([Y_i, Y_j], Y_0, ..., \hat{Y}_i, ..., \hat{Y}_j, ..., Y_q),\end{aligned}$$

where $[\quad , \quad ]$ refers to the Lie bracket of vector fields, and $\hat{Y}$ means omitting the argument $Y$ beneath. The direct sum $\Omega^*(X, E)$ of the $\Omega^q(X, E)$, $q \geq 0$, forms a complex. The group $G_\infty$ operates on $X$ and on the space $\Omega^*(X, E)$ of smooth $E$-valued forms on $X$. Given a torsion-free discrete subgroup $\Gamma$ of $G_\infty$, for example, an arithmetic subgroup of $G$, the cohomology $H^*(X/\Gamma, \tilde{E})$ of the manifold $X/\Gamma$ with coefficients in the local system defined by $(\nu, E)$ is canonically isomorphic to the cohomology $H^*(\Omega(X, E)^\Gamma)$, the deRham cohomology as it is called. Using a suitable sheaf $\tilde{E}$ this equality is still true even if $\Gamma$ has torsion elements, that is, (cf. [23, VII, 2.2])

$$H^*(X/\Gamma, \tilde{E}) = H^*(\Omega(X, E)^\Gamma).$$

5.5. **An interpretation in Lie algebra cohomology.** The deRham cohomology groups $H^*(\Omega(X, E)^\Gamma)$ are related in a natural way to relative Lie algebra cohomology groups. Foundational material on this latter cohomology theory is summarized in Appendix E to this article. It is this transition by which some questions on the cohomology of arithmetic groups are turned into questions about cohomological properties of unitary representations of the underlying Lie group $G_\infty$ and questions of the spectrum of $\Gamma$. However, since this reinterpretation in terms of relative Lie algebra cohomology only relies on Lie-theoretic data, we work in this context.

Let $G$ be a real Lie group with finitely many connected components, $\mathfrak{g}$ its Lie algebra, and let $K$ be a compact subgroup of $G$, $\mathfrak{k}$ its Lie algebra. We denote the natural projection map $G \to K\backslash G$ by $\pi$. Then the triple $(G, \pi, K\backslash G)$ is a principal bundle with structure group $K$. As above, let $(\nu, E)$ be a finite-dimensional irreducible representation of the real Lie group $G$ on a real or complex vector space $E$. We want to study the complex $\Omega^*(K\backslash G, E)$ of smooth $E$-valued differential forms in terms of representation theory. Since $G$ is a Lie group there is a natural identification of the tangent space at the point $e \in G$ with $\mathfrak{g}$. This leads to a trivialization of $q$-forms. More precisely, there is an identification of complexes

$$\Omega^*(G, E) \tilde{\longrightarrow} \mathrm{Hom}(\Lambda^*\mathfrak{g}, C^\infty(G) \otimes E). \tag{5.1}$$

The pullback map $\pi^* : \Omega^*(K\backslash G, E) \longrightarrow \Omega^*(G, E)$ of the $C^\infty$-map $\pi$ is compatible with differentials, thus an inclusion of complexes. It is not difficult to identify the image of $\Omega^*(K\backslash G, E)$ under $\pi^*$. We endow $C^\infty(G)\otimes E$ with the $G$-module structure given as the tensor product of the left regular representation $l$ of $G$ on $C^\infty(G)$ and of $(\nu, E)$. Then a $q$-form $\omega$ in $\mathrm{Hom}(\Lambda^*\mathfrak{g}, C^\infty(G)\otimes E)$ is in the image of $\pi^*$ if $\omega$ is annihilated by the interior products $i_Y, Y\in\mathfrak{k}$, and $\omega$ lies in $\mathrm{Hom}_K(\Lambda^*\mathfrak{g}, C^\infty(G)\otimes E)$, where $K$ acts on $\Lambda^*\mathfrak{g}$ by the adjoint action.

Then the space $C^\infty(G)_K$ of all $C^\infty$-vectors $f$ for which $l(K)f$ spans a finite-dimensional subspace of $C^\infty(G)$ is preserved by the action of $\mathfrak{g}$ (obtained by differentiation of $l$) and compatible with the action of $K$. Moreover $C^\infty(G)_K$ is locally finite as a $K$-module. Thus, $C^\infty(G)_K$ is a $(\mathfrak{g}, K)$-module. This concept (see Appendix F) provides the formal framework for a representational-theoretic approach to the cohomology. Indeed, there is an isomorphism of graded complexes of $\Omega^*(K\backslash G, E)$ onto $C^*(\mathfrak{g}, K, C^\infty(G)_K \otimes E)$.

Given any discrete torsion-free subgroup $\Gamma$ of $G$ the space of functions invariant by $\Gamma$ acting on the right is a $(\mathfrak{g}, K)$-submodule of $C^\infty(G)_K$. One obtains an isomorphism of $\Omega^*(K\backslash G, E)^\Gamma$ onto $C^*(\mathfrak{g}, K, C^\infty(G/\Gamma)_K \otimes E)$. Thus, there is a canonical isomorphism

$$H^*(K\backslash G/\Gamma, \tilde{E}) = H^*(\Omega(K\backslash G, E)^\Gamma) \xrightarrow{\sim} H^*(\mathfrak{g}, K, C^\infty(G/\Gamma)_K \otimes E). \tag{5.2}$$

We refer, for example, to [23, Chapter VII] or [153] for a more thorough treatment. Moreover, we draw attention to Sections 13.1, 13.2 for further development of this material.

# Part III
# Geometric cycles

## 6. Geometric constructions: generalities

In this section $G$ denotes a connected semi-simple algebraic group defined over an algebraic number field $k$, $\Gamma \subset G(k)$ an arithmetic subgroup. In general, we retain the notation of Sections 4 and 5.

A given reductive $k$-subgroup $H$ of $G$ gives rise to a natural map

$$j_{H|\Gamma} : X_H/\Gamma_H \longrightarrow X/\Gamma,$$

where $\Gamma_H = \Gamma \cap H(k)$. In this section we outline a proof of the following.

**Theorem D.** *Let $G$ be a connected semi-simple algebraic $k$-group, let $H \subset G$ be a connected reductive $k$-subgroup, and let $\Gamma$ be an arithmetic subgroup of $G(k)$. Then there exists a subgroup $\Gamma'$ of finite index in $\Gamma$ such that if $\Gamma$ is replaced by $\Gamma'$, the map*

$$j_{H|\Gamma'} : X_H/\Gamma'_H \longrightarrow X/\Gamma'$$

*is a proper, injective, closed embedding, and so that each connected component of the image is an orientable, totally geodesic submanifold of $X/\Gamma'$.*

This result is proven in various steps throughout this section. Groups $H$ which originate with the set of fixed points of a rational automorphism of finite order on $G$ are examples of special interest. In this case the injectivity is not an issue. This is discussed in Section 6.4. Non-abelian Galois cohomology turns out to be a decisive tool in analyzing these fixed point sets.

6.1. **Geometric cycles.** Let $H$ be a reductive $k$-subgroup of $G$, let $K_H$ be a maximal compact subgroup of the real Lie group $H_\infty$, and let $X_H = K_H \backslash H_\infty$ be the space associated to $H_\infty$, as in Section 5.1. If $x_0 \in X$ is fixed under the natural action of $K_H \subset G_\infty$ on $X$, then the assignment $h \mapsto x_0\, h$ defines a closed embedding

$$X_H = K_H \backslash H_\infty \longrightarrow X;$$

that is, the orbit map identifies $X_H$ with a totally geodesic submanifold of $X$. Thus, we also have a natural map

$$j_{H|\Gamma} : X_H/\Gamma_H \longrightarrow X/\Gamma,$$

where $\Gamma_H = \Gamma \cap H(k)$. We consider the composite of the inclusion $i : H_\infty/\Gamma_H \to G_\infty/\Gamma$ and the projection $\pi : G_\infty/\Gamma \to X/\Gamma$. The latter map is proper since it is the projection of a locally trivial fibration with compact fibers. The same is true for the surjective map $\pi_H : H_\infty/\Gamma_H \to X_H/\Gamma_H$. The composite $j_{H|\Gamma} \circ \pi_H$ coincides with the map $\pi \circ i$. We have

**Lemma.** *The map $j_{H|\Gamma}$ is proper.*

*Proof.* We may assume that $k = \mathbb{Q}$; otherwise, we replace $G$ by the algebraic $\mathbb{Q}$-group $\mathrm{Res}_{k/\mathbb{Q}} G$ obtained from $G$ by restriction of scalars (see Appendix C). In view of the identity $j_H \circ \pi_H = \pi \circ i$ it suffices to show that $i$ is proper because this would imply that also $j_H \circ \pi_H$ has this property, hence also $j_H$. Since the map $i$ is injective, it is proper if and only if it has closed image. In turn, this is equivalent to $H_\infty, \Gamma$ being closed in $G_\infty$, hence also to $\Gamma$ having closed image in $H_\infty \backslash G_\infty$. However, by [15, 7.7], there exist a finite-dimensional $\mathbb{Q}$-vector space $W$, a representation $G \longrightarrow GL(W)$ defined over $\mathbb{Q}$ and a point $w \in W_\mathbb{Q}$ so that the orbit of $w$ is closed and its isotropy group is $H$. By Section 4.1. we may suppose that $w$ belongs to a $\Gamma$-stable lattice in $W_\mathbb{Q}$. The $\Gamma$-orbit of $w$ is therefore discrete in $W_\mathbb{R}$. Since the $\Gamma$-orbit of $w$ identifies with the image $p(\Gamma)$ of $\Gamma$ under the projection $p : G_\infty \to H_\infty \backslash G_\infty$, $p(\Gamma)$ is closed in $H_\infty \backslash G_\infty$. □

Now we are interested in situations in which for a given subgroup $H$ and a torsion-free arithmetic subgroup $\Gamma$ of $G$, the corresponding map $j_{H|\Gamma}$ is an injective immersion. Thus, by being proper, $j_{H|\Gamma}$ is an embedding, and the image $j_H(X_H/\Gamma_H)$ of $X_H/\Gamma_H$ is a submanifold in $X/\Gamma$. This submanifold is totally geodesic, to be called a **geometric cycle** in $X/\Gamma$.

6.2. **Actions of Galois groups.** In his study of the cohomology of arithmetic subgroups of the special linear group $SL_2$ over an algebraic number field, Harder [52] laid out the nucleus for an important class of examples by taking into account the action of the Galois group $\mathrm{Gal}(k/\mathbb{Q})$ on the corresponding locally symmetric space $X/\Gamma$ and its cohomology. We find this approach brought to fruition in the work of Rohlfs [118] on arithmetic groups with Galois actions. More precisely, let $G$ be a connected semi-simple algebraic group defined over an algebraic number field $l$ and let $k/l$ be a Galois extension with Galois group $\mathcal{G} = \mathrm{Gal}(k/l)$. Let $v \in V_\infty$ be an Archimedean place of $k$ with corresponding embedding $\sigma_v : k \to \overline{k}$. A given

element $\tau \in \mathcal{G}$ defines, via the assignment $\sigma_v \mapsto \sigma_v \circ \tau = \sigma_{v'}$, an isomorphism $k_v \to k_{v'}$. Thus, there are natural actions by $\mathcal{G}$ on

$$G_\infty = \prod_{v \in V_\infty} G_v,$$

and on $X = \prod_{v \in V_\infty} X_v$, respectively.

Suppose that $\Gamma \subset G$ is a $\mathcal{G}$-stable torsion-free arithmetic subgroup. Then $\mathcal{G}$ acts on the quotient $X/\Gamma$. As proved in [118, 1.4], the connected components of the fixed point set $\mathrm{Fix}(\mathcal{G}, X/\Gamma)$ of this action can be parametrized by the non-abelian Galois cohomology set $H^1(\mathcal{G}, \Gamma)$. Each of these components $F(\gamma), [\gamma] \in H^1(\mathcal{G}, \Gamma)$, is a locally symmetric space of the form $X(\gamma)/\Gamma(\gamma)$ whose structure is completely determined by the group $G(\gamma) \subset G$ of points that are fixed under the action of $\mathcal{G}$ twisted by the cocycle representing the class $[\gamma]$. Indeed, it is a totally geodesic submanifold in $X/\Gamma$. The fundamental group of such a component is isomorphic to the group $\Gamma(\gamma)$ of elements in $\Gamma$ fixed by the $\gamma$-twisted action of $\mathcal{G}$ on $\Gamma$. It is inherent in this description that the map

$$j_{G(\gamma)} : X_{G(\gamma)}/\Gamma(\gamma) = X(\gamma)/\Gamma(\gamma) \longrightarrow X/\Gamma$$

is injective in that case.

Next we explain and prove these results in a more general context.

This construction of totally geodesic submanifolds can be partly subsumed under the general approach in which one studies fixed points of (finite abelian groups $\Theta$ of) automorphisms of finite order on $G/l$ and of the induced morphisms on $X/\Gamma$. The technical tool of non-abelian Galois cohomology sets attached to $\Theta$ is of considerable help in analyzing the corresponding fixed point sets. In particular, it provides an algebraic frame for the passage from one group $\Gamma$ to a subgroup $\Gamma'$ of finite index in $\Gamma$.

For the convenience of the reader we recall some basis facts regarding non-abelian Galois cohomology to make our exposition coherent. We refer to [136, Chap. I, 5] or [71, Chap. VII] for a more thorough treatment.

6.3. **Non-abelian Galois cohomology.** Let $\Theta$ be a group acting on a set $A$. The action of $s \in \Theta$ is written as $a \mapsto sa = {}^s a$, and we define $A^\Theta = \mathrm{Fix}(\Theta, A) = \{a \in A \mid {}^s a = a \text{ for all } s \in \Theta\}$. If $\Theta = \langle \tau \rangle$ is generated by one element $\tau$, we write $A^\Theta = A^\tau$.

If the group $\Theta$ acts on a group $A$ as a group of automorphisms, then $H^1(\Theta, A)$ denotes the first non-abelian cohomology set of $\Theta$ in $A$. It is defined as follows (cf. [136, Chap. I,5]): A cocycle of $\Theta$ in $A$ is a map $\gamma : \Theta \to A$, $s \mapsto \gamma_s$, so that $\gamma_{st} = \gamma_s \, {}^s\gamma_t$ for all $s, t \in \Theta$. One writes $\gamma = (\gamma_s)$. The set of all cocycles of $\Theta$ in $A$ will be denoted by $Z^1(\Theta, A)$. There is the trivial cocycle, defined by the constant map $\Theta \to A$, $s \mapsto 1_A$. Two cocycles $\gamma$ and $\zeta$ are said to be equivalent if there is an element $a \in A$ so that $\zeta_s = a^{-1}\gamma_s \, {}^s a$ for all $s \in \Theta$. This notion defines an equivalence relation on $Z^1(\Theta, A)$. Then $H^1(\Theta, A)$ is the set of equivalence classes. It is a pointed set, and its distinguished element is the class of the trivial cocycle to be denoted by $1_\Theta$.

Suppose the groups $\Theta$ and $A$ act on a set $M$ in a compatible way; that is, we have ${}^s(am) = {}^s a {}^s m$ for all $s \in \Theta$, $a \in A$, $m \in M$. Then, given a cocycle $\gamma = (\gamma_s)$, $s \in \Theta$, for $H^1(\Theta, A)$ there is a $\gamma$-twisted $\Theta$-action on $M$ given by $m \mapsto \gamma_s \, {}^s m$, $s \in \Theta$. We denote the fixed points of this action by $M(\gamma)$, $\gamma$ a given cocycle for $H^1(\Theta, A)$.

6.4. **Fixed points of $\mathbb{Q}$-rational automorphisms of finite order.** Let $\Theta$ be a finite abelian group of $\mathbb{Q}$-rational automorphisms of $G/\mathbb{Q}$. Choose a maximal compact subgroup $K$ of $G(\mathbb{R})$, stable under the group $\Theta$ [60, 13.5]. Then the group $\Theta$ acts on the symmetric space $X = K\backslash G(\mathbb{R})$. Given a $\Theta$-stable torsion-free arithmetic subgroup $\Gamma$ of $G(\mathbb{Q})$ there is a natural action of $\Theta$ on the locally symmetric space $X/\Gamma$. If $\gamma = (\gamma_s)$, $s \in \Theta$, is a cocycle of $\Theta$ in $\Gamma$, then there are $\gamma$-twisted actions of $\Theta$ on $G$ and on $\Gamma$, defined by $g \mapsto \gamma_s \, {}^s g \gamma_s^{-1}$, $g \in G$, $s \in \Theta$. The induced action of $\Theta$ on $X$ is given by $x \mapsto {}^s x \gamma_s^{-1}$, $x \in X$, $s \in \Theta$. The new action of $\Theta$ induced on $X/\Gamma$ coincides with the previous one. Let $\Gamma(\gamma)$ be the set of elements in $\Gamma$ fixed by the $\gamma$-twisted action of $\Theta$ and let $X(\gamma)$ be the set of fixed points of the $\gamma$-twisted $\Theta$-action on $X$. Then the natural map $\pi_\gamma : X(\gamma)/\Gamma(\gamma) \to X/\Gamma$ is injective. This can be seen as follows.

Let $x, y \in X(\gamma)$ and suppose there exists $\delta \in \Gamma$ so that $x = y \cdot \delta$. Since $x$ and $y$ are fixed under the $\gamma$-twisted $\Theta$-action we have

$$ {}^s x \cdot \gamma_s^{-1} = x \quad \text{resp.} \quad {}^s y \cdot \gamma_s^{-1} = y \quad \text{for all} \quad s \in \Theta. $$

This implies, by applying $s \in \Theta$ to the equation $x = y \cdot \delta$,

$$ {}^s x = {}^s(y \cdot \delta) = {}^s y \, {}^s \delta = y \cdot \gamma_s \, {}^s \delta $$

and hence

$$ x \gamma_s = y \cdot \gamma_s \, {}^s \delta; $$

that is,

$$ x = y \cdot \gamma_s \, {}^s \delta \gamma_s^{-1}. $$

But the group $\Gamma$ acts freely on $X$ so that $\delta = \gamma_s \, {}^s \delta \gamma_s^{-1}$ for all $s \in \Theta$. Thus, $\delta \in \Gamma(\gamma)$. The image of $\pi_\gamma$,

$$ F(\gamma) := \mathrm{Im}\pi_\gamma \cong X(\gamma)/\Gamma(\gamma), $$

lies in the fixed point set $\mathrm{Fix}(\Theta, X/\Gamma)$. Notice that $F(\gamma)$ depends only on the class in $H^1(\Theta, \Gamma)$ represented by the cocycle $\gamma$. The spaces $F(\gamma)$ are non-empty since the action of $\Theta$ on $X$ is via isometries ([60, Thm. 13.5]). Any two points of $X(\gamma)$ are joined by a unique geodesic of $X$ ([60, Lemma 14.3]). Thus, $F(\gamma)$ is a connected totally geodesic closed submanifold of $X/\Gamma$. Its fundamental group is isomorphic to $\Gamma(\gamma)$.

All fixed points of $\Theta$ acting on $X/\Gamma$ arise by this construction. Consider a point $\overline{x} \in \mathrm{Fix}(\Theta, X/\Gamma)$ represented by $x \in X$. Then there exist uniquely determined elements $\gamma_s \in \Gamma$ such that ${}^s x = x\gamma_s$, $s \in \Theta$. Then we have

$$ {}^s({}^t x) = x \gamma_{st}, $$

respectively,

$$ {}^s({}^t x) = {}^s(x\gamma_t) = {}^s x \, {}^s \gamma_t = x \gamma_s \, {}^s \gamma_t $$

so that $\gamma_{st} = \gamma_s \, {}^s \gamma_t$ for all $s, t \in \Theta$. Hence $\gamma = (\gamma_s)$, $s \in \Theta$, is a cocycle of $\Theta$ in $\Gamma$. Given another representative $x' = x \cdot c$, $c \in \Gamma$, of $\overline{x}$, the attached cocycle is determined by $\gamma'_s = c^{-1} \gamma_s \, {}^s c$, $s \in \Theta$. Thus every point in the fixed point set $\mathrm{Fix}(\Theta, X/\Gamma)$ determines a unique class in $H^1(\Theta, \Gamma)$. As a consequence, the fixed

point set is a disjoint union of the connected non-empty sets $F(\gamma)$, $\gamma \in H^1(\Theta, \Gamma)$, that is,

$$\mathrm{Fix}(\Theta, X/\Gamma) = \coprod_{\gamma \in H^1(\Theta,\Gamma)} F(\gamma).$$

All of its connected components $F(\gamma)$ are of the form $X(\gamma)/\Gamma(\gamma)$, determined by the subgroup $G(\gamma) \subset G$, and, by construction, the map $j_{G(\gamma)}$ is injective. Such a geometric cycle $X(\gamma)/\Gamma(\gamma) = X_{G(\gamma)}/\Gamma(\gamma)$ (arising from a $\mathbb{Q}$-rational automorphism of finite order) is called a **special geometric cycle**.

6.5. **A result of Raghunathan.** In this section we outline an argument due to Raghunathan by which, given a reductive $k$-subgroup $H$ in $G$, we see that the question of injectivity regarding the map $j_{H|\Gamma}$ is, up to subgroups of finite index in the arithmetic group, not an issue. More precisely,

**Theorem E.** *Let $G$ be a connected semi-simple algebraic $k$-group, let $H \subset G$ be a connected reductive $k$-subgroup, and let $\Gamma$ be an arithmetic subgroup of $G(k)$. Then there exists a subgroup $\Gamma'$ of finite index in $\Gamma$ such that if $\Gamma$ is replaced by $\Gamma'$, the map*

$$j_{H|\Gamma'} : X_H/\Gamma'_H \longrightarrow X/\Gamma'$$

*is injective.*

Since this result is somewhat hidden in a technically involved part of [35, Section 2], we sketch the argument.

We note that it suffices to prove the assertion for one specific arithmetic group $\Gamma$ and that, by replacing $G$ by $\mathrm{Res}_{k/\mathbb{Q}}G$, we may assume that $k = \mathbb{Q}$. Such a group $\Gamma$ will be chosen in the course of the proof.

With suitably chosen maximal compact subgroups $K_H \subset H(\mathbb{R})$, resp. $K \subset G(\mathbb{R})$ as in Section 6.1, we may write $X_H = K_H \backslash H(\mathbb{R})$, resp. $X = K\backslash G(\mathbb{R})$, so that $X_H \longrightarrow X$ is a closed embedding. We consider the sequence of natural maps

$$H(\mathbb{R}) \longrightarrow H(\mathbb{R})/\Gamma \longrightarrow X_H/\Gamma_H \longrightarrow X/\Gamma.$$

Let $x, y \in H(\mathbb{R})$, and let $\overline{x}, \overline{y}$ be their images in $X_H/\Gamma_H$; that is, they are of the form of double cosets

$$\overline{x} = K_H x \Gamma_H, \quad \overline{y} = K_H y \Gamma_H.$$

The condition that $j_{H|\Gamma}(\overline{x}) = j_{H|\Gamma}(\overline{y})$ is equivalent to the identity of double cosets

$$Kx\Gamma = Ky\Gamma.$$

This implies that there are elements $\gamma \in \Gamma, k^{-1} \in K$ so that

$$x = k^{-1} y \gamma$$

and therefore

$$\gamma \in H(\mathbb{R}) k H(\mathbb{R}).$$

In turn, equivalently,

$$H(\mathbb{R})\gamma \cap KH(\mathbb{R}) \neq \emptyset.$$

Thus we are reduced to showing that, for a suitably chosen group $\Gamma$, this implies that $\gamma \in H$ and hence also $k \in K_H$.

As a first step, we show that the double coset $H(\mathbb{C})kH(\mathbb{C}), k \in K$, is a closed subvariety in $G(\mathbb{C})$. Let $G_u$ be a maximal compact subgroup of $G(\mathbb{C})$ that contains a maximal compact subgroup $H_u$ of $H(\mathbb{C})$. The groups $G_u$ and $H_u$ are compact

real forms of $G(\mathbb{C})$ and $H(\mathbb{C})$, respectively; that is, the group $G(\mathbb{C})$ can be endowed with the structure of an $\mathbb{R}$-group $G'$ that induces an $\mathbb{R}$-group structure $H'$ on $H(\mathbb{C})$ such that $G'(\mathbb{R}) = G_u$ and $H'(\mathbb{R}) = H_u$. There is a natural isomorphism $\phi$ defined over $\mathbb{C}$ of $G(\mathbb{C})$ on $G'(\mathbb{C})$ which carries $H(\mathbb{C})$ on $H'(\mathbb{C})$ and induces the identity on $K$. We fix an imbedding $G' \longrightarrow GL_N$ over $\mathbb{R}$ for some $N$. We then have a natural action of $H' \times H'$ on $V := M_n(\mathbb{C})$ by left and right translations. By construction, a given element $k \in K$ lies in $V_{\mathbb{R}} = M_n(\mathbb{R})$, and the orbit $H_u k H_u$ is closed in $V_{\mathbb{R}}$ in the ordinary topology since it is compact. Now results of Birkes [14, 5.3] show, since $H_u$ is anisotropic over $\mathbb{R}$, that $H'kH'$ is Zariski closed in $G'$. This implies that $H(\mathbb{C})kH(\mathbb{C})$ is a closed subvariety in $G(\mathbb{C})$ for any $k \in K$ because $\phi$ is the identity on $K$.

In the second step, we fix a realization of $G$ as a $\mathbb{Q}$-subgroup of some $GL_N$. For any subring $R \subset \mathbb{C}$, let $R[G]$ denote the $R$-algebra of polynomials in the matrix entries $a_{ij}(g), i, j = 1, ..., N$, of $G$ by means of this embedding. The group $H(\mathbb{Q})$ acts on $\mathbb{Q}[G]$ via left and right translations. Let $\mathbb{Q}[G]_0$ denote the subalgebra of elements in $\mathbb{Q}[G]$ that are bi-invariant under this action of $H(\mathbb{Q})$. Since $H(\mathbb{Q})$ is Zariski dense in $H$, $\mathbb{C}[G]_0 = \mathbb{Q}[G]_0 \otimes_{\mathbb{Q}} \mathbb{C}$ is the $\mathbb{C}$-algebra of $H$-bi-invariant elements of $\mathbb{C}[G]$. We define $\mathbb{Z}[G]_0 = \mathbb{Z}[G] \cap \mathbb{Q}[G]_0$. Then $\mathbb{Q}[G]_0 = \mathbb{Z}[G]_0 \otimes_{\mathbb{Z}} \mathbb{Q}$, and we obtain

$$\mathbb{C}[G]_0 = \mathbb{Z}[G]_0 \otimes_{\mathbb{Z}} \mathbb{C}. \tag{6.1}$$

The $\mathbb{C}$-algebra $\mathbb{C}[G]_0$ is finitely generated, [15, 7.6]. Thus, we can find a set $S = \{f_1, ..., f_s\}$ of finitely many elements $f_i, i = 1, ..., s$, in $\mathbb{Z}[G]_0$ which generate $\mathbb{C}[G]_0$ as a $\mathbb{C}$-algebra. We may assume that $f_i(1) = 0$ for all $i = 1, ..., s$. As a consequence the elements in the generating set $S$ are polynomials in the variables $a_{ij}(g) - \delta_{ij}, 1 \leq i, j \leq N$, but without constant term.

By the very definition, the group $G_{\mathbb{Z}} = GL_N(\mathbb{Z}) \cap G(\mathbb{Q})$ is an arithmetic subgroup as well as the principal congruence subgroups $\Gamma(q) \subset G_{\mathbb{Z}}, q \in \mathbb{N}, q \geq 1$. The $s$-tuple $(f_1, ..., f_s)$ defines a $\mathbb{Q}$-morphism $F : G \longrightarrow \mathbb{C}^s$. The map $F$ is constant on the double cosets $HxH$. Moreover, if $\gamma \in \Gamma(q)$, then $a(g) \equiv 1 \bmod q$; that is, $F$ maps the principal congruence subgroup $\Gamma(q)$ into $(q\mathbb{Z})^s \subset \mathbb{C}^s$. Since the group $K$ is compact, the subset $F(K) \subset \mathbb{C}^s$ is bounded. Due to the discreteness of $F(\Gamma(q))$, this permits us to choose a large number $q_0 \in \mathbb{N}$ so that the intersection $F(K) \cap F(\Gamma(q_0))$ only consists of the origin $(0, ..., 0)$ in $\mathbb{C}^s$.

In our analysis of the map $j_H$, we now consider this specific group $\Gamma(q_0) =: \Gamma$. For given elements $x, y \in \Gamma$ with $j_H(\overline{x}) = j_H(\overline{y})$ we have seen above that there are elements $\gamma \in \Gamma, k^{-1} \in K$ so that $\gamma \in HkH$. By our first step in the proof, the double cosets $HkH$ and $H1H$ are both Zariski closed in $G$, and, by [15, 7.6 (ii)], the elements of $\mathbb{C}[G]_0$ separate the closed orbits of $H \times H$, that is, the closed double cosets with respect to $H$. In particular, if $HkH \neq H1H$, one can find $f \in \mathbb{C}[G]_0$ with $f(1) = 0$ but $f(k) \neq 0$. However, by construction, $F(k)$ is the origin in $\mathbb{C}^s$. This implies, because $S = \{f_1, ..., f_s\}$ generates $\mathbb{C}[G]_0$ as an algebra over $\mathbb{C}$, that every element of $\mathbb{C}[G]_0$ is zero on $k$. Therefore we may conclude that $HkH = H1H$; in turn,

$$k \in K_H = K \cap H(\mathbb{R})$$

and

$$\gamma \in \Gamma_H = \Gamma \cap H(\mathbb{Q}).$$

This proves $\overline{x} = \overline{y}$ and thus our claim.

6.6. **Orientability.** Given a connected reductive $\mathbb{Q}$-subgroup $H$ of a connected semi-simple algebraic $\mathbb{Q}$-group $G$ as above, the real Lie group $H(\mathbb{R})$ of real points of $H$ may fail to be connected. Thus, with regard to the action of $H(\mathbb{R})$ on the totally geodesic submanifold $X_H$ in $X$, the group $H(\mathbb{R})$ may contain elements which do not act in an orientation-preserving manner on $X_H$. Thus, in general, even if $\Gamma$ is torsion free, it is not the case that the manifold $X_H/\Gamma_H$ carries a natural orientation. However, in order to analyze the contribution of $X_H/\Gamma_H$ to the (co)homology of $X/\Gamma$, for example, via its fundamental class, one has to deal with this problem. The question of orientability arises in an even stronger form if one actually wants to determine the intersection number (if it is defined) of the cycle $X_H/\Gamma_H$ with a suitable submanifold of complementary dimension in $X/\Gamma$. One needs that all connected components of this intersection are orientable. The following result, to be found in [122], shows that this situation can always be achieved, up to a subgroup of finite index.

**Theorem F.** *Let $G$ be a connected semi-simple algebraic $\mathbb{Q}$-group, and let $G_{\mathbb{Z}}$ be the group of integral points relative to some given embedding $\rho : G \longrightarrow GL_n$ over $\mathbb{Q}$. Let $\{H^j\}, j = 1, ..., h$, be a finite family of reductive algebraic $\mathbb{Q}$-subgroups of $G$ which will be viewed via $\rho$ as subgroups of $GL_n$. Then there exists a non-zero ideal $\mathfrak{q} \subset \mathbb{Z}$ such that*

$$\Gamma(\mathfrak{q}) \cap H^j_{\mathbb{Z}} \subset H^j(\mathbb{R})^0, \quad j = 1, ..., h.$$

If the groups $H^j$, $j = 1, ..., h$, are semi-simple, we can choose the ideal $\mathfrak{q}$ to be coprime to any given fixed non-zero ideal $\mathfrak{r} \subset \mathbb{Z}$.

## 7. Invariant differential forms

Let $\Gamma$ be a torsion-free arithmetic subgroup of a semi-simple algebraic $\mathbb{Q}$-group $G$. Let $I^*_{G(\mathbb{R})}$ be the algebra of $G(\mathbb{R})$-invariant forms on $X$. Such a differential form $\omega$ is both closed and coclosed (with respect to any $G(\mathbb{R})$-invariant Riemannian metric), hence harmonic, so it passes to a harmonic differential form, and hence a cohomology class on $X/\Gamma$. In this section we will explain the following results.

**Theorem G.** *The algebra $I^*_{G(\mathbb{R})}$ can be naturally identified with the real cohomology $H^*(X_u, \mathbb{R})$ of the compact dual symmetric space $X_u$ of $X$ and there is a homomorphism*

$$\beta^*_\Gamma : H^*(X_u, \mathbb{R}) \tilde{\longrightarrow} I^*_{G(\mathbb{R})} \longrightarrow H^*(\Omega(X, \mathbb{R})^\Gamma) \tilde{\longrightarrow} H^*(X/\Gamma, \mathbb{R}).$$

*Moreover, if $X/\Gamma$ is compact, then $\beta^*_\Gamma$ is injective.*

Note that in the case of a compact quotient $X/\Gamma$, the part Image$\beta^*_\Gamma$ of the cohomology is easily understood since the compact dual symmetric spaces are familiar objects as Grassmannians and flag manifolds.

Besides giving a detailed outline of this construction we study this map in the cases where $X$ is a Riemannian symmetric space of real rank one (and negative curvature). Our eventual aim is to show in the following section how this map can be used to detect non-vanishing (co)homology classes carried by geometric cycles.

7.1. **The compact dual symmetric space.** Let $G$ be a connected semi-simple real Lie group, $\mathfrak{g}$ its Lie algebra, and let $K \subset G$ be a maximal compact subgroup with Lie algebra $\mathfrak{k}$. Let

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

be the corresponding Cartan decomposition. There is an associated involutive automorphism $\theta : \mathfrak{g} \to \mathfrak{g}$, called the Cartan involution, which acts as the identity Id on the subalgebra $\mathfrak{k}$ and as $-(\mathrm{Id})$ on $\mathfrak{p}$. In other words, $(G, K)$ is a Riemannian symmetric pair of non-compact type associated with the orthogonal symmetric Lie algebra $(\mathfrak{g}, \theta)$ in the sense of [60, Chap. VI]. Let $G_u$ be a maximal compact subgroup of the complexification $G_{\mathbb{C}}$ of $G$ which contains $K$. We may suppose that the Lie algebra $\mathfrak{g}_u$ of $G_u$ admits the Cartan decomposition

$$\mathfrak{g}_u = \mathfrak{k} \oplus \mathfrak{p}_u \quad \text{with } \mathfrak{p}_u = i\mathfrak{p}.$$

Note that $\mathfrak{g}_u$ is a compact real form of the complexification $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}$. If $B$ denotes the Killing form of $\mathfrak{g}_{\mathbb{C}}$, its restrictions to $\mathfrak{g} \times \mathfrak{g}$ and $\mathfrak{g}_u \times \mathfrak{g}_u$ are the Killing forms of $\mathfrak{g}$ and $\mathfrak{g}_u$, respectively. The homogeneous space $X_u := K\backslash G_u$ endowed with the unique Riemannian structure induced by the restriction of $-B$ to $i\mathfrak{p} \times i\mathfrak{p}$ is a compact symmetric Riemannian space, the compact dual symmetric space of $X = K\backslash G$. Note that there is a natural identification of the tangent space $T_{Ke}(X_u)$ at the origin with the Lie algebra $\mathfrak{p}_u = i\mathfrak{p}$, where $i\mathfrak{p}$ is viewed as a real subspace of $\mathfrak{p}_{\mathbb{C}}$.

Let $I_G := \Omega_I(K\backslash G)$ be the space of $G$-invariant $\mathbb{R}$-valued $C^\infty$-forms on $X$. By evaluating a form at the origin we obtain an isomorphism

$$I_G \xrightarrow{\sim} \mathrm{Hom}_K(\Lambda(\mathfrak{p}), \mathbb{R}) = (\Lambda(\mathfrak{g}/\mathfrak{k})^*)^K \tag{7.1}$$

of $I_G$ on the complex $C(\mathfrak{g}, \mathfrak{k}, \mathbb{R}) = \mathrm{Hom}_K(\Lambda(\mathfrak{p}), \mathbb{R})$ underlying the relative Lie algebra cohomology $H(\mathfrak{g}, \mathfrak{k}, \mathbb{R})$ (see Appendix E). Similiarly, there is an analogous isomorphism

$$I_{G_u} \xrightarrow{\sim} \mathrm{Hom}_K(\Lambda(\mathfrak{p}_u), \mathbb{R}) = (\Lambda(\mathfrak{g}_u/\mathfrak{k})^*)^K \tag{7.2}$$

of the space $I_{G_u}$ of $G_u$-invariant $\mathbb{R}$-valued $C^\infty$-forms on $X_u$ onto the complex $C(\mathfrak{g}_u, \mathfrak{k}, \mathbb{R}) = \mathrm{Hom}_K(\Lambda(\mathfrak{p}_u), \mathbb{R}) = (\Lambda(\mathfrak{g}_u/\mathfrak{k})^*)^K$. Let $K_{\mathbb{C}}$ denote the complexification of $K$ in $G_{\mathbb{C}}$, $\mathfrak{k}_{\mathbb{C}}$ its Lie algebra. Then we have

$$K_{\mathbb{C}} \cap G_u = K_{\mathbb{C}} \cap G = K.$$

Therefore $X$ and $X_u$ are embedded in the space $X_{\mathbb{C}} := K_{\mathbb{C}}\backslash G_{\mathbb{C}}$ as orbits of the coset $K_{\mathbb{C}}$.

The inclusions $\mathfrak{p} \longrightarrow \mathfrak{p}_{\mathbb{C}}$ and $\mathfrak{p}_u = i\mathfrak{p} \longrightarrow \mathfrak{p}_{\mathbb{C}}$ induce natural $\mathbb{R}$-linear maps

$$\alpha : (\Lambda\mathfrak{p}^*)^K \longrightarrow (\Lambda\mathfrak{p}_{\mathbb{C}}^*)^{K_{\mathbb{C}}}$$

and

$$\alpha_u : (\Lambda\mathfrak{p}_u^*)^K \longrightarrow (\Lambda\mathfrak{p}_{\mathbb{C}}^*)^{K_{\mathbb{C}}},$$

respectively. These maps are injective. By restriction we also obtain $\mathbb{R}$-linear maps in the reverse direction, to be denoted

$$\rho : (\Lambda\mathfrak{p}_{\mathbb{C}}^*)^{K_{\mathbb{C}}} \longrightarrow (\Lambda\mathfrak{p}^*)^K \quad \text{and} \quad \rho_u : (\Lambda\mathfrak{p}_{\mathbb{C}}^*)^{K_{\mathbb{C}}} \longrightarrow (\Lambda\mathfrak{p}_u^*)^K.$$

The composition $\rho_u \circ \alpha$ provides $\mathbb{R}$-linear isomorphisms

$$(\Lambda\mathfrak{p}^*)^K \longrightarrow (\Lambda\mathfrak{p}_u^*)^K.$$

Now we take into account the identifications $I_G \xrightarrow{\sim} (\Lambda\mathfrak{p}^*)^K$ and $I_{G_u} \xrightarrow{\sim} (\Lambda\mathfrak{p}_u^*)^K$, respectively. Let $\omega$ be a multilinear form of degree $q$ on $\mathfrak{p}_{\mathbb{C}}$ whose restriction to $\mathfrak{p}_u$ is real-valued. Then the form $i^q\omega$ (with $i = \sqrt{-1}$) has a real-valued restriction on

$\mathfrak{p}$. Thus, in terms of invariant $\mathbb{R}$-valued $q$-forms, the assignment $\omega \mapsto i^q\omega$ provides isomorphisms

$$I^q_{G_u} \xrightarrow{\sim} I^q_G, \tag{7.3}$$

for each $q \in \mathbb{N}$.

In other words, it is the fact that a $G_{\mathbb{C}}$-invariant form on $X_{\mathbb{C}}$ restricts to a $G$-invariant form on the embedded space $X$ and to a $G_u$-invariant form on $X_u$, respectively, which lies behind this isomorphism.

7.2. **Invariant forms and cohomology.** Given a connected semi-simple real Lie group $G$, $K \subset G$ a maximal compact subgroup, as before, let $\Gamma \subset G$ be a torsion-free discrete subgroup. It acts properly discontinuously on the space $X = K\backslash G$ of maximal compact subgroups of $G$. The space $X/\Gamma$ is a complete Riemannian manifold. Let $X_u = K\backslash G_u$ denote the compact dual symmetric space of $X$. Since $X_u$ is the quotient of a compact connected group modulo a closed connected subgroup, the cohomology $H^*(X_u, \mathbb{R})$ admits a natural interpretation as the space of $G_u$-invariant $\mathbb{R}$-valued $C^\infty$-forms on $X_u$, that is, $H^*(X_u, \mathbb{R}) \xrightarrow{\sim} I^*_{G_u}$ as explained in appendix E.2. In view of the isomorphism of the latter space with the space of $G$-invariant $\mathbb{R}$-valued smooth forms on $X$, we obtain an isomorphism

$$H^*(X_u, \mathbb{R}) \xrightarrow{\sim} I^*_{G_u} \xrightarrow{\sim} I^*_G. \tag{7.4}$$

A $G$-invariant $\mathbb{R}$-valued form $\omega$ on $X$ naturally descends to a form on $X/\Gamma$; thus there is a map

$$H^*(X_u, \mathbb{R}) \xrightarrow{\sim} I^*_G \longrightarrow \Omega^*(X, \mathbb{R})^\Gamma. \tag{7.5}$$

The space $I^*_G$ consists of closed (even harmonic) forms; thus we get a homomorphism

$$\beta^*_\Gamma : H^*(X_u, \mathbb{R}) \longrightarrow H^*(\Omega(X, \mathbb{R})^\Gamma) \xrightarrow{\sim} H^*(X/\Gamma, \mathbb{R}). \tag{7.6}$$

We may rewrite this homomorphism in terms of relative Lie algebra cohomology. First, recall the isomorphisms $H^*(X_u, \mathbb{R}) \tilde{=} I^*_{G_u} \tilde{=} C^*(\mathfrak{g}_u, \mathfrak{k}, \mathbb{R})$. This latter complex is naturally isomorphic to the complex $C^*(\mathfrak{g}, \mathfrak{k}, \mathbb{R})$. Since $(\mathfrak{g}, \theta)$ is an orthogonal symmetric Lie algebra, the differential of the latter complex vanishes identically; hence $C^*(\mathfrak{g}, \mathfrak{k}, \mathbb{R}) = H^*(\mathfrak{g}, \mathfrak{k}, \mathbb{R})$. Now recall that the deRham cohomology group $H^*(\Omega(X, \mathbb{R})^\Gamma)$ may be viewed as the relative Lie algebra cohomology $H^*(\mathfrak{g}, \mathfrak{k}, C^\infty(G/\Gamma))$. Then the inclusion $\mathbb{R} \to C^\infty(G/\Gamma)$ of $\mathbb{R}$ on the space of constant functions induces a homomorphism in cohomology so that we obtain the reinterpretation of $\beta_\Gamma$ by

$$C^*(\mathfrak{g}_u, \mathfrak{k}, \mathbb{R}) \tilde{\to} C^*(\mathfrak{g}, \mathfrak{k}, \mathbb{R}) \tilde{\to} H^*(\mathfrak{g}, \mathfrak{k}, \mathbb{R}) \to H^*(\mathfrak{g}, \mathfrak{k}, C^\infty(G/\Gamma)). \tag{7.7}$$

Now we suppose that the discrete group $\Gamma \subset G$ gives rise to a compact quotient $X/\Gamma$, or, equivalently, that $G/\Gamma$ is compact. A $G$-invariant $\mathbb{R}$-valued form $\omega \in I^*_G$ is closed and coclosed, therefore harmonic. As a harmonic form, $\omega \neq 0$ cannot be a coboundary; the homomorphism $\beta_\Gamma$ is injective. Thus, the injectivity of the homomorphism $\beta_\Gamma$ is a consequence of Hodge theory. Results of Matsushima [95] show the existence of a constant $m(G)$, only depending on $\mathfrak{g}$, so that $\beta_\Gamma$ is surjective in degrees up to $m(G)$.

In [16], an analogue of this latter result is given in the case of an arithmetic subgroup $\Gamma$ of a semi-simple algebraic group defined over $\mathbb{Q}$. In this case the group $G(\mathbb{R})$ of real points of $G$ plays the role of our real Lie group above. Note that in such a case the quotient $G(\mathbb{R})/\Gamma$ has finite invariant volume but is not necessarily

compact. Furthermore, the homomorphism $\beta_\Gamma$ is injective only in degrees up to a constant $c(G/\mathbb{Q})$. This constant $c(G/\mathbb{Q})$ is slightly smaller than half of the $\mathbb{Q}$-rank of $G$; see also [162].

7.3. **Real rank one symmetric spaces: an example.** Let $G$ be a connected semi-simple real Lie group with finite center, and let $(G, K)$ be a Riemannian symmetric pair of non-compact type associated with the orthogonal symmetric Lie algebra $(\mathfrak{g}, \theta)$. By definition, the rank of the corresponding symmetric space $X = K\backslash G$ is the maximal dimension of a flat totally geodesic submanifold of $X$.

By the classification of Riemannian globally symmetric spaces, rank one symmetric spaces of negative curvature are real, complex and quaternionic hyperbolic spaces and the Cayley hyperbolic plane. One can construct the complex and quaternionic hyperbolic $n$-spaces, to be denoted $H^n_{\mathbb{C}}$ and $H^n_{\mathbb{H}}$ respectively in the sequel, in a way that resembles to a large extent the definition of real hyperbolic $n$-space $H^n := H^n_{\mathbb{R}}$ as given in Section 3 (see [106, Section 19], or [26, chapter II. 10]). Recall that real hyperbolic $n$-space $H^n_{\mathbb{R}}$ is described by the pair $(G, K) = (SO_0(n, 1), SO(n))$.

In the following, let $F$ denote either $\mathbb{R}$, $\mathbb{C}$ or the division algebra $\mathbb{H}$ of Hamilton quaternions. Viewed as a vector space over $\mathbb{R}$, the central simple algebra $\mathbb{H}$ has a basis $1, i, j, k$ subject to the relations $i^2 = -1, j^2 = -1, ij = k = -ji$. If $x = x_0 1 + x_1 i + x_2 j + x_3 k \in \mathbb{H}$, then we write $\overline{x} = x_0 1 - x_1 i - x_2 j - x_3 k$ to denote its quaternionic conjugate. The $\mathbb{R}$-linear map $\tau_c : \mathbb{H} \to \mathbb{H}$, defined by $x \mapsto \overline{x}$, satisfies $\tau_c^2 = \mathrm{Id}$ and $\tau_c(xy) = \tau_c(y)\tau_c(x)$ for all $x, y \in \mathbb{H}$; that is, $\tau_c$ is an involution. The norm $N : \mathbb{H} \to \mathbb{R}$, given by $x \mapsto x\tau_c(x)$, is multiplicative. If $F = \mathbb{C}$, we write $z \mapsto \overline{z}, z \in \mathbb{C}$, to denote the usual complex conjugation.

Let $p$ and $q$ be non-negative integers with $p + q > 0$, and let $F^{p,q}$ be the vector space $F^{p+q}$ endowed with the non-degenerate form $b_{p,q}$ defined by

$$b_{p,q}(x, y) = \sum_{1 \le i \le p} x_i \, \overline{y_i} \; - \sum_{p+1 \le i \le p+q} x_i \, \overline{y_i}. \tag{7.8}$$

Let $O_F(b_{p,q}) = \{g \in GL_{p+q}(F) \mid b_{p,q}(gv, gv') = b_{p,q}(v, v') \text{ for all } v, v' \in F^{p+q}\}$ denote the subgroup of elements in $GL_{p+q}(F)$ which leave the form $b_{p,q}$ invariant.

Traditionally, if $F = \mathbb{R}$, $O_{\mathbb{R}}(b_{p,q})$ is denoted by $O(p, q)$. Clearly, $O(p) := O(p, 0)$ is compact. We put $SO(p, q) = O(p, q) \cap SL_{p+q}(\mathbb{R})$, $SO(p) := SO(p, 0)$. If $F = \mathbb{C}$, $O_{\mathbb{C}}(b_{p,q})$ is denoted by $U(p, q)$. The group $U(p) := U(p, 0)$ is compact. We put $SU(p, q) = U(p, q) \cap SL_{p+q}(\mathbb{C})$, $SU(p) = SU(p, 0)$. We have that $SU(p, q) \cap U(p + q)$ coincides with the group $S(U(p) \times U(q))$ of diagonal block matrices $\mathrm{diag}(u_p, u_q), u_p \in U(p), u_q \in U(q)$ with $\det(u_p)\det(u_q) = 1$ . The group $O_{\mathbb{H}}(b_{p,q})$ is usually denoted by $Sp(p, q)$. Write $Sp(p) := Sp(p, 0)$.

In the case of interest for us, i.e., $p = n$, $q = 1$, the group $O_F(b_{n,1})$ acts by isometries on hyperbolic $n$-space $H^n_F$. The stabilizer of a point in $H^n_F$ is conjugate to $O(n) \times O(1)$ if $F = \mathbb{R}$, to $U(n) \times U(1)$ if $F = \mathbb{C}$ and to $Sp(n) \times Sp(1)$ if $F = \mathbb{H}$. The subgroup $Z$ in $O_F(b_{n,1})$ of elements which act trivially on $H^n_F$ consists of $+1$ and $-1$ if $F = \mathbb{R}$ or $\mathbb{H}$, and is isomorphic to $U(1)$ if $F = \mathbb{C}$. Usually, the quotient group $O_F(b_{n,1})/Z$ is denoted by $PO_F(b_{n,1})$.

As a final result, complex hyperbolic $n$-space $H^n_{\mathbb{C}}$ is described by the pair $G = SU(n, 1), K = S(U(n) \times U(1))$. It can be shown that the isometry group of $H^n_{\mathbb{C}}$ is generated by $PO_{\mathbb{C}}(b_{n,1})$ and an involution induced by complex conjugation.

Quaternionic hyperbolic $n$-space $H^n_{\mathbb{H}}$ is given by the symmetric pair $G = Sp(n,1)$, $K = Sp(n) \times Sp(1)$. If $n > 1$, the isometry group of $H^n_{\mathbb{H}}$ is $PO_{\mathbb{H}}(b_{n,1})$.

For the sake of completeness, we mention in passing that the Cayley hyperbolic plane is described by a rank one real form of exceptional type $F_4$, to be denoted $F_{4(-20)}$, with maximal compact subgroup $K = Spin(9)$.

We now explicitly describe the compact dual symmetric space $H^n_{F,u}$ that corresponds to the $n$-dimensional hyperbolic space $H^n_F$. By definition, these are given as the quotient space $G_u/K$; that is,

$$\begin{aligned} H^n_{\mathbb{R},u} &= SO(n+1)/(SO(n) \times SO(1)), \\ H^n_{\mathbb{C},u} &= SU(n+1)/S(U(n) \times U(1)), \\ H^n_{\mathbb{H},u} &= Sp(n+1)/(Sp(n) \times Sp(1)). \end{aligned}$$

If $F = \mathbb{R}$, clearly $H^n_{\mathbb{R},u}$ is the $n$-sphere $S^n$. In the other cases, it turns out that

$$H^n_{\mathbb{C},u} = P_n\mathbb{C}, \quad \text{resp.} \quad H^n_{\mathbb{H},u} = P_n\mathbb{H};$$

that is, if $F \neq \mathbb{R}, H^n_{F,u}$ is the $n$-dimensional projective space $P_nF$ over $F$. We think of $P_nF$, $n \geq 0$, as the orbit space of $F^{n+1} - (0)$ under the natural action of $F^* = F - (0)$ given by $y \mapsto y\mu, \mu \in F^*$. Let $\pi : F^{n+1} - (0) \to P_nF$ be the canonical projection that maps $y = (y_0, ..., y_n)$ on its class $[y]$ modulo $F^*$. One may also view $[y]$ as a one-dimensional subspace in $F^{n+1}$. We call $(y_0, ..., y_n)$ the homogeneous coordinates for $[y]$.

Now suppose that $F = \mathbb{H}$. The group $GL_{n+1}(\mathbb{H})$ acts naturally on $F^{n,1} = \mathbb{H}^{n,1}$ and induces a transitive action of the subgroup $O_{\mathbb{H}}(b_{n+1}) = Sp(n+1)$ on $P_n\mathbb{H}$. The stabilizer in $Sp(n+1)$ of the point with homogeneous coordinates $(0, ..., 0, 1)$ is isomorphic to $Sp(n) \times Sp(1)$. Thus, we obtain $H^n_{\mathbb{H},u} = P_n\mathbb{H}$.

A similar consideration applies to the complex projective space $P_n\mathbb{C}$.

There exists a finite cellular decomposition of $P_nF$ into disjoint cells of dimension $0, d, 2d, ..., nd$ where $d = \dim_{\mathbb{R}} F$. More precisely, for each $0 \leq k \leq n$, there exists a single $k$-cell $e_k := \{[y] \in P_nF \quad | \quad y_i = 0 \quad \text{for } i > k\}$ so that $P_nF = \bigcup_{0 \leq k \leq n} e_k$. For a given $m \leq n$, the $m$-skeleton of this cellular decomposition is the union of the cells $e_k, k = 0, ..., m$. It may be identified with the image of $P_mF$ under the natural inclusion $P_mF \longrightarrow P_nF$, defined by $[y_0, ..., y_m] \mapsto [y_0, ..., y_m, 0, ..., 0]$. Obviously, this is a cellular map. If $F = \mathbb{C}$ or $\mathbb{H}$, there are no cells of odd dimension in this cellular decomposition. Thus, in determining the cohomology groups of $P_nF$ with integral coefficients in these two cases, we obtain the final result

$$H^i(P_nF, \mathbb{Z}) = \mathbb{Z} \quad \text{if } i = 0, d, 2d, ..., nd$$

and $H^i(P_nF, \mathbb{Z}) = 0$ otherwise. This assertion is also true in the case $F = \mathbb{R}$ if one replaces the coefficients $\mathbb{Z}$ by $\mathbb{Z}/2\mathbb{Z}$. Furthermore, the inclusions $P_mF \longrightarrow P_nF$, $m \leq n$, induce isomorphisms

$$H^i(P_mF, R) \xleftarrow{\sim} H^i(P_nF, R) \quad \text{for } i \leq md,$$

with coefficients $R = \mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z}$ accordingly.

## 8. Non-vanishing cycles

Let $X = K\backslash G$ be a Riemannian symmetric space given by a Riemannian symmetric pair $(G, K)$ of non-compact type. A semi-simple Lie subgroup $H \subset G$ gives rise to a totally geodesic submanifold $X_H \subset X$. In fact, it is the symmetric space attached to the Riemannian symmetric pair $(H, K_H)$, $K_H = H \cap K$, of non-compact

type. Let $\Gamma$ be a torsion-free discrete subgroup $\Gamma$ in $G$ so that $X_H/\Gamma_H$ and $X/\Gamma$ are compact manifolds and the map

$$j_{H|\Gamma} : X_H/\Gamma_H \longrightarrow X/\Gamma$$

is a smooth embedding. Thus, we may view $j_{H|\Gamma}(X_H/\Gamma_H)$ as a totally geodesic submanifold in $X/\Gamma$. Due to Venkataramana [149] or later on independently Lafont-Schmidt [81], there is a criterion in terms of the homomorphism induced in cohomology by the natural map $\mathrm{inv}_H : X_{H,u} \longrightarrow X_u$ of the compact dual symmetric spaces which ensures that the fundamental class $o_{X_H/\Gamma_H}$ of the geometric cycle $X_H/\Gamma_H$ is a non-trivial class in $H_*(X/\Gamma, \mathbb{R})$. As an application, we discuss the case of arithmetic quotients attached to quaternionic hyperbolic $n$-space.

8.1. **Construction.** Let $G$ be a connected semi-simple real Lie group with finite center, and let $(G, K)$ be a Riemannian symmetric pair of non-compact type associated with the orthogonal symmetric Lie algebra $(\mathfrak{g}, \theta)$. We denote the corresponding Riemannian symmetric space by $X$. The choice of a maximal compact subgroup $K \subset G$ corresponds to the choice of a distinguished point $x_0 \in X$. Suppose that $H$ is a connected semi-simple Lie subgroup of $G$ such that $H \cap K =: K_H$ is a maximal compact subgroup of $H$. Via the assignment $h \mapsto x_0 h$ the space $X_H = K_H \backslash H$ may be viewed as a totally geodesic Riemannian submanifold in $X$. Thus, given a torsion-free discrete subgroup $\Gamma$ of $G$, we also have a natural map

$$j_{H|\Gamma} : X_H/\Gamma_H \longrightarrow X/\Gamma \tag{8.1}$$

with $\Gamma_H := \Gamma \cap H$. We suppose that the manifolds $X/\Gamma$ and $X_H/\Gamma_H$ are both oriented in a coherent way. Furthermore, in view of our application in the situation of arithmetically defined groups, we may suppose that the map $j_{H|\Gamma} : X_H/\Gamma_H \to X/\Gamma$ is a smooth embedding. Consequently, we may view $X_H/\Gamma_H$ as a totally geodesic submanifold in $X/\Gamma$, to be denoted by $X_H/\Gamma_H$ as well.

8.2. **A simple criterion.** Now suppose that $X$ and $X_H$ are both symmetric spaces of non-compact type and that the corresponding quotients $X/\Gamma$ and $X_H/\Gamma_H$ are compact spaces of dimension $d(G)$ and $d_H$, respectively. As $K_H \backslash H = X_H$ is a totally geodesic submanifold of $X$ there is a totally geodesic embedding

$$inv_H : X_{H,u} \longrightarrow X_u \tag{8.2}$$

of the associated compact dual symmetric spaces. Then, for a suitable choice of the discrete group $\Gamma$, the embedding $j_{H|\Gamma}$ gives rise to a commutative diagram

$$\begin{array}{ccc} H^*(X_u, \mathbb{R}) & \xrightarrow{\beta^*_\Gamma} & H^*(X/\Gamma, \mathbb{R}) \\ \Big\downarrow{\scriptstyle inv^*_H} & & \Big\downarrow{\scriptstyle j^*_{H|\Gamma}} \\ H^*(X_{H,u}, \mathbb{R}) & \xrightarrow{\beta^*_{\Gamma_H}} & H^*(X_H/\Gamma_H, \mathbb{R}), \end{array} \tag{8.3}$$

where the horizontal homomorphisms $\beta^*_\Gamma$ and $\beta^*_{\Gamma_H}$, defined in Section 7.2, are injective. In degree $d_H$, since $X_H/\Gamma_H$ is compact, $H^{d_H}(X_H/\Gamma_H, \mathbb{R})$ is one-dimensional. As $H^{d_H}(X_{H,u}, \mathbb{R}) = I^{d_H}_{H(\mathbb{R})} = \mathbb{R}$, the injective homomorphism $\beta^{d_H}_{\Gamma_H}$ is an isomorphism. Thus, if $inv^{d_H}_H$ is not identically zero, the morphism $j^{d_H}_{H|\Gamma}$ is non-trivial; that is, $j^{d_H}_{H|\Gamma}$ is surjective. By use of the Kronecker product, one obtains that the

fundamental class $o_{X_H/\Gamma_H}$ of the totally geodesic submanifold $X_H/\Gamma_H$ is a non-trivial class in $H_*(X/\Gamma, \mathbb{R})$. This simple purely geometric argument can be found in [149] or [81].

8.3. **Examples.** This criterion gives a simplified approach to some non-vanishing results previously proved in a slightly more complicated way. We discuss the case of arithmetic quotients attached to quaternionic hyperbolic $n$-space.

Let $k$ be a totally real algebraic number field of degree $d > 1$. As before, let $\sigma_i : k \to \mathbb{R}$, $1 \leq i \leq d$, be the distinct field embeddings. Let $D$ be a quaternion division algebra over $k$, with canonical involution $\tau_c$, also called quaternionic conjugation. Let $(E, h)$ be a finite-dimensional vector space of $D$ endowed with a non-degenerate $D$-valued $k$-bilinear form on $E$. We suppose that the form $h$ is Hermitian; that is, we have

$$h(\lambda u, \mu v) = \lambda h(u, v)\tau_c(\mu), \tag{8.4}$$

$$h(u, v) = \tau_c(h(v, u)) \tag{8.5}$$

for all $u, v \in E, \lambda, \mu \in D$. Note that necessarily $h(v, v) \in k$ for all $v \in E$. The associate isometry group of $h$, defined by

$$G = \{g \in GL_D(E) \mid h(gu, gv) = h(u, v) \text{ for all } u, v \in E\}, \tag{8.6}$$

is a connected, simple and simply connected algebraic group defined over $k$. By definition, the arithmetic subgroups of $G(k)$ are the subgroups commensurable with $G_{\mathcal{O}_k}$. Let $G' = \mathrm{Res}_{k/\mathbb{Q}}\, G$ be the algebraic $\mathbb{Q}$-group obtained from $G$ by restriction of scalars (see Appendix C). If $D$ is split over $\mathbb{R}$, then the group $G'(\mathbb{R})$ of real points of $G'$ is of the form $\prod_{1\leq i\leq d} Sp(p_i, \dim E - p_i)$ where $(p_i, \dim E - p_i)$ is the signature of $h^{\sigma_i}$ , $1 \leq i \leq d$.

In the specific case of arithmetic quotients of quaternionic hyperbolic $n$-space there is the following result.

**Theorem H.** *Let $k \neq \mathbb{Q}$ be a totally real algebraic number field, and let $D$ be a quaternion division algebra over $k$. Let $(E, h)$ be a non-degenerate Hermitian space over $D$ of signature $(n, 1)$ all of whose conjugates are positive definite. If $\Gamma$ is a torsion-free arithmetic subgroup of the isometry group $G$ of $(E, h)$, then the corresponding arithmetic quotient $X/\Gamma$ of the quaternionic hyperbolic $n$-space $X = H^n_{\mathbb{H}}$ is compact and its cohomology $H^i(X/\Gamma, \mathbb{R})$ contains a non-trivial cohomology class if $i = 4, 8, ..., 4(n-1)$. More precisely, given a degree $i = 4m, 1 \leq m < n$, the fundamental class $o_{X_H/\Gamma_H}$ of the totally geodesic submanifold $X_H/\Gamma_H = H^m_{\mathbb{H}}/\Gamma(\sigma_m)$ obtained as a component of the fixed point set $\mathrm{Fix}(\sigma_m, X/\Gamma)$ of a suitable involution $\sigma_m$ detects this cohomology class by duality.*

Since $h$ has signature $(n, 1)$ and all its conjugates $h^{\sigma_i}$ , $2 \leq i \leq d$, have signature $(n + 1, 0)$, the group $G'(\mathbb{R})$ is of the form $Sp(n, 1).C$ with $C$ the compact group $\prod_{1<i\leq d} Sp(n+1)$, and the associated symmetric space is quaternionic $n$-space $H^n_{\mathbb{H}}$. An arithmetic torsion-free subgroup $\Gamma$ of $G(k)$ projects to the real Lie group $Sp(n, 1)$ as a discrete subgroup. The corresponding quotient is compact. For simplicity, by choosing a basis $\{e_i\}_{1\leq i\leq n+1}$ of $E$, we may suppose that $E = D^{n+1}$. For a given integer $m$, $1 \leq m < n$, we define an involution $\sigma_m$ on $G$ by the assignment

$$\sigma_m : G \longrightarrow G, \tag{8.7}$$

$$g \mapsto \Sigma_m\, g\, \Sigma_m, \tag{8.8}$$

where $\Sigma_m$ denotes the $(n+1)\times(n+1)$ matrix

$$\begin{pmatrix} \mathrm{Id}_m & 0 & 0 \\ 0 & -\mathrm{Id}_{n-m} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The group $G(\sigma_m) = \{g \in G \mid \sigma_m(g) = g\}$ of fixed points will serve as our group $H$ in our previous general discussion. Thus there is a totally geodesic submanifold $F(\sigma_m) := X(\sigma_m)/\Gamma(\sigma_m)$ of $H^n_{\mathbb{H}}/\Gamma$ where the symmetric space, being determined by the real Lie group $Sp(m,1)$, is quaternionic hyperbolic $m$-space $H^m_{\mathbb{H}}$. Since we interpret $X(\sigma_m)/\Gamma(\sigma_m)$ as a connected component of the fixed point set $\mathrm{Fix}(\sigma_m, X/\Gamma)$, the injectivity of the map

$$j_{G(\sigma_m)|\Gamma} : H^m_{\mathbb{H}}/\Gamma(\sigma_m) \longrightarrow H^n_{\mathbb{H}}/\Gamma \tag{8.9}$$

for a torsion-free arithmetic subgroup $\Gamma$ of $G(k)$ follows immediately from Section 6.4. In other words, we do not have to pass over to a subgroup of finite index to obtain this result.

As pointed out in Section 7.3, the homomorphism in the cohomology of the corresponding dual symmetric spaces

$$inv^i_{G(\sigma_m)} : H^i(H^n_{\mathbb{H},u}, \mathbb{R}) \longrightarrow H^i(H^m_{\mathbb{H},u}, \mathbb{R}) \tag{8.10}$$

is an isomorphism for all $i \leq 4m$. In particular, $inv^{4m}_{G(\sigma_m)}$ is an isomorphism; thus we obtain the assertion of the theorem.

In the case of compact arithmetic quotients $X/\Gamma$ attached to complex hyperbolic $n$-space $H^n_{\mathbb{C}}$, the same line of arguments leads to an analogous result. By use of the totally geodesic embedding $H^m_{\mathbb{C}} \longrightarrow H^n_{\mathbb{C}}$, $0 \leq m < n$, and the corresponding isomorphism $H^{2m}(H^n_{\mathbb{C},u}, \mathbb{R}) \xrightarrow{\sim} H^{2m}(H^m_{\mathbb{C},u}, \mathbb{R})$ for the cohomology of the dual symmetric spaces one obtains non-trivial cohomology classes in $H^i(X/\Gamma, \mathbb{R})$ for $i = 2m$, $1 \leq m < n$, detected by a geometric cycle of the form $H^m_{\mathbb{C}}/\Gamma'$, $\Gamma'$ a suitable subgroup of $Iso(H^m_{\mathbb{C}})$.

Since the dual symmetric space of real hyperbolic $n$-space $H^n_{\mathbb{R}}$ is the $n$-sphere $S^n$, with non-trivial cohomology only in degrees 0 and $n$, this approach does not lead to a construction of non-trivial cohomology classes in $H^i(H^n_{\mathbb{R}}/\Gamma, \mathbb{R})$. However, as first shown in [102], an assertion of this type is correct, but it needs a different method of proof. In fact, it is based on an analysis of intersection numbers of geometric cycles to be discussed in the next section.

## 9. Intersection numbers of geometric cycles

9.1. **Intersection numbers.** Let $G$ be a connected reductive algebraic $\mathbb{Q}$-group, and let $\Gamma \subset G(\mathbb{Q})$ be a torsion-free arithmetic subgroup. We consider a geometric cycle $Y$ in $X/\Gamma$ which originates with a reductive $\mathbb{Q}$-subgroup $H$ of $G$. In particular, there is an embedding

$$j_{H|\Gamma} : X_H/\Gamma_H \longrightarrow X/\Gamma$$

whose image, a totally geodesic submanifold, is $Y$. For example, such geometric cycles arise as fixed point components of a rational $\mathbb{Q}$-automorphism $\mu$ of finite order on $X/\Gamma$. As discussed in Section 6.4, the connected components of the fixed point set $\mathrm{Fix}(\langle\mu\rangle, X/\Gamma)$ are totally geodesic closed submanifolds in $X/\Gamma$ of the form $F(\gamma) = X(\gamma)/\Gamma(\gamma)$, where $\gamma$ is a cocycle representing a class in the non-abelian

cohomology set $H^1(\langle\mu\rangle,\Gamma)$; that is, we have as a disjoint union of connected non-empty sets

$$\mathrm{Fix}(\langle\mu\rangle, X/\Gamma) = \coprod_{\gamma\in H^1(\langle\mu\rangle,\Gamma)} F(\gamma).$$

In general, we are interested in cases where

- a geometric cycle $Y$ is orientable and
- its fundamental class is not homologous to zero in $X/\Gamma$, in singular homology or homology with closed supports, as necessary.

As stated in Theorem F [Section 6.6], there exists a subgroup of finite index in $\Gamma$ such that the corresponding cycles are orientable. Thus we may suppose that the geometric cycles we consider have this property.

One way to go about the second question is to construct an orientable submanifold $Y'$ of complementary dimension such that the intersection product (if defined) of its fundamental class with that of $Y$ is non-zero. In doing so, if $X/\Gamma$ is non-compact, we have to assume that at least one of the cycles $Y, Y'$ is compact, while the other need not be. In order to find a non-zero intersection product, if at all possible, it is often necessary to replace the arithmetic group $\Gamma$ by a suitable subgroup of finite index.

For the sake of convenience we recall the definition of the intersection number of two closed oriented submanifolds $Y, Y'$ in an oriented manifold $V$ of dimension $\dim V = m$. We suppose that $Y\cap Y'$ is compact and that the submanifolds have complementary dimensions, that is, $\dim Y + \dim Y' = \dim V$. Let $N = Y$ or $Y'$, and let $U$ be a tubular neighborhood of $N$ in $V$. Then there is an isomorphism

$$H^{m-\dim N}(U, U-N;\mathbb{Z}) \xrightarrow{\sim} H^{m-\dim N}(V, V-N;\mathbb{Z}).$$

Since the normal bundle of $N$ in $V$ is oriented in a natural way, there is, given a point $x\in N$, an isomorphism

$$j_x : H^{m-\dim N}(U_x, (U-N)_x;\mathbb{Z}) \xrightarrow{\sim} \mathbb{Z},$$

where $U_x$ denotes the fiber of the normal bundle in $U$ over $x\in N$. Then there is a uniquely determined class $\omega_N \in H^{m-\dim N}(U, U-N;\mathbb{Z})$ such that $\omega_N$ is mapped under the composite of maps

$$H^{m-\dim N}(U, U-N;\mathbb{Z}) \xrightarrow{\sim} H^{m-\dim N}(U_x, (U-N)_x;\mathbb{Z}) \xrightarrow{\sim} \mathbb{Z}$$

to the element 1 in $\mathbb{Z}$. Then the intersection number of $Y$ and $Y'$ is defined by the expression

$$[Y][Y'] = (\omega_Y \smile \omega_{Y'})(o_V); \tag{9.1}$$

that is, the integer $[Y][Y']$ is obtained by evaluating the cup-product $(\omega_Y \smile \omega_{Y'})$ on the fundamental class $o_V \in H^{cl}_m(V,\mathbb{R})$ of $V$.

The reader may wish to consult Appendix D for some results on products in the (co)homology of manifolds which are needed in the text.

9.2. **Excess bundles.** We retain the setting of the previous paragraph. Suppose that the two closed oriented submanifolds $Y, Y'$ of the oriented manifold $V$ intersect *perfectly*; that is, the connected components of the intersection $Y\cap Y'$ are immersed submanifolds of $V$ and for each of the components $F$ of $Y\cap Y'$ the tangent bundle

$TF$ of $F$ coincides with the intersection of the restriction of the tangent bundles of $Y$ and $Y'$ to $F$,

$$TF = TY_{|F} \cap TY'_{|F}. \tag{9.2}$$

In such a case, each of these components $F$, say of dimension $f \geq 1$, gives rise to the so-called *excess bundle* $\eta_F$ of $F$ (see [108, Section 3]). More precisely, let $TY_{|F} + TY'_{|F}$ be the bundle over $F$ whose fiber over a point $x \in F$ consists of the span of the fibers $T_xY$ and $T_xY'$ in the fiber $T_xV$ of the tangent bundle $TV$ of $V$. Then the following sequence,

$$0 \to TF \to TY_{|F} \oplus TY'_{|F} \to TY_{|F} + TY'_{|F} \to 0, \tag{9.3}$$

is an exact sequence of bundles. Since $Y$ and $Y'$ intersect perfectly there is a well-defined $f$-dimensional vector bundle $\eta_F$ over $F$ so that the sequence

$$0 \to TY_{|F} + TY'_{|F} \to TV_{|F} \to \eta_F \to 0 \tag{9.4}$$

is an exact sequence of bundles. The intersection is transversal in $F$ if and only if the excess bundle attached to $F$ is the zero bundle.

Incidentally, excess intersections have been considered by algebraic geometers since the early 1800s; see the discussion in [39, Chap. 9].

Suppose that the connected component $F$ we deal with of the intersection $Y \cap Y'$ of the two closed oriented submanifolds of the oriented manifold $V$ is orientable. By choosing an orientation of $F$, the excess bundle $\eta_F$ is an oriented vector bundle over $F$ in a natural way. We denote the uniquely determined Euler class of the excess bundle $\eta_F$ by $e(\eta_F) \in H^f(F, \mathbb{Z})$ [103, chap. 9, 10]. By definition, the Euler number $e(\eta)_{[F]}$ of the excess bundle $\eta_F$ is the integer obtained by evaluating the Euler class on the fundamental class $o_F$ of $F$, that is,

$$\langle e(\eta_F), o_F \rangle =: e(\eta)_{[F]}. \tag{9.5}$$

In accordance with the classical theory of intersection numbers, if $F = \{x\}$ consists of just one point, we define $e(\eta)_{[F]} = 1$ if $T_xY + T_xY'$ equals $T_xV$ as oriented vector spaces and $e(\eta)_{[F]} = -1$ otherwise. In general, the Euler number of the excess bundle does not depend on the choice of the orientation of $F$.

Then the following result ([122], 3.3) holds true.

**Theorem I.** *Let $Y, Y'$ be two closed immersed oriented submanifolds of an oriented manifold $V$. Suppose that $Y, Y'$ are of complementary dimension, that is,* $\dim Y + \dim Y' = \dim V$ *and that $Y$ and $Y'$ intersect perfectly, and that the intersection is compact consisting of (necessarily finitely many) orientable connected components $F$. Then the intersection number $[Y][Y']$ can be expressed as the sum of the Euler numbers of the corresponding excess bundles, that is,*

$$[Y][Y'] = \sum e(\eta)_{[F]},$$

*where the sum ranges over the connected components $F$ of the intersection $Y \cap Y'$.*

9.3. **Intersection numbers of special geometric cycles and non-abelian Galois cohomology.** Let $\sigma, \tau$ be two $\mathbb{Q}$-rational automorphisms of $G$ of finite order which commute with one another. Let $\Theta = \langle \sigma, \tau \rangle$ denote the abelian group generated by $\sigma$ and $\tau$. As explained in Section 6.4, given a $\Theta$-stable torsion-free arithmetic subgroup $\Gamma$ of $G(\mathbb{Q})$, there is a natural action of $\Theta$ on the locally symmetric space $X/\Gamma$ as well as one of $\sigma$ and $\tau$, respectively. Recall that

for $\mu = \sigma, \tau$, the fixed point set $\mathrm{Fix}(\langle\mu\rangle, X/\Gamma)$ is a disjoint union of connected components $F(\gamma) = X(\gamma)/\Gamma(\gamma)$ with $\gamma \in H^1(\langle\mu\rangle, \Gamma)$. By using the natural map $H^1(\langle\mu\rangle, \Gamma) \longrightarrow H^1(\langle\mu\rangle, G(\mathbb{R}))$, we can view a representing cocycle $\gamma$ of $\langle\mu\rangle$ in $\Gamma$ as a representative for a class in $H^1(\langle\mu\rangle, G(\mathbb{R}))$. If two cocycles $\gamma$ and $\gamma'$ which are not equivalent give rise to the same class in $H^1(\langle\mu\rangle, G(\mathbb{R}))$, then $X(\gamma)$ is a translate of the totally geodesic Riemannian submanifold $X(\gamma')$ under an element in $G(\mathbb{R})$. In particular, $X(\gamma)$ and $X(\gamma')$, and hence $F(\gamma)$ and $F(\gamma')$, have the same dimension.

We call the connected component corresponding to the base point $1_{\langle\mu\rangle}$ in the non-abelian Galois cohomology set $H^1(\langle\mu\rangle, \Gamma)$ a *special cycle*, to be denoted $C(\langle\mu\rangle, \Gamma)$. It is worth noting that each of the connected components of $\mathrm{Fix}(\langle\mu\rangle, X/\Gamma)$, say $F(\gamma_0)$, may be viewed as a special cycle. This is achieved by replacing the automorphism $\mu$ with the automorphism obtained by twisting $\mu$ with the cocycle $\gamma_0$ of $\langle\mu\rangle$ in $\Gamma$ (see Section 6.4). Let

$$\mathrm{res}_{(\sigma,\tau)} : H^1(\Theta, \Gamma) \longrightarrow H^1(\langle\sigma\rangle, \Gamma) \times H^1(\langle\tau\rangle, \Gamma) \tag{9.6}$$

be the natural map induced by the map that assigns to a given cocycle $\gamma = (\gamma_s)$, $s \in \Theta$, the uniquely determined pair $(\gamma_\sigma, \gamma_\tau)$ of cocycles $\gamma_\sigma = (\gamma_s), s \in \langle\sigma\rangle$, and $\gamma_\tau = (\gamma_s), s \in \langle\tau\rangle$. If $\gamma = (\gamma_s), s \in \Theta$, is a cocycle representing a class in the kernel of the map $\mathrm{res}_{(\sigma,\tau)}$, then

$$X(\gamma) = X(\gamma_\sigma) \cap X(\gamma_\tau). \tag{9.7}$$

By [122, Section 1], the special cycles $C(\langle\sigma\rangle, \Gamma)$ and $C(\langle\tau\rangle, \Gamma)$ in $X/\Gamma$ intersect perfectly, and the set of connected components of this intersection is parametrized by the set $\ker \mathrm{res}_{(\sigma,\tau)}$.

As discussed in Section 6.6, by replacing $\Gamma$ with a suitable subgroup of finite index, we may suppose that $X/\Gamma$, the special cycles $C(\langle\sigma\rangle, \Gamma)$ and $C(\langle\tau\rangle, \Gamma)$ and the connected components of their intersection are oriented. In addition we suppose that the cycles are of complementary dimension and that their intersection $C(\langle\sigma\rangle, \Gamma) \cap C(\langle\tau\rangle, \Gamma)$ is compact. By Theorem I [Section 9.2] the intersection number of these two cycles is given as the sum of the Euler numbers $e(\eta)_{[F(\gamma)]}$ of the excess bundles attached to the connected components $F(\gamma)$, $\gamma \in \ker\ \mathrm{res}(\sigma, \tau)$, that is,

$$[C(\langle\sigma\rangle, \Gamma)]\ [C(\langle\tau\rangle, \Gamma)] = \sum_{\gamma \in \ker\ \mathrm{res}_{(\sigma,\tau)}} e(\eta)_{[F(\gamma)]}.$$

Subject to an additional assumption, one can replace the Euler numbers of the excess bundles by the Euler characteristics of the components $F(\gamma)$. Then one has

**Theorem J.** *Let $G$ be a connected semi-simple algebraic $\mathbb{Q}$-group, and let $\sigma, \tau$ be a pair of $\mathbb{Q}$-rational automorphisms of $G$ of finite order commuting with one another. Let $\Gamma \subset G(\mathbb{Q})$ be a $\langle\sigma, \tau\rangle$-stable torsion-free arithmetic subgroup so that the corresponding locally symmetric manifold $X/\Gamma$ is compact. Suppose that*

- *the real Lie groups $G(\mathbb{R}), G(\sigma)(\mathbb{R})$, and $G(\tau)(\mathbb{R})$ act in an orientation-preserving manner on their respective spaces $X, X(\sigma)$, and $X(\tau)$,*
- *the spaces $X(\sigma)$ and $X(\tau)$ intersect in exactly one point with positive intersection number (or, equivalently, the group $G(\langle\sigma, \tau\rangle)(\mathbb{R})$ is compact),*
- $\dim X(\sigma) + \dim X(\tau) = \dim X$.

*Then there exists a $\langle\sigma,\tau\rangle$-stable normal subgroup $\Gamma' \subset \Gamma$ of finite index so that*

$$[C(\langle\sigma\rangle,\Gamma)]\,[C(\langle\tau\rangle,\Gamma)] = \sum_{\gamma\in\ker\ \mathrm{res}_{(\sigma,\tau)}} \chi(F(\gamma)) \neq 0.$$

*Moreover, all connected components $F(\gamma)$ have the same dimension modulo* 4.

The proof of this result can be found in [122, Section 4]. However, some comments are in order.

Later on we will describe situations in which the first assumption is fulfilled. One can easily choose orientations on $X, X(\sigma)$, and $X(\tau)$, respectively, so that the intersection number of $X(\sigma)$ and $X(\tau)$ is positive. By use of an extension of Hirzebruch's proportionality principle to bundles, one can express the Euler number of the excess bundle of a given connected component of the intersection in terms of an invariant volume. More precisely, given such a component $F(\gamma) = X(\gamma)/\Gamma(\gamma)$ of dimension $f(\gamma)$, let $\omega$ be an invariant positive measure on the corresponding real Lie group $G(\gamma)(\mathbb{R})$. Then one has, as proved in [122, Prop. 4.2],

$$e(\eta)_{[F(\gamma)]} = (-1)^{f(\gamma)/2} p(\gamma) \int_{G(\gamma)(\mathbb{R})^0/\Gamma} \omega. \tag{9.8}$$

The proportionality factor $p(\gamma)$ is given as the ratio of the Euler number of a certain bundle $e(\eta_{u,F(\gamma)})$ over the compact dual space $X(\gamma)_u$ determined by the excess bundle $\eta_{F(\gamma)}$ to the volume of the compact real form $G_u(\gamma)^0$ with respect to a measure $\omega_u$ on $G_u(\gamma)(\mathbb{R})$ determined by $\omega$, i.e.,

$$p(\gamma) = e(\eta_{u,F(\gamma)})_{[X(\gamma)_u]}\,[\mathrm{vol}_{\omega_u}(G_u(\gamma)(\mathbb{R})^0]^{-1}. \tag{9.9}$$

The Euler number $e(\eta)_{[F(\gamma)]}$ vanishes if the corresponding group $G(\gamma)(\mathbb{R})^0$ does not contain a compact Cartan subgroup or if the dimension $f(\gamma)$ of the component $F(\gamma)$ is odd. These conditions resemble very much the conditions under which the Euler characteristic of an arithmetic quotient vanishes. We note that there is a similar formula for the Euler characteristic of a fixed point component $F(\gamma)$, namely,

$$\chi(F(\gamma)) = (-1)^{f(\gamma)/2}\chi(X(\gamma)_u)\,[\mathrm{vol}_{\omega_u}(G_u(\gamma)(\mathbb{R})^0]^{-1} \int_{G(\gamma)(\mathbb{R})^0/\Gamma} \omega. \tag{9.10}$$

Then, under the given two hypotheses, the Euler number of the bundle $e(\eta_{u,F(\gamma)})$ over $X(\gamma)_u$ is the Euler characteristic of $X(\gamma)_u$, that is, $e(\eta_{u,F(\gamma)}) = \chi(X(\gamma)_u)$. Thus, by comparing the formula for $e(\eta)_{[F(\gamma)]}$ with the one for $\chi(F(\gamma))$, we obtain the formula for the intersection number of the two cycles in question as stated.

In a second step, by replacing $\Gamma$ with a suitable subgroup $\Gamma'$ of finite index, we can achieve that all connected components $F(\gamma)$, $\gamma \in \ker\ \mathrm{res}_{\sigma,\tau} \subset H^1(\Theta,\Gamma')$, have the same dimension $f(\gamma)$ modulo 4 and such that $\chi(F(\gamma)) \neq 0$. Since the sign of $\chi(F(\gamma))$ is $(-1)^{f(\gamma)/2}$, all connected components $F(\gamma)$ contribute with the same sign to the intersection number. Thus $[C(\langle\sigma\rangle,\Gamma')]\,[C(\langle\tau\rangle,\Gamma')] \neq 0$.

## 10. Geometric construction of homology

This general analysis of the intersections and intersection numbers of geometric cycles allows us to construct non-vanishing (co)homology classes for the arithmetic quotients in question. In this and the following section, we exhibit some families of examples.

We start off with the work of Millson-Raghunathan [102] in which they dealt with special cycles in specific cases. Their approach differs from the one described in Section 9 in so far as their object of concern is a real Lie group, say $SO(p,q)$ or $SU(p,q)$, without specifying an underlying rational structure. Then, in order to match the various conditions as orientability et cetera, they have to do a case-by-case construction of cocompact discrete subgroups. However, our previous general results replace this discussion and go beyond.

10.1. **Real, complex and quaternionic hyperbolic $n$-space.** First, as an easy implementation of the general scheme discussed in Section 9, we resume the discussion in Section 8.3 of arithmetic quotients attached to quaternionic hyperbolic $n$-space. The situation is as follows: Let $D$ be a quaternion division algebra over a totally real algebraic number field $k \neq \mathbb{Q}$, and let $(E,h)$ be an $(n+1)$-dimensional Hermitian space over $D$. The isometry group $G$ of $(E,h)$ is a connected, simple and simply connected $k$-group. Suppose that $h$ has signature $(n,1)$ and that all its conjugates (under $\mathrm{Gal}(k/\mathbb{Q})$) are of signature $(n+1,0)$. An arithmetic torsion-free subgroup $\Gamma$ of $G(k)$ gives rise to a compact arithmetic quotient $X/\Gamma$ of $X = H^n_{\mathbb{H}}$. For a given integer $m, 1 \leq m < n$, there is a rational involution $\sigma_m : G \to G$ (see Section 8.3. for a definition). Given a rational point $x$ in the special cycle $C(\langle\sigma_m\rangle,\Gamma)$ represented by $x \in X(\langle\sigma_m\rangle)$, there is the Cartan involution $\theta_x$ corresponding to $x$. The rationality of $x$ means that $\theta_x$ is an automorphism of $G$ defined over $k$. Then $\tau_m := \sigma_m\theta_x$ is a rational involution of $G$ which commutes with $\sigma_m$. The spaces $X(\sigma_m)$ and $X(\tau_m)$ are totally geodesic submanifolds of $X$ of complementary dimensions $4m$ and $4(n-m)$, meeting only at $x$. By arranging the orientations of $X$, $X(\sigma_m)$ and $X(\tau_m)$ suitably it can be achieved that the intersection number of $X(\sigma_m)$ and $X(\tau_m)$ is positive. Thus, by Theorem J [Section 9.3], there exists a $\langle\sigma_m,\tau_m\rangle$-stable subgroup $\Gamma'$ of finite index in $\Gamma$ such that the Euler characteristics $\chi(F(\gamma)),\gamma \in \ker\ \mathrm{res}_{(\sigma_m,\tau_m)}$ of the connected components of the intersection of the special cycles $C(\langle\sigma_m\rangle,\Gamma')$ and $C(\langle\tau_m\rangle,\Gamma')$ are all positive. Hence, the cycles contribute non-trivially to cohomology. More precisely, they detect non-vanishing classes in $H^j(H^n_{\mathbb{H}}/\Gamma',\mathbb{R})$, $j = 4m, 4(n-m)$. Since the integer $m$ ranges over the set $\{1 \leq m < n\}$ we get classes in degrees $i = 4, 8, ..., 4(n-1)$ in this way. Thus, we see that this investigation of intersection numbers of geometric cycles gives another proof of Theorem H [Section 8.3].

However, Millson and Raghunathan discovered an additional feature in this approach; namely, one can pass to a subgroup $\Delta$ of finite index in $\Gamma'$ so that the fundamental classes $o_{C(\langle\sigma_m\rangle,\Delta)}$ and $o_{C(\langle\tau_m\rangle,\Delta)}$ are not dual to an invariant form on the locally symmetric space $X/\Delta$, that is, to a class in the image of the injective homomorphism

$$\beta^*_\Delta : H^*(X_u,\mathbb{R}) \longrightarrow H^*(\Omega(X,\mathbb{R})^\Delta) \xrightarrow{\sim} H^*(X/\Delta,\mathbb{R}).$$

Classes in the image of the map $\beta^*_\Delta$ are occasionally referred to as "continuous classes".

This is contained in the following general result of Millson-Raghunathan [102, Thm. 2.1] (where we essentially retain the notation of Section 9.3).

**Theorem K.** *Let $\sigma,\tau$ be two $\mathbb{Q}$-rational involutions of a semi-simple $\mathbb{Q}$-group $G$ which commute with one another, and let $\Gamma$ be a $\langle\sigma,\tau\rangle$-stable torsion-free arithmetically defined subgroup of $G$ so that the locally symmetric space $X/\Gamma$ is compact.*

*Suppose that the corresponding special cycles $C(\langle\sigma\rangle,\Gamma)$ and $C(\langle\tau\rangle,\Gamma)$ in $X/\Gamma$ are of complementary dimension and that their intersection number*

$$[C(\langle\sigma\rangle,\Gamma)]\,[C(\langle\tau\rangle,\Gamma)] \neq 0.$$

*Then there exists a subgroup $\Delta$ of finite index in $\Gamma$ so that the cohomology classes corresponding to the fundamental classes $o_{C(\langle\sigma\rangle,\Delta)}$ and $o_{C(\langle\tau\rangle,\Delta)}$ by Poincaré duality cannot be represented by a $G(\mathbb{R})$-invariant $\mathbb{R}$-valued $C^\infty$-form on $X$.*

In the case of arithmetic quotients attached to real or complex hyperbolic $n$-space, a construction analogous to the one above provides non-vanishing cohomology classes in $H^j(H^n_F/\Gamma,\mathbb{R})$, $F=\mathbb{R},\mathbb{C}$, in degrees $j=d,2d,3d,...,(n-1)d$, with $d=\dim_{\mathbb{R}}F$, $\Gamma$ a suitably chosen arithmetically defined subgroup relative to a rational structure on the real Lie group being dealt with. This latter rational structure has to be carefully chosen as well. Again, these classes are not detected by invariant forms. In Theorem C [Section 2.4], this result is stated in the case of real hyperbolic $n$-space. In this case, the arithmetic groups are standard arithmetic unit groups acting on hyperbolic $n$-space.

In the case that the Betti number $\dim_{\mathbb{R}}H^i(X/\Gamma,\mathbb{R})$ of the compact locally symmetric space $X/\Gamma$ is larger than $\dim_{\mathbb{R}}H^i(X_u,\mathbb{R})$ it can be made arbitrarily large by passing over to a subgroup of finite index. We refer to [17] for a proof in the case of congruence subgroups, and to [1] for a general argument.

10.2. **Classical groups.** This approach applies as well to groups of units of certain quadratic or Hermitian forms, defined over suitably algebraic number fields, which do not represent zero rationally over their field of definition. As described in Section 6, in particular, Theorem F, non-abelian Galois cohomology is a powerful technical aid in determining the orientability of geometric cycles, and it considerably simplifies the arguments in [102], where the case of the orthogonal group $O(p,q)$ was first studied.

For example, as in Appendix B, let $k\neq\mathbb{Q}$ be a totally real algebraic number field, $f$ a non-degenerate quadratic form on $k^n$, $n\geq 2$, and let $G=SO(f)$ be the special orthogonal group of $f$. Suppose that $f$ as a form over $k_v, v\in V_\infty$, has Witt index $p$ at some Archimedean place $v_0$ and is positive definite at all other Archimedean places. A torsion-free arithmetic subgroup $\Gamma$ of $SO(f)$ gives rise to a compact locally symmetric space $X/\Gamma$ of dimension $p(n-p)$. As an application of Theorem J [Section 9.3], we get the following result, which we give in its original form as proved (but not explicitly stated) in [102, Section 4]. We write $n=p+q$, and we assume $p\geq q$.

**Theorem L.** *There exists a uniform discrete arithmetic subgroup $\Gamma\subset SO(p,q)$ so that $H^j(X/\Gamma,\mathbb{R})$ contains a cohomology class which is not the restriction of a continuous class from $SO(p,q)$ for any integer $j$ strictly between $0$ and $pq$ and divisible by either $p$ or $q$.*

The analogous statements for $SU(p,q)$ and $Sp(p,q)$ also hold, provided $0<j<d(pq)$ and either $dp|j$ or $dq|j$, where $d=\dim_{\mathbb{R}}\mathbb{C}$ or $d=\dim_{\mathbb{R}}\mathbb{H}$, respectively.

## 11. Geometric construction of homology: exceptional groups

In this section we discuss the construction of geometric cycles and corresponding (co)homology classes for compact arithmetically defined quotients attached to some exceptional groups over algebraic number fields.

### 11.1. Geometric cycles and groups of type $G_2$.

*Composition algebras.* Let $C$ be a composition algebra over a field $k$; that is, $C$ is a finite-dimensional $k$-algebra $(C, +, \cdot)$ with identity element $e$ and endowed with a non-degenerate quadratic form $N$ which is multiplicative (or permits composition, as one says), $N(xy) = N(x)N(y)$ for all $x, y \in C$. The quadratic form is often referred to as the norm of $C$; it is already uniquely determined by the structure of $(C, +, \cdot)$ as a $k$-algebra. Let $b_C(x, y) = N(x + y) - N(x) - N(y)$, $x, y \in C$, be the associated bilinear form. By definition, a composition subalgebra $D$ of $C$ is a non-singular linear subspace $D$ of $(C, +)$ which contains $e$ and is closed under multiplication. A composition algebra $C$ is endowed with a conjugation defined by the assignment $x \mapsto \overline{x} := b_C(x, e)e - x, x \in C$. A composition subalgebra of $C$ is necessarily closed under conjugation.

As a consequence of the structure theory for composition algebras ([71, 33.17], or [145, 1.6.2]) there exist only composition algebras of dimension 1, 2, 4 or 8. Composition algebras of dimension 1 or 2 are commutative and associative; those of dimension 4 are the quaternion algebras over $k$ (these are associative but not commutative) and those of dimension 8 are neither associative nor commutative. The latter ones, to be called Cayley algebras (or octonion algebras) over $k$ can be constructed by a doubling process from a quaternion compositon subalgebra. This construction is obtained by gluing together two copies of a quaternion algebra $(Q, N_Q)$ in the following way. Let $\lambda \in k^*$ be an arbitrary non-zero element. Then $C = Q \oplus Q$, endowed with the multiplication

$$(x, y) \cdot (x', y') := (xx' + \lambda\overline{y'}y, y'x + y\overline{x'}), \tag{11.1}$$

for $x, y, x'$, and $y' \in Q$ and the quadratic form

$$N(x, y) := N_Q(x) - \lambda N_Q(y), x, y \in Q, \tag{11.2}$$

is a composition algebra of dimension 8, to be denoted $CD(Q, \lambda)$. It contains $Q$ as a composition subalgebra.

Given a Cayley algebra over $k$ there exist $a, b, c \in C$ with $N(a)N(b)N(c) \neq 0$ so that the elements $e, a, b, ab, c, ac, bc, (ab)c$ form an orthogonal basis of $C$ with respect to $b_C$. We call such a triple $(a, b, c)$ a basic triple for $(C, N)$. If we represent the norm form of $C$ with respect to this basis in coordinates, we obtain the following expression (if $\mathrm{char}(k) \neq 2$):

$$x_0^2 - \alpha x_1^2 - \beta x_2^2 + \alpha\beta x_3^2 + \gamma(x_4^2 - \alpha x_5^2 - \beta x_6^2 + \alpha\beta x_7^2)$$

with non-zero constants $\alpha, \beta$ and $\gamma$.

We note that a Cayley algebra (or, more generally, a composition algebra) is determined by its norm form; that is, two Cayley algebras $C$ and $C'$ are isomorphic if the corresponding quadratic spaces $(C, N_C)$ and $(C', N_{C'})$ are isometric. Thus, the classification of Cayley algebras over the field $k$ is reduced to the determination of equivalence classes of quadratic forms of the given type.[8]

[8] Forms of this type are also called 3-fold Pfister quadratic forms.

Let $C$ be a Cayley algebra over $k$ with norm form $N$. If $N$ is isotropic, i.e., if there exists $x \in C, x \neq 0$, with $N(x) = 0$, then $C$ contains zero divisors. In this case the norm form has maximal Witt index 4. However, in such a case, $C$ is uniquely determined up to $k$-isomorphism. A representative of this unique isomorphism class is the Cayley algebra $CD(M_2(k), -1)$, to be called the split Cayley algebra over $k$. If the norm form $N$ is anisotropic, i.e., $N(x) \neq 0$ for all $x \in C, x \neq 0$, or, equivalently, the Witt index of $N$ is 0, then each $x$ has an inverse element with respect to multiplication. In such a case, $C$ is called an octonion divison algebra.

Suppose that $C$ is a Cayley algebra over the field $k = \mathbb{C}$. Since a non-degenerate form over $C$ is isotropic, $C$ is $\mathbb{C}$-isomorphic to the split Cayley algebra over $\mathbb{C}$. If $k = \mathbb{R}$, the norm form of a Cayley algebra over $\mathbb{R}$ is isotropic or positive definite. Consequently, there are exactly two possibilities up to $\mathbb{R}$-isomorphism, the split case, and the algebra $\mathbb{O} = CD(\mathbb{H}, -1)$ of Graves-Cayley octonions, repectively.

If $k$ is a perfect field of cohomological dimension $\leq 2$, e.g., a totally imaginary number field or a $p$-adic field, then any Cayley algebra $C$ over $k$ is isomorphic to the split Cayley algebra over $k$ because the norm form of $C$ is isotropic in this case [145, 1.10].

*The automorphism group of a Cayley algebra.* Let $(C, N)$ be a Cayley algebra over an algebraic number field $k$. The group $G(k) = \mathrm{Aut}_k(C)$ of $k$-algebra automorphisms of $C$ is the group of $k$-points of a connected simple $k$-group $G$, of type $G_2$, with trivial center. Since a $k$-algebra automorphism of $C$ leaves the norm form $N$ invariant, the algebraic group $G$ is a closed subgroup of the orthogonal group $O(C, N)$ of the quadratic space $(C, N)$.

Let $\mathcal{O}_k$ denote the ring of integers of $k$. An octonion $x \in C$ is *integral* if $N(x) \in \mathcal{O}_k$ and $tr(x) = b_C(x, e) \in \mathcal{O}_k$ are integral elements in $k$. By definition, an *order* $\Lambda$ in $C$ is a subring of $C$ containing $e$ which is a finitely generated free $\mathcal{O}_k$-submodule of $C$ with $k\Lambda = C$ and which consists of integral elements. The corresponding group of $\Lambda$-units is defined by

$$G_\Lambda = \{g \in G(k) \mid g(\Lambda) = \Lambda\}.$$

It is an arithmetic subgroup, as is every subgroup $\Gamma \subset G(k)$ that is commensurable with a group of $\Lambda$-units.

*Geometric cycles in compact arithmetic quotients attached to Cayley algebras.* We now suppose that $k$ is a totally real algebraic number field of degree $d = [k : \mathbb{Q}] \geq 2$. Given a Cayley algebra $C$ over $k$, let $G(k) = \mathrm{Aut}_k(C)$ be as above and set

$$G' = \mathrm{Res}_{k/\mathbb{Q}}\, G,$$

the algebraic $\mathbb{Q}$-group obtained from $G$ by restriction of scalars as in Appendix C. There is an isomorphism $G'(\mathbb{R}) \longrightarrow \prod_{v \in V_\infty} G_v$ for the group of real points of $G'$, where $G_v = G^{\sigma_v}(k_v)$. Let $X = K \backslash G'(\mathbb{R})$, $K \subset G'(\mathbb{R})$ a maximal compact subgroup, be the associated symmetric space; it is of dimension $8m$, where $m$ denotes the number of places $v \in V_\infty$ at which $C$ is split. If $\Gamma \subset G(k)$ is a torsion-free arithmetic subgroup, the quotient $X/\Gamma$ is a compact locally symmetric space. In his thesis [157], Waldner discusses the geometric construction of cohomology classes in this case.

**Theorem M.** *Let $C$ be a Cayley algebra over a totally real algebraic number field $k$ of degree $d = [k : \mathbb{Q}] \geq 2$. Suppose that $C$ is split at one place $v_1 \in V_\infty$ and is isomorphic to the octonion algebra $\mathbb{O} = CD(\mathbb{H}, -1)$ at the remaining places $v \in V_\infty$. Let $\Gamma \subset G(k)$ be an arithmetically defined subgroup of the $k$-group $G$ attached to $\mathrm{Aut}_k(C)$. Then, possibly replacing $\Gamma$ by a subgroup $\Gamma' \subset \Gamma$ of finite index, there exist non-zero geometric cycle classes in $H^i(X/\Gamma, \mathbb{C})$ where $i = 3, 4$, and $5$. Moreover, non-zero invariant classes exist in degrees $i = 0, 4$, and $8$. The groups $H^i(X/\Gamma, \mathbb{C})$, $i = 1, 2, 6$, and $7$, all vanish.*

The geometric cycle classes arise in two ways. If $(a, b, c)$ is a basic triple for $(C, N)$, subject to the conditions in the previous subsection, then the assignments

$$\sigma : a \mapsto -a, b \mapsto b, c \mapsto c \quad \text{and} \quad \tau : a \mapsto a, b \mapsto -b, c \mapsto c \tag{11.3}$$

give rise to two rational involutions $\sigma, \tau$ on $G$. These automorphisms commute, and the corresponding groups $G(\sigma)$, $G(\tau)$ of fixed points are of type $SO(f)$, $f$ a 4-dimensional quadratic form. By a careful choice of $(a, b, c)$, one can pass over to a subgroup $\Gamma'$ and apply Theorem J [Section 9.3] to obtain special cycles $C(\sigma, \Gamma')$ and $C(\tau, \Gamma')$ with non-zero intersection number. Hence they represent non-trivial homology classes in $H_4(X/\Gamma', \mathbb{C})$.

Starting with a basic triple $(a, b, c)$ for the composition algebra $C$, in [157, Section 4], C. Waldner constructs geometric cycles $C_a(\Gamma), C_{b,a}(\Gamma)$ of dimension 5 and 3, respectively, which intersect perfectly (but not transversally), with positive intersection number. Their intersection consists of finitely many connected components of dimension 2. By use of Theorem I [Section 9.2], a computation of Euler numbers leads to the non-vanishing result as stated.

Using the fact that $H^i(X_u, \mathbb{R}) = \mathbb{R}$ for $i = 0, 4, 8$, and vanishes otherwise, we obtain invariant cohomology classes in $H^i(X/\Gamma, \mathbb{R})$ in these degrees.

Finally, the vanishing of $H^i(X/\Gamma, \mathbb{C})$, $i = 1, 2, 6$, and 7, is a consequence of the representation theory of the real Lie group of type $G_2$. This will be discussed in Section 15.

11.2. **Geometric cycles and groups of type $F_4$.** Algebraic groups of type $F_4$ arise as groups of automorphisms of exceptional simple Jordan algebras of dimension 27. The structure of these algebras, also called Albert algebras, and the corresponding automorphism groups were systematically studied by Albert and Jacobson [2, 64] and Springer [145]. We are going to discuss some results on geometric cycles in arithmetic quotients originating with arithmetically defined subgroups in algebraic groups of type $F_4$ defined over a totally real algebraic number field.

*Albert algebras.* Let $k$ be a field with $\mathrm{char}(k) \neq 2, 3$, and let $(C, N)$ be a Cayley algebra over $k$. For $A = (x_{i,j}) \in M_3(C)$, let $\overline{A} = (\overline{x}_{i,j})$, where $x \mapsto \overline{x}$ denotes conjugation in $C$. Given a diagonal matrix $\delta = \mathrm{diag}(d_1, d_2, d_3) \in GL_3(k)$, the assignment $A \mapsto \delta^{-1}\overline{A}^t\delta$ defines an involution on $M_3(C)$, to be denoted $\Delta$. We may replace $\delta$ by $\mathrm{diag}(td_1, td_2, td_3)$ for any $t \in k, t \neq 0$, without changing this involution. Let

$$\mathcal{H}(C, \Delta) = \{A \in M_3(C) \mid A = A^\Delta\} \tag{11.4}$$

be the set of all $\Delta$-Hermitian matrices in $M_3(C)$. Endowed with the usual matrix addition, scalar multiplication and with the product

$$A \cdot B = \tfrac{1}{2}(AB + BA), \tag{11.5}$$

where $AB$ is the usual matrix product in $M_3(C)$, $\mathcal{H}(C,\Delta)$ is a commutative, non-associative algebra of dimension 27 over $k$. It is a central simple algebra with the identity matrix as unit element. We call an algebra of the form $\mathcal{H}(C,\Delta)$ or a twisted form of $\mathcal{H}(C,\Delta)$ an Albert algebra. This algebra is naturally equipped with a non-degenerate quadratic form $q$, the quadratic trace, to be defined by

$$q(A) = \tfrac{1}{2}\mathrm{trace}(A^2).$$

It is worth noting that the multiplication $(A,B) \mapsto A \cdot B$ satisfies the identity $((A\cdot A)\cdot B)\cdot A = (A\cdot A)\cdot(B\cdot A)$ for all $A, B \in \mathcal{H}(C,\Delta)$. In fact, the algebra $\mathcal{H}(C,\Delta)$ and twisted forms of $\mathcal{H}(C,\Delta)$ are central simple exceptional Jordan algebras. In turn, as proved by Albert, any central simple exceptional Jordan algebra is a twisted form of $\mathcal{H}(C,\Delta)$ for some Cayley algebra $C$ over $k$. By [2, Thm. 3], two algebras $\mathcal{H}(C,\Delta)$ and $\mathcal{H}(C',\Delta')$ are isomorphic only if $C$ and $C'$ are isomorphic. Thus, one usually calls $C$ the coordinate algebra of $\mathcal{H}(C,\Delta)$. This result is supplemented by a set of necessary and sufficient conditions on $\Delta$ and $\Delta'$ under which there exists an isomorphism between $\mathcal{H}(C,\Delta)$ and $\mathcal{H}(C,\Delta')$. As a consequence, two algebras $\mathcal{H}(C,\Delta)$ and $\mathcal{H}(C',\Delta')$ with isomorphic coordinate algebras $C \cong C'$ are isomorphic if the quadratic forms $q$ on $\mathcal{H}(C,\Delta)$ and $q'$ on $\mathcal{H}(C',\Delta')$ are equivalent over $k$ [145, 5.8.1].

This result provides a constructive approach to enumerate (up to isomorphism) all Albert algebras over specific fields. Of interest to us are the cases $k = \mathbb{R}$, a local field, or an algebraic number field.

If $k = \mathbb{R}$, there are (up to isomorphism) exactly two Cayley algebras over $\mathbb{R}$, the split Cayley algebra $CD(M_2(\mathbb{R}), -1) =: C_s$ and the algebra $\mathbb{O}$ of Graves-Cayley octonions. In the former case, there is only one isomorphism class of Albert algebras whose coordinate algebra is $C_s$. A representative of this class is $\mathcal{H}(C_s,\Delta_s)$, where $\Delta_s$ is the involution given by $\delta = \mathrm{diag}(1,-1,1)$. In the latter case, there are two isomorphism classes with $\mathbb{O}$ as coordinate algebra. These classes can be represented by $\mathcal{H}(C_a,\Delta_0)$ and $\mathcal{H}(C_a,\Delta_1)$, respectively, where $\delta_0 = \mathrm{diag}(1,1,1)$ and $\delta_1 = \mathrm{diag}(1,-1,1)$ for any non-split Cayley algebra $C_a$ over $\mathbb{R}$.

Let $k$ be an algebraic number field of degree $n = s + 2t$, where $s$, resp. $t$, denotes the number of real, resp. complex, places $v \in V$ of $k$. Let $A = \mathcal{H}(C,\Delta)$ be an Albert algebra defined over $k$. Given a place $v \in V$ there is the local analogue

$$A_v = A \otimes_k k_v$$

of $A$ given as the tensor product over $k$ of $A$ with the local field $k_v$. If $v$ is a non-Archimedean place there is only one isomorphism class, the one determined by the split Cayley algebra over $k_v$. The same assertion is true if $v$ is a complex place. If $v$ is a real place, there are three isomorphism classes. As a result, there are $3^s$ different isomorphism classes of Albert algebras over $k$. Given two Albert algebras $A = \mathcal{H}(C,\Delta)$ and $A' = \mathcal{H}(C,\Delta')$ with the same coordinate algebra there are conditions on the matrices $\Delta$ and $\Delta'$ under which the algebras $A$ and $A'$ are isomorphic [2, Thm. 5].

*The automorphism group of an Albert algebra.* Let $A$ be an Albert algebra defined over some field $k$ with $\mathrm{char}(k) \neq 2, 3$. Then the group $\mathrm{Aut}_k(A)$ of $k$-algebra automorphisms of $A$ is the group of $k$-rational points of an algebraic group $G$ defined over $k$. This group is a connected simple algebraic group of type $F_4$.

Suppose that $k$ is an algebraic number field. Given an Albert algebra $A$ over $k$, the corresponding $k$-group of automorphisms is denoted by $G$. Let $G' = \mathrm{Res}_{k/\mathbb{Q}}(G)$ be the algebraic $\mathbb{Q}$-group obtained from $G$ by restriction of scalars. Then $G'(\mathbb{R})$ is isomorphic to the product of real Lie groups $G_v = G^{\sigma_v}(k_v)$ of type $F_4$, $v \in V_\infty$. According to the classification of Albert algebras over $\mathbb{R}$ [2, Section 13], the following three possibilities can occur for $G_v$ if $v \in V_\infty, v$ real:

- If $A_v = \mathcal{H}(C_s, \Delta_s)$ (where $\Delta_s$ is the involution given by $\delta = \mathrm{diag}(1, -1, 1)$) is the split Albert algebra over $\mathbb{R}$, the corresponding Lie group $G_v$ is a simple Lie group of real rank 4, to be denoted $F_{4(4)}$, following the notation of Cartan.
- If $A_v = \mathcal{H}(C_a, \Delta_1)$, where $\Delta_1$ is the involution given by $\delta = \mathrm{diag}(1, -1, 1)$ for any non-split Cayley algebra $C_a$ over $\mathbb{R}$, then the Lie group $G_v$ is of real rank 1, to be denoted $F_{4(-20)}$. Recall that $C_a$ is isomorphic to the algebra $\mathbb{O}$ of Graves-Cayley octonions.
- If $A_v = \mathcal{H}(C_a, \Delta_0)$, where $\Delta_0$ is the involution given by $\delta = \mathrm{diag}(1, 1, 1)$ for any non-split Cayley algebra $C_a$ over $\mathbb{R}$, then the Lie group $G_v$ is of real rank 0, to be denoted $F_{4(-52)}$. This group is compact.

The real Lie group of type $F_{4(4)}$ gives rise to an irreducible symmetric space of dimension 28 whereas the symmetric space corresponding to $F_{4(-20)}$ is of dimension 16. In fact, it is the Cayley hyperbolic plane [106, Section 19].

If $v \in V_\infty, v$ complex, there is only the split Cayley algebra over $\mathbb{C}$ in this case. Thus, Albert algebras over $\mathbb{C}$ form one isomorphism class, and $G_v$ is uniquely determined.

*Geometric cycles for groups of type $F_4$.* Let $k$ be a totally real algebraic number field of degree $d = [k : \mathbb{Q}] \geq 2$. Given an Albert algebra $A$ over $k$ we denote by $G'$ the algebraic $\mathbb{Q}$-group $\mathrm{Res}_{k/\mathbb{Q}}\, G$ obtained from the group $G$ of $k$-algebra automorphisms of $A$ by restriction of scalars. There is an isomorphism $G'(\mathbb{R}) \longrightarrow \prod_{v \in V_\infty} G_v$ for the group of real points of $G'$, where $G_v = G^{\sigma_v}(k_v)$. We suppose that $G$ satisfies the following assumptions.

- The real Lie group $G_{v_1}$ is isomorphic to $F_{4(-20)}$ for the place $v_1$ corresponding to $\sigma_1 = \mathrm{Id}$.
- For all other places $v \in V_\infty, v \neq v_1$, the group $G_v$ is isomorphic to the compact group $F_{4(-52)}$.

Then $G'(\mathbb{R})$ is the product of the real split simple Lie group of type $F_{4(-20)}$ of $\mathbb{R}$-rank 1 and a finite product of compact Lie groups. The associated symmetric space $X = K \backslash G'(\mathbb{R})$, $K \subset G'(\mathbb{R})$ a maximal compact subgroup, is of dimension 16. If $\Gamma \subset G(k)$ is a torsion-free arithmetic subgroup, then the quotient $X/\Gamma$ is a compact locally symmetric space.

**Proposition.** *Given an arithmetically defined subgroup $\Gamma$ in the algebraic $\mathbb{Q}$-group $\mathrm{Res}_{k/\mathbb{Q}}\, G$ obtained from the group $G$ of $k$-algebra automorphisms of the Albert algebra $A$ by restriction of scalars, its cohomology $H^*(X/\Gamma, \mathbb{R})$ contains (up to a subgroup $\Gamma' \subset \Gamma$ of finite index) a non-trivial cohomology class of degree 8. By duality, this class is detected by the fundamental class of a totally geodesic submanifold $X_H/\Gamma_H$ in $X/\Gamma$ with $H$ a reductive subgroup of $G$ defined over $k$. We note that $H_{v_1} = Spin(8, 1)$. This geometric cycle arises as a fixed point component under a suitable rational involution on the group $G$.*

Taking into account the classification of irreducible unitary representations of the real Lie group $F_{4(-20)}$, one can show that there are no cohomology classes in degrees $1, 2, 3$ and $13, 14, 15$. The $G_{v_1}$-invariant forms on $X$ account, via the injective homomorphism

$$\beta_\Gamma^* : H^*(X_u, \mathbb{R}) \longrightarrow H^*(\Omega(X, \mathbb{R})^\Gamma) \tilde{\longrightarrow} H^*(X/\Gamma, \mathbb{R}),$$

for classes in degrees $0, 8$ and $16$. The class constructed above by a geometric cycle is not in the image of the map $\beta_\Gamma^*$. As in the case of groups of type $G_2$, there are other geometric cycles in the arithmetic quotient which contribute to cohomology [133]. They originate in certain subalgebras of the Albert algebra $A$.

11.3. **Other exceptional groups.** Along the same lines, one can deal with special cycles in compact arithmetic quotients attached to exceptional groups of other types. As an application of Theorem J [Section 9.3] we find among these $k$-forms of the exceptional Lie groups (up to compact factors) $E_{6(-14)}, E_{6(-26)}, E_{7(-28)}$. This relies on the following general result in Lie theory [122, 4.7], which allows us to resolve orientability questions: Suppose that $\sigma$ is an automorphism of finite order (defined over $\mathbb{R}$) of a connected simply connected semi-simple algebraic $\mathbb{R}$-group. If the group $G(\mathbb{R})$ of real points of $G$ is simply connected as a topological group, then the group $G(\sigma)(\mathbb{R})$ of fixed points is connected. It is worth noting that all split exceptional groups as well as $E_{6(2)}, E_{7(-5)}, E_{8(-24)}$ do not fall in the class addressed in this assertion.

However, though some work has been done in the cases $E_{6(-14)}, E_{6(-26)}, E_{7(-28)}$ as well as in the case $F_{4(-20)}$, general geometric cycles still offer a variety of interesting open questions.

## 12. Modular symbols

Given a connected semi-simple algebraic $\mathbb{Q}$-group $G$ of $\mathrm{rk}_{\mathbb{Q}} G \geq 1$, let $\Gamma$ be an arithmetically defined subgroup of $G$. By Section 5.3, our assumption on the $\mathbb{Q}$-rank of $G$ implies that $X/\Gamma$ is not compact. In this section our main focus is on geometric cycles

$$j_{L|\Gamma} : X_L/\Gamma_L \longrightarrow X/\Gamma$$

which originate with a Levi subgroup $L$ of a proper parabolic $\mathbb{Q}$-subgroup $P$ of $G$. By passing over to a suitable subgroup of finite index, the fundamental class of such a non-compact cycle $C(x_0, L|\Gamma)$ detects a non-trivial class in homology with closed supports, and, by duality, a non-trivial cohomology class in $H^*(X/\Gamma, \mathbb{R})$. The method of proof for this result relies on analyzing the intersection of this cycle with a compact cycle $C(x_0, N|\Gamma)$ of complementary dimension attached to the unipotent radical $N$ of $P$. By means of Raghunathan's Theorem E [Section 6.5], we extend the scope of the original works of Ash [4] and Ash-Borel [7], at the same time, simplifying their approach.

The cycles $C(x_0, L|\Gamma)$, usually called modular symbols, are natural generalizations of the "classical" modular symbols. The latter ones whose study was initiated by Birch in his work on elliptic curves have served as an indispensable tool linking geometry and arithmetic [96].

In general, the adjunction of corners $\overline{X}/\Gamma$ of the non-compact locally symmetric space $X/\Gamma$ provides a suitable framework to understand the geometric significance of both the modular symbols and the related compact cycles.[9] Thus, we first briefly review the construction of the compactification $\overline{X}/\Gamma$ of $X/\Gamma$. In particular, we describe in some detail a single face in $\overline{X}/\Gamma$ corresponding to a $\Gamma$-conjugacy class of parabolic $\mathbb{Q}$-subgroups of $G$.[10] Which portion of the cohomology $H^*(X/\Gamma, \mathbb{R})$ is generated by modular symbols or what their arithmetic meaning is, in particular, with regard to special values of $L$-functions [54], are still open questions.

12.1. **Parabolic subgroups.** [*In this subsection we do not necessarily assume that $G$ is even a reductive algebraic $k$-group.*] A parabolic $k$-subgroup of an algebraic group $G$ is, by definition, a parabolic subgroup of the connected component $G^0$ of the identity in the Zariski topology, that is, a closed subgroup $P$ of $G^0$ such that $G^0/P$ is a projective variety. Let $\mathcal{P}(G)$ denote the set of parabolic $k$-subgroups of $G$. As usual, we denote the unipotent radical of a connected $k$-group $G$ by $R_uG$, and $R_dG$ denotes the $k$-split radical of $G$. If $G$ is not connected, then its unipotent radical (resp. split radical) is the one of $G^0$, to be denoted by the same letter as before. Recall that an algebraic $k$-group $H$ is reductive if $R_uH = \{1\}$. Any reductive $k$-subgroup of the given group $G$ is contained in a maximal one. One calls these latter groups the Levi $k$-subgroups of $G$; they are conjugate under the unipotent radical $R_uG$ of $G$. A Levi $k$-subgroup $L$ of $G$ gives rise to a decomposition $G = LR_uG$ as a semidirect product.

We choose a maximal $k$-split torus $S$ in $G^0/R_uG$, that is, $S(k) = (k^*)^n$ with $n$ maximal. All such tori are conjugate under $G(k)$. Let $\Phi_k = \Phi_k(G/R_uG, S)$ be the set of $k$-roots of $G^0/R_uG$ with respect to $S$ and let $\Delta$ be a basis of $\Phi_k$; that is, we choose a set of (positive) simple roots for $S$. This gives rise to a Dynkin diagram. By definition, the $k$-rank of $G$, to be denoted $\mathrm{rk}_kG$, is the dimension $n$ of $S$, equivalently, the number of elements in $\Delta$. The conjugacy classes with respect to $G^0(k)$ in the set $\mathcal{P}(G)$ of parabolic $k$-subgroups are in one-to-one correspondence with the subsets of $\Delta$.

Corresponding to $J \subset \Delta$ there is the class represented by the standard parabolic subgroup $P_J$. We let $S_J = (\bigcap_{\alpha\in J} \ker\alpha)^0$, and we denote the centralizer of $S_J$ by $Z(S_J)$. Then the image of $P_J$ under the projection $G \to G/R_uG$ is the semidirect product of its unipotent radical $U_J$ by $Z(S_J)$, a so-called Levi decomposition of $P_J$. The group $L_J := Z(S_J)$ is reductive, a Levi subgroup of $P_J$. Notice that the characters of $S$ in $U_J$ are exactly the positive roots which contain at least one simple root not in $J$. Since any $P$ in $\mathcal{P}(G)$ is $G^0(k)$-conjugate to a unique $P_J$, the corresponding subset $J \subset \Delta$ is called the type of $P$, to be denoted $J(P)$. The groups $P_J$ are called the standard parabolic $k$-subgroups of $G$ determined by the choice of $S$ and the set $\Delta$ of simple roots.

Let $P$ be a parabolic $k$-subgroup of $G$ with unipotent radical $R_u(P)$. We denote by $R_d(P)$ the $k$-split radical of $P$. Then the quotient $S_P = R_d(P)/R_u(P)R_dG$

[9] We refer to [43] for a comprehensive survey of this and other compactifications of $X/\Gamma$ in the case of modular varieties. These include the Baily-Borel compactification, the family of toroidal compactifications and the reductive Borel-Serre compactification.

[10] In the case of classical modular curves $X_0(N)$, this amounts to understanding the cusps. Modular symbols are 1-dimensional cycles, namely, certain paths in the Riemann surface which connect cusps.

is a $k$-split torus in $P$. Let $A_P$ denote the identity component $S_P(\mathbb{R})^0$ of $S_P(\mathbb{R})$; $A_P$ is called the split component of $S_P$. If $Q$ is another parabolic $k$-subgroup of $G$ with $Q \subset P$, the inclusion $R_d(P) \subset R_d(Q)$ induces an injective morphism of $S_P$ into $S_Q$. This gives rise to a decomposition $A_Q = A_{Q,P} \times A_P$, where $A_{Q,P} = (\bigcap_{\alpha \in \Delta - J(P)} \ker\alpha) \cap A_Q$.

As usual we put, given a connected algebraic $k$-group $H$, ${}^0H = \bigcap_{\chi \in X_k(H)} \ker\chi^2$. The group ${}^0H$ is normal in $H$ and is defined over $k$. Let $S_H$ be a maximal $k$-split torus of the radical of $H$. Then we have $H(\mathbb{R}) = S_H(\mathbb{R})^0.{}^0H(\mathbb{R})$ as a semi-direct product for the group of real points of $H$. The group ${}^0H(\mathbb{R})$ contains every compact subgroup of $H(\mathbb{R})$, and, if $k = \mathbb{Q}$, every arithmetic subgroup of $H$. In the case of a given parabolic subgroup $P$ of $G$ with Levi subgroup $L$, we obtain $L(\mathbb{R}) = {}^0L(\mathbb{R})A_P$, and thus also $P(\mathbb{R}) = {}^0L(\mathbb{R})A_P R_u(P)(\mathbb{R})$. The latter decomposition is called a Langlands decomposition of $P$.

12.2. **Corners.** For the sake of simplicity we suppose that $G$ is a connected reductive algebraic group defined over $\mathbb{Q}$ with positive $\mathbb{Q}$-rank $\mathrm{rk}_{\mathbb{Q}}G$. Let $\Gamma$ be a torsion-free arithmetic subgroup of $G$, and let $P$ be a proper parabolic $\mathbb{Q}$-subgroup of $G$. The associated homogeneous space $X$ is of the form $A_G K \setminus G(\mathbb{R})$ with $K$ a maximal compact subgroup of $G(\mathbb{R})$ and $A_G$ the identity component of the group of real points of a maximal $\mathbb{Q}$-split torus in the center of $G$. It is a principal $A_P$-bundle under the geodesic action ([22, Section 3]). Indeed, the group $P(\mathbb{R})$ also acts transitively on $X$; hence $X = K_P A_G \setminus P(\mathbb{R})$, where $K_P = K \cap P(\mathbb{R})$. Moreover, $K_P$ is contained in the unique $\theta$-stable Levi subgroup $L \subset P$ (where $\theta$ is the Cartan involution corresponding to $K$). So the center of $L(\mathbb{R})$ commutes with $K_P A_G$ and it therefore acts (from the left) on $X$, and hence it passes to an action of $A_P$ on $X$. This is the geodesic action of Borel and Serre. By definition, the corner $X(P)$ associated to $P$ is the total space $X(P) = \overline{A_P} \times_{A_P} X$ of the associated bundle with fibre $\overline{A_P}$. By use of the identification $A_P \xrightarrow{\sim} (\mathbb{R}_+^*)^m$, $m := \#|\Delta - J(P)|$, $\overline{A_P}$ denotes the closure of $A_P$ in $\mathbb{R}^m$. For a given $Q \in \mathcal{P}(G)$, we put $e(Q) = A_Q \backslash X$. Note that $X = e(G^0)$. Then the adjunction of corners $\overline{X}$ is the disjoint union of the sets $e(P)$, $P \in \mathcal{P}(G)$. In particular, we have

$$X(P) = \coprod_{P \subset Q} e(Q). \tag{12.1}$$

Furthermore, given $P, Q \in \mathcal{P}(G)$, $X(P) \cap X(Q) = X(R)$ with $R$ the smallest parabolic $\mathbb{Q}$-subgroup of $G$ which contains $P$ and $Q$. There exists a uniquely determined structure as a manifold with corners on $\overline{X}$ so that $X(P)$, $P \in \mathcal{P}(G)$, is an open submanifold with corners of $\overline{X}$. The action of $\Gamma$ on $X$ extends to a proper action on $\overline{X}$. The faces $e(P)$, $P \in \mathcal{P}(G)$, are permuted under this action. The quotient $\overline{X}/\Gamma$ is a compact manifold with corners [22, Thm. 9.3], homotopically equivalent to its interior $X/\Gamma$. We denote by $e'(P)$ the image of $e(P)$ under the natural projection $\overline{X} \longrightarrow \overline{X}/\Gamma$. Two faces $e'(P)$ and $e'(Q)$ intersect non-trivially if and only if $e'(P) = e'(Q)$. In such a case the corresponding parabolic subgroups are conjugate under $\Gamma$. There are only finitely many $\Gamma$-conjugacy classes of elements in $\mathcal{P}(G)$. Thus the boundary $\partial(\overline{X}/\Gamma)$ is the disloint union of a finite number of faces $e'(P)$ which correspond bijectively to the set $\mathcal{P}(G)/\Gamma$.

The closure of the face $e'(P)$ in $\overline{X}/\Gamma$ is of the form

$$CL_{\overline{X}/\Gamma}(e'(P)) = \coprod_{Q\in\mathcal{P}(P)/\Gamma\cap P} e'(Q). \tag{12.2}$$

In other words, there is a neighborhood in $\overline{X}$ for which $\Gamma$-equivalence and $(\Gamma\cap P)$-equivalence coincide. This assertion is an essential consequence of reduction theory.

The closures of the faces form a closed cover of $\partial(\overline{X}/\Gamma)$ whose nerve is the quotient under the action of $\Gamma$ of the Tits building $\mathcal{T}$ of proper parabolic $\mathbb{Q}$-subgroups of $G$. The maximal parabolic $\mathbb{Q}$-subgroups of $G$ are the vertices of $\mathcal{T}$. The Leray spectral sequence attached to the nerve of this covering abuts to the cohomology of the boundary $\partial(\overline{X}/\Gamma)$. This latter cohomology appears naturally in the long exact cohomology sequence of the pair $(\overline{X}, \partial(\overline{X}/\Gamma))$. Therefore, it makes sense to analyze the families of restriction maps

$$r^*_{G,P} : H^*(\overline{X}/\Gamma, E) \longrightarrow H^*(e'(P), E) \tag{12.3}$$

and their interplay. This relates to an investigation of the Leray spectral sequence.

12.3. **Construction of modular symbols.** Now we are in a position to describe, given a proper parabolic $\mathbb{Q}$-subgroup in $G$, the construction of related geometric cycles in the corresponding non-compact arithmetic quotient $X/\Gamma$. Their significance is best understood in the framework of the adjunction of corners $\overline{X}/\Gamma$. As before, $P$ denotes a parabolic $\mathbb{Q}$-subgroup of $G$, $N = R_u(P)$ its unipotent radical and $L$ a Levi subgroup of $P$. Let $S_P$ be a maximal $\mathbb{Q}$-split torus in the center of $L$ and $A_P = S_P(\mathbb{R})^0$ the identity component of $S_P(\mathbb{R})$. Let $\Delta(P, A_P)$ be a set of simple elements of the set $\Phi(P, A_P)$ of roots of $P$ with respect to $A_P$. An element $a \in A_P$ is said to be "far out" (in the quotient $\overline{X}/\Gamma$) if the elements in $\Delta(P, A_P)$ have sufficiently large values on $a$. Given $t \in \mathbb{R}, t > 0$, we write $A_{P,t} := \{a \in A_P \mid \alpha(a) \geq t \text{ for all } \alpha \in \Delta(P, A_P)\}$. We choose a point $x_0 \in X$, that is, a corresponding maximal compact subgroup $K$ in $G(\mathbb{R})$, such that $x_0$ is fixed under a maximal compact subgroup $K_L$ of $L(\mathbb{R})$. Then, as defined in section 6, we have closed embeddings

$$N(\mathbb{R}) \longrightarrow X,$$

defined by $n \mapsto x_0 an$, $n \in N(\mathbb{R})$, for a given $a \in A_P$, and

$$X_L \longrightarrow X,$$

induced by $h \mapsto x_0 h$, $h \in L(\mathbb{R})$, respectively. These give rise to natural maps

$$j_{N,a|\Gamma} : N(\mathbb{R})/N(\mathbb{R}) \cap \Gamma \longrightarrow X/\Gamma \tag{12.4}$$

and

$$j_{L|\Gamma} : X_L/\Gamma_L \longrightarrow X/\Gamma. \tag{12.5}$$

Notice that the submanifolds $x_0 aN(\mathbb{R})$ and $x_0 X_L$ have complementary dimensions in $X$ and intersect only at $x_0$. By Theorem E [Section 6.5], we can find a subgroup $\Gamma'$ of finite index in $\Gamma$ so that the corresponding maps $j_{N,a|\Gamma'}$ and $j_{L|\Gamma'}$ are injective.

However, in the case of the map $j_{N,a|\Gamma}$ we can do slightly better if the given arithmetic group $\Gamma$ is neat. In such a case, there exists $t_0 \in \mathbb{R}, t_0 \geq 0$, so that for all $a \in A_t$ with $t \geq t_0$ the map $j_{N,a|\Gamma}$ is injective. This can be derived from reduction theory, as done in [7], 1.2. It relies on the fact that "at infinity" close to the face $e(P)$ the action of $\Gamma$ on $X$ can be replaced by the one of $\Gamma \cap P$.

Still, in general, we may conclude the following result.

**Theorem N.** *There exists a torsion-free arithmetic group $\Gamma$ in $G$ so that*

(a) *the images of the manifolds $N(\mathbb{R})/(N(\mathbb{R}) \cap \Gamma)$ and $X_L/\Gamma_L$ under the respective maps $j_{N,a|\Gamma}$ and $j_{L|\Gamma}$ are closed oriented submanifolds in $X/\Gamma$, the former compact, the latter non-compact, and*

(b) *these submanifolds, to be denoted $C(x_0a, N|\Gamma)$ and $C(x_0, L|\Gamma)$, respectively, intersect transversally at finitely many points, each with intersection number one.*

*The second assertion, that is, the non-vanishing of the intersection product follows again from reduction theory,* [7]*, 2.5. There are the following consequences of this result in terms of the fundamental classes in homology and cohomology.*

First, the fundamental class $[x_0a, N]_\Gamma$ of the compact submanifold $C(x_0a, N|\Gamma)$ is not homologous to zero, in singular homology. By going over to suitable subgroups $\Gamma'$ of finite index in $\Gamma$ one may get an arbitrarily large number of linearly independent classes of this type in $H_{d_N}(X/\Gamma, \mathbb{R})$, where $d_N$ is the dimension of the cycle $C(x_0a, N|\Gamma)$. Second, in the case of the non-compact cycle $C(x_0, L|\Gamma)$ attached to a Levi subgroup $L$ of $P$, its fundamental class $[x_0, L]_\Gamma$ is non-trivial, in homology with closed supports, and, by duality, gives rise to a non-trivial cohomology class in $H^{d(G)-d_L}(X/\Gamma, \mathbb{R})$, where $d(G) = \dim X$ and $d_L$ is the dimension of the cycle. Usually one calls the cycles $C(x_0, L|\Gamma)$ "modular symbols" of type $L$.[11] Essentially, the cohomology classes in $H^*(X/\Gamma, \mathbb{R})$ attached to modular symbols $C(x_0, L|\Gamma)$, $L$ a Levi subgroup of a proper parabolic $\mathbb{Q}$-subgroup of $G$, have a non-trivial image under the restriction map $H^*(\overline{X}/\Gamma, \mathbb{R}) \longrightarrow H^*(\partial(\overline{X}/\Gamma), \mathbb{R})$ to the cohomology of the boundary of the compactification $\overline{X}/\Gamma$ of $X/\Gamma$. More precisely, they restrict non-trivially under the restriction map

$$r^*_{G,P} : H^*(\overline{X}/\Gamma, E) \longrightarrow H^*(e'(P), E). \tag{12.6}$$

When the given group $P$ is a minimal parabolic $\mathbb{Q}$-subgroup of $G$, $L$ a Levi subgroup of $P$, then the corresponding cycle $C(x_0, L|\Gamma)$ is of dimension $\mathrm{rk}_{\mathbb{Q}}G$. By [5], the dual classes of modular symbols of this type, to be called minimal modular symbols, generate the cohomology group $H^{cd(\Gamma)}(X/\Gamma, \mathbb{R})$ in degree $\mathrm{cd}(\Gamma) = \dim X - \mathrm{rk}_{\mathbb{Q}}G$, the cohomological dimension of $\Gamma$. However, it is important for arithmetic applications to have a finite set of generators for $H^{cd(\Gamma)}(X/\Gamma, \mathbb{R})$ in terms of minimal modular symbols. In specific cases as, for example, $G = \mathrm{Res}_{k/\mathbb{Q}}(SL_n)$ or $\mathrm{Res}_{k/\mathbb{Q}}(Sp_n)$, $k$ an algebraic number field with a Euclidean ring of integers, finite spanning sets are exhibited in [10] and [51]. The algorithmic approach taken there is closely tied up with questions in the geometry of numbers.

With regard to the general question to which extent and in which way modular symbols contribute non-trivially to the cohomology groups $H^*(X/\Gamma, E)$ the case of minimal parabolic subgroups is quite exceptional. First, by the construction of cuspidal automorphic representations and related cohomology classes for arithmetic subgroups of $GL_n$ over an algebraic number field ([80], resp. [21]), there are non-vanishing results for the cohomology in degrees which cannot be detected by any modular symbol of type $L$, $L$ a Levi subgroup of $G$. Second, modular symbols of a different type can contribute to cohomology in the same cohomological degree.

[11] Some authors use this notion even in cases where $L$ is replaced by an arbitrary reductive $\mathbb{Q}$-subgroup of $G$. We prefer to stick to the narrow sense.

# Part IV
# Towards automorphic forms

## 13. Automorphic forms and cohomology of arithmetic groups

Let $\Gamma$ be a torsion-free arithmetic subgroup of a semi-simple algebraic $\mathbb{Q}$-group $G$. As seen in Section 5, the lifting of differential forms by the projection $G(\mathbb{R})/\Gamma \longrightarrow X/\Gamma$ induces an isomorphism of the space $\Omega^*(X,E)^\Gamma$ of $\Gamma$-invariant smooth $E$-valued forms on $X$ and the cochain complex $C^*(\mathfrak{g}, K, C^\infty(G(\mathbb{R})/\Gamma)_K \otimes E)$ of relative Lie algebra cohomology. Thus, there is a canonical isomorphism

$$H^*(X/\Gamma, \tilde{E}) = H^*(\Omega(X,E)^\Gamma) \tilde{\longrightarrow} H^*(\mathfrak{g}, K, C^\infty(G(\mathbb{R})/\Gamma)_K \otimes E).$$

If $V$ is a subspace of $C^\infty(G(\mathbb{R})/\Gamma)$, stable under $\mathfrak{g}$ and $K$, the inclusion $V \otimes E \to C^\infty(G(\mathbb{R})/\Gamma) \otimes E$ induces a homomorphism

$$j_V : H^*(\mathfrak{g}, K, V \otimes E) \longrightarrow H^*(\mathfrak{g}, K, C^\infty(G(\mathbb{R})/\Gamma)_K \otimes E).$$

This map has been studied for various choices of $V$. It is particularly interesting to replace $C^\infty(G(\mathbb{R})/\Gamma)_K$ by a smaller space without altering the cohomology.

The following result [36] pertaining to the $(\mathfrak{g}, K)$-module $\mathcal{A}(G,\Gamma) \to C^\infty(G/\Gamma)_K$ of automorphic forms with respect to $\Gamma$ (as defined in Section 13.3) is decisive for the close relation between geometry and the theory of automorphic representations.

**Theorem.** *In the case of a congruence subgroup $\Gamma \subset G(\mathbb{Q})$ there are isomorphisms in cohomology*

$$H^*(\mathfrak{g}, K, \mathcal{A}(G,\Gamma) \otimes E) \tilde{\to} H^*(\mathfrak{g}, K, C^\infty(G(\mathbb{R})/\Gamma)_K \otimes E) \tilde{\to} H^*(X/\Gamma, \tilde{E}).$$

The $(\mathfrak{g}, K)$-module $\mathcal{A}(G,\Gamma)$ of automorphic forms permits a decomposition (as a direct sum of $(\mathfrak{g}, K)$-modules) along their cuspidal support. This gives rise to a decomposition

$$H^*(X/\Gamma, E) = \bigoplus_{\{P\} \in \mathcal{C}} H^*(\mathfrak{g}, K, \mathcal{A}(G,\Gamma)_{\{P\}} \otimes E),$$

where the sum ranges over the set $\mathcal{C}$ of classes of associate parabolic $\mathbb{Q}$-subgroups of $G$. The summand indexed by the associate class $\{G\}$ of the full group is called the *cuspidal cohomology*. The other summands are called the *Eisenstein cohomology*. Thus, in the obvious notation, one has a sum decomposition of the cohomology of $\Gamma$,

$$H^*(X/\Gamma, E) = H^*_{cusp}(X/\Gamma, E) \oplus H^*_{Eis}(X/\Gamma, E),$$

into the subspace of classes represented by cuspidal automorphic forms and the Eisenstein cohomology [38]. Each summand in the Eisenstein cohomology is built up by Eisenstein series or residues of such attached to suitable cuspidal automorphic forms on the Levi components of the elements in $\{P\}$. The space generated by these automorphic forms is denoted by $\mathcal{A}(G,\Gamma)_{\{P\}}$. For a precise definition of these spaces, we refer to [36] or [38].

For an arbitrary arithmetic group $\Gamma \subset G(\mathbb{Q})$ there is the following approach to the notion of cuspidal cohomology. The space $L^2(G(\mathbb{R})/\Gamma)$ of complex-valued square-integrable functions on $G(\mathbb{R})/\Gamma$, viewed as a unitary $G(\mathbb{R})$-module under left translations, contains as a $G(\mathbb{R})$-invariant subspace the space $L^2_{cusp}(G(\mathbb{R})/\Gamma)$ of

cuspidal automorphic forms [58]. This space is a direct Hilbert sum of irreducible unitary representations $(\pi, H_\pi)$ of $G(\mathbb{R})$ with finite multiplicities

$$L^2_{cusp}(G/\Gamma) = \widehat{\bigoplus_{\pi\in\hat{G}}} m_{cusp}(\pi,\Gamma)\ H_\pi.$$

Up to infinitesimal equivalence, there are only finitely many irreducible unitary representations $(\pi, H_\pi)$ with non-zero cohomology $H^*(\mathfrak{g}, K, H_\pi \otimes E)$ for a given coefficient system $E$. Thus, $H^*(\mathfrak{g}, K, L^2_{cusp}(G(\mathbb{R})/\Gamma)_K \otimes E)$ decomposes as a finite direct algebraic sum of spaces $H^*(\mathfrak{g}, K, H_\pi \otimes E)$. It turns out that the map

$$j_{cusp} : H^*(\mathfrak{g}, K, L^{2,\infty}_{cusp}(G(\mathbb{R})/\Gamma)_K \otimes E) \longrightarrow H^*(\mathfrak{g}, K, C^\infty(G(\mathbb{R})/\Gamma)_K \otimes E)$$

is injective. The image of $j_{cusp}$, to be denoted by $H^*_{cusp}(X/\Gamma, E)$, will be called the cuspidal cohomology of $X/\Gamma$. In the case of a congruence subgroup it coincides with the cuspidal cohomology of $\Gamma$ as defined above. It is the injectivity of the map $j_{cusp}$ which relates questions about the cuspidal spectrum $L^2_{cusp}(G(\mathbb{R})/\Gamma)$ of $\Gamma$ to questions on the cohomology of this group [131]. The theory of irreducible unitary representations of the underlying real Lie group $G(\mathbb{R})$ is an indispensable tool in this study.

If $G(\mathbb{R})/\Gamma$ is compact, we may replace the map $j_{cusp}$ by the map $j_V$ with $V = L^2(G(\mathbb{R})/\Gamma)$. We obtain that $j_V$ is an isomorphism, and this gives rise to a decomposition of the cohomology of $\Gamma$,

$$H^*(X/\Gamma, \tilde{E}) \tilde{\longrightarrow} \bigoplus_{\pi\in\hat{G}} m(\pi,\Gamma)\ H^*(\mathfrak{g}, K, H_{\pi,K} \otimes E),$$

as a finite algebraic sum, a result originally due to Matsushima [95]; see below.

13.1. **A result of Matsushima.** Suppose $G$ is a real reductive Lie group with finitely many connected components, $K \subset G$ is a maximal compact subgroup and $\Gamma \subset G$ is a torsion-free discrete subgroup so that the quotient $G/\Gamma$ is compact. In that case the left regular representation of $G$ on the space $L^2(G/\Gamma)$ of square integrable functions (modulo the center) on $G/\Gamma$ decomposes as a direct Hilbert sum of irreducible unitary representations $(\pi, H_\pi)$ of $G$ with finite multiplicities

$$L^2(G/\Gamma) = \widehat{\bigoplus_{\pi\in\hat{G}}} m(\pi,\Gamma)\ H_\pi. \tag{13.1}$$

Here $\hat{G}$ denotes the unitary dual of $G$, and the multiplicity $m(\pi,\Gamma)$ with which $(\pi, H_\pi)$ occurs in $L^2(G/\Gamma)$ is a non-negative integer for each $\pi$. Given such an irreducible unitary Hilbert space representation $(\pi, H_\pi)$ the space $H_{\pi,K}$ of all $C^\infty$-vectors $v \in H_\pi$ such that $\pi(K)v$ spans a finite-dimensional subspace of $H_\pi$ carries a natural $(\mathfrak{g}, K)$-module structure (see Appendix F for this notion and related material in representation theory). By a result of Harish-Chandra, since $(\pi, H_\pi)$ is irreducible, $H_{\pi,K}$ is irreducible as a $(\mathfrak{g}, K)$-module. The space $H_{\pi,K}$ of $K$-finite vectors is dense in the space $H^\infty_\pi$ of $C^\infty$-vectors for $H_\pi$, and there is an inclusion

$$\widehat{\bigoplus_{\pi\in\hat{G}}} m(\pi,\Gamma)\ H_{\pi,K} \longrightarrow C^\infty(G/\Gamma)_K. \tag{13.2}$$

Let $(\nu, E)$ be a finite-dimensional irreducible real or complex representation of $G$. Then this inclusion induces an isomorphism

$$H^*(K\backslash G/\Gamma, \tilde{E}) = H^*(\mathfrak{g}, K, C^\infty(G/\Gamma)_K \otimes E) \xrightarrow{\sim} \bigoplus_{\pi\in\hat{G}} m(\pi,\Gamma)\ H^*(\mathfrak{g}, K, H_{\pi,K}\otimes E),$$

where the right-hand side is a finite direct algebraic sum. This fundamental result is due to Matsushima [95]. The representations which can possibly contribute to the sum on the right-hand side are usually called representations with non-vanishing (Lie algebra) cohomology. In view of an observation of D. Wigner, the cohomology $H^*(\mathfrak{g}, K, H_{\pi,K}\otimes E)$ can only be non-zero if the center of the enveloping algebra of $\mathfrak{g}$ acts on $H_\pi \otimes E$ as in the trivial representation. As a consequence, given $(\nu, E)$, there are (up to infinitesimal equivalence) only finitely many irreducible representations $(\pi, H_\pi)$ of $G$ with non-vanishing relative Lie algebra cohomology $H^*(\mathfrak{g}, K, H_{\pi,K} \otimes E)$. Only those $(\pi, H_\pi)$ might occur whose infinitesimal character $\chi_\pi$ coincides with the one of the contragredient representation of $(\nu, E)$. Thus, cohomology isolates a finite set (depending on $(\nu, E)$) of irreducible unitary representations $(\pi, H_\pi)$ of $G$ occurring in the spectrum $L^2(G/\Gamma)$ with finite multiplicities.

If one drops the assumption that the quotient $G/\Gamma$ is compact, these results are not true any more. However, by replacing the coefficient module $C^\infty(G/\Gamma)$ by appropriate spaces of functions which satisfy certain growth conditions, they hold in a modified form.

13.2. **Cuspidal cohomology and square-integrable cohomology.** For the sake of simplicity, we suppose that our real Lie group $G$ is the group of real points of a connected semi-simple algebraic $\mathbb{Q}$-group of $\mathbb{Q}$-rank not zero and that $\Gamma$ is an arithmetically defined subgroup of this group. In this case, the space $L^2(G/\Gamma)$ of complex-valued square-integrable functions, viewed as a unitary $G$-module via left translations, is a direct sum of the discrete spectrum $L^2_{dis}(G/\Gamma)$ and the continuous spectrum $L^2_{cont}(G/\Gamma)$ as it is called. The latter space is a Hilbert direct sum of continuous integrals each of which is a continuous sum of unitarily induced representations. The discrete spectrum $L^2_{dis}(G/\Gamma)$ decomposes into a direct Hilbert sum of irreducible unitary representations $(\pi, H_\pi)$ of $G$ with finite multiplicities

$$L^2_{dis}(G/\Gamma) = \widehat{\bigoplus_{\pi\in\hat{G}}} m(\pi,\Gamma)\ H_\pi. \tag{13.3}$$

Let $L^2_{cusp}(G/\Gamma) \subset L^2_{dis}(G/\Gamma)$ be the $G$-invariant subspace of cuspidal automorphic forms for $G$ with respect to $\Gamma$ as, for example, defined in [58]. The inclusion of the space of $C^\infty$-vectors in the discrete spectrum and the inclusion of the cuspidal spectrum into $C^\infty(G/\Gamma)$ induce natural homomorphisms

$$j_{dis} : H^*(\mathfrak{g}, K, L^{2,\infty}_{dis}(G/\Gamma)_K \otimes E) \longrightarrow H^*(\mathfrak{g}, K, C^\infty(G/\Gamma)_K \otimes E) \tag{13.4}$$

and

$$j_{cusp} : H^*(\mathfrak{g}, K, L^{2,\infty}_{cusp}(G/\Gamma)_K \otimes E) \longrightarrow H^*(\mathfrak{g}, K, C^\infty(G/\Gamma)_K \otimes E). \tag{13.5}$$

By definition, the cuspidal cohomology $H^*_{cusp}(X/\Gamma, E)$ of $X/\Gamma$ is the image of the homomorphism $j_{cusp}$. This map is injective, and the decomposition of the cuspidal spectrum in irreducible subspaces $H_\pi$, analogous to the one of $L^2_{dis}(G/\Gamma)$, yields a

decomposition of the cuspidal cohomology as a finite algebraic sum

$$H^*_{cusp}(X/\Gamma, \tilde{E}) \tilde{\longrightarrow} \bigoplus_{\pi \in \hat{G}} m_{cusp}(\pi, \Gamma)\ H^*(\mathfrak{g}, K, H_{\pi,K} \otimes E). \tag{13.6}$$

This relation serves as a gateway to a rich interplay between the cuspidal automorphic spectrum of an arithmetically defined subgroup $\Gamma$ of an algebraic $\mathbb{Q}$-group and the cohomology of the corresponding arithmetic quotient $X/\Gamma$. We refer to [131], where an expository account of some results that illustrate this rich interaction is given.

The homomorphism $j_{dis}$ is not injective in general. However, the image of $j_{dis}$ in $H^*(X/\Gamma, E) = H^*(\Omega^*(X, E)^\Gamma)$ is the subspace spanned by classes which can be represented by square-integrable forms. One calls this subspace the square-integrable cohomology of $X/\Gamma$, to be denoted $H^*_{(2)}(X/\Gamma, E)$. This notion of square-integrable cohomology of an arithmetically defined group differs from the general concept of $L^2$-cohomology of locally symmetric manifolds of finite volume. Nevertheless, in our case, the latter one is either infinite dimensional or equal to $H^*_{(2)}(X/\Gamma, E)$.

One should notice that the interior cohomology $H^*_!(X/\Gamma, E)$, defined as the image of the cohomology with compact support under the natural map (see Section 5.3), contains the cuspidal cohomology and is contained in $H^*_{(2)}(X/\Gamma, E)$.

13.3. **Cohomology and spaces of automorphic forms.** In 1971, Harder initiated in the case of the group $SL_2$ over some number field $k$ (more generally, for groups of $k$-rank 1) a program in order to understand those phenomena in the cohomology $H^*(X/\Gamma, E) = H^*(\Omega^*(X, E)^\Gamma)$ which are due to the non-compactness of the locally symmetric space $X/\Gamma$. In other words, he dealt with the problem to describe a natural complement to the interior cohomology $H^*_!(X/\Gamma, E)$ in $H^*(X/\Gamma, E)$. By use of Langlands' theory of Eisenstein series, Harder constructed in this case a complement to the interior cohomology $H^*_!(X/\Gamma, E)$ as a subspace of $H^*(X/\Gamma, E) = H^*(\Omega^*(X, E)^\Gamma)$ whose elements are obtained by taking either analytic continuation of suitable Eisenstein series or residues of such [52, 53]. This result, that every cohomology class in $H^*(X/\Gamma, E)$ can be represented by an automorphic form, pointed at deep conjectural relations between the cohomology of an arithmetic group and the theory of automorphic forms in the general case.

We briefly review the notion of an automorphic form in our context. Let $U(\mathfrak{g})$ be the universal enveloping algebra over $\mathbb{C}$. A function $f \in C^\infty(G/\Gamma)$ is said to be of moderate growth if for a given norm $\| \quad \|$ on $G$ there are constants $c, r \in \mathbb{R}, c, r > 0$ such that $|f(g)| \leq r \parallel g \parallel^c$ for all $g \in G$. By definition, a $C^\infty$-function $f : G/\Gamma \mapsto \mathbb{C}$ is of uniform moderate growth if $f$ and all its derivatives $Df$, $D \in U(\mathfrak{g})$, are of moderate growth with the same exponent. Then it is possible, following Borel's regularization theorem, to replace the $(\mathfrak{g}, K)$-module $C^\infty(G/\Gamma)_K$ in

$$H^*(K\backslash G/\Gamma, \tilde{E}) = H^*(\mathfrak{g}, K, C^\infty(G/\Gamma)_K \otimes E) \tag{13.7}$$

by the smaller space $C^\infty_{umg}(G/\Gamma)$ of functions of uniform moderate growth without altering the cohomology.

A $C^\infty$-function $f : G \mapsto \mathbb{C}$ is an automorphic form (with respect to $\Gamma$) if

- $f$ is invariant under right translation by $\Gamma$,
- $f$ is of uniform moderate growth,
- $f$ is $K$-finite, that is, the left translates of $f$ under $K$ span a finite-dimensional vector space,

- $f$ is $Z(\mathfrak{g})$-finite, that is, the functions $z \cdot f$, where $z$ runs through the elements in the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$, span a finite-dimensional space.

The space $\mathcal{A}(G,\Gamma)$ of automorphic forms with respect to $\Gamma$ carries in a natural way a $(\mathfrak{g},K)$-module structure [20, Section 2]

As proved by Franke [36] in the case of congruence subgroups of the underlying algebraic $\mathbb{Q}$-group, the inclusion $\mathcal{A}(G,\Gamma) \longrightarrow C^{\infty}_{umg}(G/\Gamma)$ induces an isomorphism in cohomology

$$H^*(\mathfrak{g},K,\mathcal{A}(G,\Gamma)\otimes E) \longrightarrow H^*(\mathfrak{g},K,C^{\infty}_{umg}(G/\Gamma)_K \otimes E). \tag{13.8}$$

He essentially proved that there is a sum decomposition of the cohomology of $\Gamma$,

$$H^*(X/\Gamma,E) = H^*_{cusp}(X/\Gamma,E) \oplus H^*_{Eis}(X/\Gamma,E), \tag{13.9}$$

into the subspace of classes represented by cuspidal automorphic forms and the so-called Eisenstein cohomology. More precisely, in terms of Lie algebra cohomology, one has the decomposition

$$H^*(X/\Gamma,E) = \bigoplus_{\{P\}\in\mathcal{C}} H^*(\mathfrak{g},K;\mathcal{A}(G,\Gamma)_{\{P\}}\otimes E), \tag{13.10}$$

where the sum ranges over the set $\mathcal{C}$ of classes of associate parabolic $\mathbb{Q}$-subgroups of $G$. The summand indexed by the associate class $\{G\}$ of the full group is precisely the cuspidal cohomology. Each summand in the Eisenstein cohomology is built up by Eisenstein series or residues of such attached to suitable cuspidal automorphic forms on the Levi components of the elements in $\{P\}$. The space generated by these automorphic forms is denoted by $\mathcal{A}(G,\Gamma)_{\{P\}}$. We refer to [36] or [38]. In the latter article, a refined decomposition of the Eisenstein cohomology has been given by taking into account the cuspidal support of the Eisenstein series involved in the construction of the corresponding class. One should observe that the geometric situation which is encoded in the adjunction of corners $\overline{X}/\Gamma$ very much resonates in this decomposition of the space of automorphic forms and the corresponding one in cohomology.

## 14. Unitary representations with non-vanishing cohomology: construction and examples

Given a compact locally symmetric space $K\backslash G/\Gamma$ as in Section 13.1, the decomposition of $H^*(X/\Gamma,E)$ as a finite algebraic sum of cohomology spaces $H^*(\mathfrak{g},K,H_{\pi,K}\otimes E)$ (or the analogous decomposition of the cuspidal cohomology in the general case in Section 13.2) leads to the problem of determining (up to infinitesimal equivalence) all irreducible unitary representations $(\pi,H_\pi)$ of $G$ with non-vanishing Lie algebra cohomology. We briefly review in this section the constructive approach to this classification problem due to Vogan–Zuckerman [154]. An outgrowth of this is the computation of the cohomology spaces $H^*(\mathfrak{g},K,H_{\pi,K}\otimes E)$. Various vanishing results are consequences of this classification. One of these is the vanishing of the cuspidal cohomology with respect to a coefficient system given by an irreducible representation $(\nu,E)$ with regular highest weight if the degree is outside a range of length rk $G-$ rk $K$ centered around the middle dimension $(1/2)\dim X$.

We conclude this section by making explicit the classification in the cases where $G$ is the exceptional split real Lie group of type $G_2$ or a special orthogonal group $SO(n,1)$ of real rank one. This enumeration allows us to interpret the geometric

construction of non-vanishing cohomology classes (in Sections 10.1 and 11.1) as an existence result pertaining to specific automorphic forms in these cases.

*In this section, $G$ denotes a connected real reductive Lie group, $K \subset G$ a maximal compact subgroup. Write $\mathfrak{g}$ for the Lie algebra of $G$, and write $\mathfrak{g}_{\mathbb{C}}$ for its complexification. We denote by $W_G$ (resp. $W_K$) the Weyl group of $G$ (resp. $K$).*

14.1. **The classification up to infinitesimal equivalence.** Suppose $(\nu, F)$ is an irreducible finite-dimensional representation of $G$. We briefly recall the Vogan–Zuckerman classification [154] of irreducible unitary representations $(\pi, H_\pi)$ of $G$ with non-vanishing cohomology

$$H^*(\mathfrak{g}, K, H_\pi \otimes F)$$

with coefficients in $H_\pi \otimes F$. A consequence of this classification is the computation of these cohomology groups. Their constructive approach is algebraic in nature. Given such a representation $(\pi, H_\pi)$, we denote the Harish-Chandra module of $H_\pi$ (i.e., the set of $K$-finite vectors in the space of $C^\infty$-vectors of $H_\pi$; see Appendix F) by the same letter or by $H_{\pi,K}$.

If this cohomology space does not vanish, then, by [23, I, 4.1], the representation $(\pi, H_\pi)$ has the same infinitesimal character as the contragredient of $F$. Thus, implied by [23, I, 5.3], there is only a finite set of infinitesimal equivalence classes of such representations $(\pi, V_\pi)$ with non-zero cohomology with respect to $F$.

Let $\theta_K$ be the Cartan involution corresponding to the maximal compact subgroup $K \subset G$, and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the corresponding Cartan decomposition. For simplicity we may assume that $G$ is connected. Otherwise, by use of [23, I, 5.1], the general case can be reduced to this one. Given an irreducible unitary representation $(\pi, H_\pi)$ of $G$ with non-vanishing cohomology with respect to $F$, there is a $\theta_K$-stable parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}$. By definition, $\mathfrak{q}$ is a parabolic subalgebra of $\mathfrak{g}_{\mathbb{C}}$ such that $\theta_K \mathfrak{q} = \mathfrak{q}$, and $\overline{\mathfrak{q}} \cap \mathfrak{q} = \mathfrak{l}_{\mathbb{C}}$ is a Levi subalgebra of $\mathfrak{q}$, where $\overline{\mathfrak{q}}$ refers to the image of $\mathfrak{q}$ under complex conjugation with respect to the real form $\mathfrak{g}$ of $\mathfrak{g}_{\mathbb{C}}$. Write $\mathfrak{u}$ for the nilradical of $\mathfrak{q}$. Then $\mathfrak{l}_{\mathbb{C}}$ is the complexification of a real subalgebra $\mathfrak{l}$ of $\mathfrak{g}$. The normalizer of $\mathfrak{q}$ in $G$ is connected since $G$ is, and it coincides with the connected Lie subgroup $L$ of $G$ with Lie algebra $\mathfrak{l}$. Then $F/\mathfrak{u}F$ is a one-dimensional unitary representation of $L$. Write $-\lambda : \mathfrak{l} \to \mathbb{C}$ for its differential. Via cohomological induction, the data $(\mathfrak{q}, \lambda)$ determine a unique irreducible unitary representation $A_{\mathfrak{q}}(\lambda)$ of $G$ so that the Harish-Chandra module of $(\pi, H_\pi)$ is equivalent to the one of $A_{\mathfrak{q}}(\lambda)$.

It is worth noting that the Levi subgroup $L$ has the same rank as $G$, is preserved by the Cartan involution $\theta_K$, and the restriction of $\theta_K$ to $L$ is a Cartan involution. Moreover, the group $L$ contains a maximal torus $T \subset K$. This result [70, Chap. V] serves as a guideline to construct all possible $\theta_K$-stable parabolic subalgebras $\mathfrak{q}$ in $\mathfrak{g}$ up to conjugation by $K$. There are only finitely many $K$-conjugacy classes of $\theta_K$-stable parabolic subalgebras $\mathfrak{q}$ in $\mathfrak{g}$.

If the infinitesimal character of the contragredient of $F$ coincides with the one of a given irreducible unitary representation $A_{\mathfrak{q}}(\lambda)$, we have

$$H^j(\mathfrak{g}, K, A_{\mathfrak{q}}(\lambda) \otimes F) = \mathrm{Hom}_{L \cap K}(\wedge^{j-R}(\mathfrak{l} \cap \mathfrak{p}), \mathbb{C}), \tag{14.1}$$

where $R = R(\mathfrak{q}) := \dim(\mathfrak{u} \cap \mathfrak{p}_{\mathbb{C}})$. In particular, note that the Lie algebra cohomology with respect to the representation $A_{\mathfrak{q}}(\lambda)$ vanishes in degrees below $\dim(\mathfrak{u} \cap \mathfrak{p}_{\mathbb{C}})$ and above $\dim(\mathfrak{u} \cap \mathfrak{p}_{\mathbb{C}}) + \dim(\mathfrak{l} \cap \mathfrak{p}_{\mathbb{C}})$. If the infinitesimal characters of the contragredient

representation of $F$ and $A_{\mathfrak{q}}(\lambda)$ do not match, we have

$$H^j(\mathfrak{g}, K, A_{\mathfrak{q}}(\lambda) \otimes F) = 0$$

for all $j$.

We summarize some properties of the irreducible unitary representations $A_{\mathfrak{q}}(\lambda)$. The representation $A_{\mathfrak{q}}(\lambda)$ is a discrete series representation if and only if the group $L$ is compact. It is a tempered representation[12] if and only if $L$ contains no non-compact simple factors. If the group $G$ contains a compact Cartan subgroup, then these two conditions are equivalent.

Suppose that $G$ has discrete series representations or, equivalently, suppose $\mathrm{rk}(G) = \mathrm{rk}(K)$. We denote by $G_{d,F}$ the set of equivalence classes of irreducible discrete series representations of $G$ whose infinitesimal character coincides with the one of the contragredient representation $F^*$. This set contains exactly $|W_G/W_K|$ elements, where $W_G$ denotes the Weyl group of $G$, $W_K$ the one of $K$. If $V$ denotes the $(\mathfrak{g}, K)$-module of $K$-finite vectors in one of these discrete series representations, then $H^j(\mathfrak{g}, K, V \otimes F) = \mathbb{C}$ if $j = (1/2)\dim(K\backslash G)$ and $H^j(\mathfrak{g}, K, V \otimes F)$ vanishes otherwise. This simple result on the cohomology of a discrete series representation reflects the basic role these representations play in the representation theory of real reductive Lie groups. The cohomology of the trivial representation of $G$ (with trivial coefficients $F$) is a much more complicated issue. It is given as the cohomology of the compact dual of the symmetric space which corresponds to the group $G$; see Appendix E and Section 7.

14.2. **Some vanishing results in cohomology.** One consequence of the Vogan–Zuckerman classification is the following result regarding the Lie algebra cohomology of an irreducible unitary tempered representation of a real Lie group.

We need the following general definition: Suppose that $M$ is a real Lie group with finitely many connected components, and let $K_M$ be a maximal compact subgroup of $M$. Assume that the Lie algebra of $M$ is reductive. We denote by $X_M = K_M\backslash M$ the corresponding symmetric space of maximal compact subgroups; its dimension is given as $2q(M) := \dim M - \dim K_M$. Set $\ell_0(M) := \mathrm{rk}(M) - \mathrm{rk}(K_M)$, where rk denotes the absolute rank, and write

$$q_0(M) := \tfrac{1}{2}\,(2q(M) - \ell_0(M)) = \tfrac{1}{2}\,(\dim X_M - \ell_0(M)).$$

This value $q_0(M)$ is an integer since the dimension and the absolute rank of a reductive Lie algebra are congruent modulo 2.

Then there is the following result regarding the Lie algebra cohomology of an irreducible unitary tempered representation of $M$ [90, Prop. 4.4] :

**Proposition.** *Suppose that $F$ is an irreducible finite-dimensional representation of $M$. Let $(\pi, H_\pi)$ be an irreducible unitary tempered representation of $M$ whose infinitesimal character coincides with that of $F^*$. Then $H^j(\mathfrak{m}, K, H_{\pi,K} \otimes F) = 0$ if $j \notin [q_0(M), q_0(M) + \ell_0(M)]$.*

This purely representation-theoretical result has various applications in our context. We mention two of them at this point, both relying on the same basic fact in the representation theory of real reductive groups. The first one is a vanishing

[12] We refer to [158] for the notion of temperedness. An irreducible unitary tempered representation is equivalent to a summand of a representation induced from a discrete series representation of a Levi subgroup of a proper parabolic subgroup; see [83].

result for the cuspidal cohomology of arithmetically defined groups. We retain the context of subsection 13.2.

**Proposition.** *Suppose that $G$ is a connected reductive algebraic $\mathbb{Q}$-group so that $\mathrm{rk}_{\mathbb{Q}} G > 0$, and suppose that the highest weight of a given finite-dimensional representation $(\nu, E)$ of $G(\mathbb{R})$ is regular. Then the cuspidal cohomology $H^*_{cusp}(X/\Gamma, E)$ of a torsion-free arithmetically defined subgroup $\Gamma \subset G$ vanishes outside the range $[q_0(G(\mathbb{R})), q_0(G(\mathbb{R})) + \ell_0(G(\mathbb{R}))]$.*

This result is a consequence of the fact that an irreducible unitary representation of $G(\mathbb{R})$ with non-vanishing Lie algebra cohomology with respect to the coefficient system given by $(\nu, E)$ is necessarily tempered if the highest weight of $(\nu, E)$ is regular (see [90, Prop. 4.2 and 5.2] or [127, Prop. 2.2]).

*Remark.* It is proved in [90] under the same regularity condition for the highest weight of the representation $(\nu, E)$ that the Eisenstein cohomology groups $H^j_{Eis}(X/\Gamma, E)$ vanish as well in degrees $j < q_0(G(\mathbb{R}))$. Thus, one has a corresponding vanishing result for $H^j(X/\Gamma, E)$ in these degrees as well.

14.3. **The constant $r_G$.** We consider an irreducible unitary representation $A_{\mathfrak{q}}(\lambda)$ of $G$ with non-vanishing cohomology $H^*(\mathfrak{g}, K, A_{\mathfrak{q}}(\lambda) \otimes F)$ with respect to the finite-dimensional representation $F$ of $G$. Suppose that $A_{\mathfrak{q}}(\lambda)$ is not the trivial representation. We note that, by the actual computation of the cohomology, $H^j(\mathfrak{g}, K, A_{\mathfrak{q}}(\lambda) \otimes F)$ vanishes for $j < R(\mathfrak{q}) = \dim(\mathfrak{u} \cap \mathfrak{p}_{\mathbb{C}})$. It is worth remarking that whenever $\mathfrak{q}$ is a proper $\theta_K$-stable parabolic subalgebra of $\mathfrak{g}$, $\dim(\mathfrak{u} \cap \mathfrak{p}_{\mathbb{C}}) \geq \mathrm{rk}_{\mathbb{R}} \mathfrak{g}$. We denote by $r_G$ the minimum of the values $R(\mathfrak{q}) = \dim(\mathfrak{u} \cap \mathfrak{p}_{\mathbb{C}})$ taken over all proper $\theta_K$-stable subalgebras $\mathfrak{q} = \mathfrak{l}_{\mathbb{C}} \oplus \mathfrak{u}$ of $\mathfrak{g}$. Clearly one has a vanishing result in degrees below $r_G$. For complex groups, the constant $r_G$ is tabulated in [34]. If $G$ is simple over $\mathbb{R}$ (and $G$ has no complex Lie group structure), a tabulation in a case-by-case list is given in [154], Section 8. In both cases, it is not clear if one could derive a formula of the constant $r_G$ in terms of the structure of the Lie algebra $\mathfrak{g}$. In general, it is greater than or equal to the split rank of $G$. It coincides with the split rank for $G = SL_n(\mathbb{R}), SU(p,q), SO(p,q)$ or $Sp_{2n}(\mathbb{R})$. We note that these vanishing results can be considerably sharpened if, for example, the given finite-dimensional representation $(\nu, F)$ of $G$ has regular highest weight.

14.4. **Examples.** By making explicit the construction of Vogan-Zuckerman discussed above, we indicate in a series of examples which irreducible unitary representations of $G$ have non-vanishing relative Lie algebra cohomology and may actually contribute to the cohomology of arithmetic groups.

*The Lie group $G_2$.* Let $G$ be the split simple real Lie group of type $G_2$. (We refer to Section 11.1 for a thorough treatment of groups of type $G_2$.) It is a connected group of $\mathrm{rk}_{\mathbb{R}} G = 2$. The Weyl group of $G$ is isomorphic to the dihedral group $D_6$ of order 12. Let $K$ be a maximal compact subgroup of $G$; its Lie algebra is isomorphic to $\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$. Let $\theta = \theta_K$ be the Cartan involution, and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the corresponding Cartan decomposition. Fix non-zero elements $ix, iy$ in $\mathfrak{k}$, the first one belonging to the first summand, the second to the second, and let $i\mathfrak{t}$ be the real vector space spanned by $ix, iy$. Then $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{k}$, and $\mathfrak{t}_{\mathbb{C}} \cong \mathbb{C}^2$. We denote the evaluation in the first and second coordinates by $e_1$ and $e_2$, respectively, and we write $\alpha_1 = e_2 - e_1$ and $\alpha_2 = 3e_1 - e_2$. Taking $\alpha_i, i = 1, 2,$

as simple roots, the set $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ of positive roots of $\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathfrak{t}_{\mathbb{C}}$ is given as the set

$$\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) = \Delta^+(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) \cup \Delta^+(\mathfrak{p}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}), \tag{14.2}$$

where

$$\Delta^+(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) = \{\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2\}, \quad \Delta^+(\mathfrak{p}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) = \{\alpha_1, \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\}.$$

Starting off from an element $z \in \mathfrak{t}$, there is an associated $\theta$-stable parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}_{\mathbb{C}}$ with Levi decomposition $\mathfrak{q} = \mathfrak{l}_{\mathbb{C}} \oplus \mathfrak{u}_{\mathbb{C}}$ defined by $\mathfrak{q}_{\mathbb{C}}$ = sum of non-negative eigenspaces of $ad(z)$, $\mathfrak{l}_{\mathbb{C}}$ = centralizer of $z$, and $\mathfrak{u}_{\mathbb{C}}$ = sum of positive eigenspaces of $ad(z)$. Let $\lambda$ be the differential of a unitary character of $L$, the connected subgroup of $G$ with Lie algebra $\mathfrak{l}_{\mathbb{C}} \cap \mathfrak{g}$, such that $\langle \alpha, \lambda_{|\mathfrak{t}_{\mathbb{C}}} \rangle \geq 0$ for each root $\alpha$ of $\mathfrak{u}$ with respect to $\mathfrak{t}_{\mathbb{C}}$. One refers to such a one-dimensional representation $\lambda : \mathfrak{l}_{\mathbb{C}} \longrightarrow \mathbb{C}$ as an admissible character. A pair $(\mathfrak{q}, \lambda)$ of a $\theta$-stable parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}_{\mathbb{C}}$ and an admissible character $\lambda$ determines a unique irreducible unitary representation $A_{\mathfrak{q}}(\lambda)$ of $G$ with non-vanishing cohomology with respect to a suitable finite-dimensional representation $(\nu, F)$ of $G$.

Up to infinitesimal equivalence, if $\mathfrak{l}_{\mathbb{C}} \subset \mathfrak{k}_{\mathbb{C}}$, one obtains discrete series representations, and there are exactly three of them up to infinitesimal equivalence having the same infinitesimal character for a given admissible character $\lambda$. Recall that this number is generally given as the ratio $|W_G/W_K|$.

Consider two elements $z_j \in \mathfrak{t}$, $j = 1, 2$, with $\alpha_j(z_j) > 0$ and $\alpha_k(z_j) = 0$ for $k \neq j$. We denote the corresponding $\theta$-stable parabolic subalgebra as constructed by $\mathfrak{q}_j, j = 1, 2$. The connected subgroup $L_j$, $j = 1, 2$, is isomorphic to $SL_2(\mathbb{R}) \times U(1)$. Let $\lambda : \mathfrak{l}_j \longrightarrow \mathbb{C}$ be an admissible character. Then the corresponding irreducible unitary representation $A_{\mathfrak{q}_j}(\lambda)$ of $G$ is non-tempered. We summarize this classification result in the case of the coefficient system given by the trivial representation $F = \mathbb{C}$; that is, the admissible character $\lambda$ is trivial.

**Proposition.** *Let $G$ be the split simple real Lie group of type $G_2$, $\mathfrak{g}$ its Lie algebra, and $K \subset G$ a maximal compact subgroup. Up to infinitesimal equivalence there are exactly the following irreducible unitary representations $(\pi, H_\pi)$ of $G$ with non-vanishing relative Lie algebra cohomology $H^*(\mathfrak{g}, K, H_\pi \otimes \mathbb{C})$:*

- *the trivial representation $\pi = \mathbb{C}$ with*

$$H^j(\mathfrak{g}, K, \mathbb{C}) = \begin{cases} \mathbb{C}, & \text{if } j = 0, 4, 8, \\ 0, & \text{otherwise}; \end{cases}$$

- *three discrete series representations $\pi_i, i = 1, 2, 3,$ with*

$$H^j(\mathfrak{g}, K, H_{\pi_i} \otimes \mathbb{C}) = \begin{cases} \mathbb{C}, & \text{if } j = 4, \\ 0, & \text{otherwise}; \end{cases}$$

- *the two non-tempered representations $A_{\mathfrak{q}_i}(0)$, $i = 1, 2$, as constructed above with*

$$H^j(\mathfrak{g}, K, A_{\mathfrak{q}_i}(0) \otimes \mathbb{C}) = \begin{cases} \mathbb{C}, & \text{if } j = 3, 5, \\ 0, & \text{otherwise.} \end{cases}$$

*The Lie group* $SO(n,1)$. Let $G = SO_0(n,1)$ be the connected component of the identity of the group of isometries of hyperbolic $n$-space. Its Lie algebra $\mathfrak{g}$ can be identified with the algebra $\mathfrak{so}(n,1)$ of real matrices of order $n+1$ which are skew symmetric with respect to the form $f$ of Witt index 1 represented by the matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & E_{n-1} & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Let $\theta : \mathfrak{g} \longrightarrow \mathfrak{g}$ be the Cartan involution defined by the assignment $Y \mapsto fYf$, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the corresponding Cartan decomposition, and let $K$ be the maximal compact subgroup of $G$ with Lie algebra $\mathfrak{k}$. Fix a maximal torus $T \subset K$, its centralizer $H := C_G(T)$ is a Cartan subgroup of $G$. The Lie algebra $\mathfrak{h}$ has a Cartan decomposition $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$ with $\mathfrak{a}$ the centralizer of $T$ in $\mathfrak{p}$.

The complex Lie algebra $\mathfrak{g}_{\mathbb{C}}$ is isomorphic to $\mathfrak{so}(n+1,\mathbb{C})$. Thus, we have to distinguish two cases given by the parity of $n$. If $n = 2k$ is even, then $G$ contains a compact Cartan subgroup, and the root system of $G$ is of type $B_k$. If $n = 2k-1$ is odd, the root system has type $D_k$.

In the first case, that is, $n = 2k$, the possible $\theta$-stable parabolic subalgebras $\mathfrak{q}$ in $\mathfrak{g}$ containing $\mathfrak{t}_{\mathbb{C}}$ can be represented up to $K$-conjugacy by the elements in the set $\{\mathfrak{q}_0, \ldots, \mathfrak{q}_{k-1}, \mathfrak{q}_k^{\pm}\}$, where, given $\mathfrak{q} = \mathfrak{l}_{\mathbb{C}} \oplus \mathfrak{u}$, the corresponding Levi subgroup $L$ is isomorphic to

$$\begin{aligned} L_0 &= SO_0(2k,1), \\ L_i &= SO(2)^i \times SO_0(2(k-i),1), \quad 0 < i < k, \\ L_k &= SO(2)^k. \end{aligned}$$

Note that there are two $\theta$-stable parabolic subalgebras in this set which both give rise to the same $L_k$.

**Proposition.** *Let $(\nu, F)$ be an irreducible finite-dimensional representation of $G = SO_0(n,1)$, $n = 2k$, with highest weight $\lambda$. If $(\pi, H_\pi)$ is an irreducible unitary representation of $G$ with non-zero cohomology with respect to $F$, then the Harish-Chandra module of $(\pi, H_\pi)$ is one of the representations*

$$A_{\mathfrak{q}_i}(\lambda^i), \quad i = 0, \ldots, k-1,$$

*or*

$$A_{\mathfrak{q}_k^{\pm}}(\lambda^k)$$

*with $\lambda^i_{|\mathfrak{t}_{\mathbb{C}}} = \lambda$.*

*We have the following non-vanishing result:*

$$H^j(\mathfrak{g}, K, A_{\mathfrak{q}_i}(\lambda^i) \otimes F) = \begin{cases} \mathbb{C}, & \text{if } j = i, n-i, \\ 0, & \text{otherwise}; \end{cases}$$

$$H^j(\mathfrak{g}, K, A_{\mathfrak{q}_k^{\pm}}(\lambda^k) \otimes F) = \begin{cases} \mathbb{C}, & \text{if } j = k, \\ 0, & \text{otherwise}. \end{cases}$$

In the second case, that is, $n = 2k-1$, the possible $\theta$-stable parabolic subalgebras $\mathfrak{q}$ in $\mathfrak{g}$ containing $\mathfrak{t}_{\mathbb{C}}$ can be represented up to $K$-conjugacy by the elements in the

set $\{\mathfrak{q}_0, \ldots, \mathfrak{q}_{k-1}\}$ where, given $\mathfrak{q} = \mathfrak{l}_{\mathbb{C}} \oplus \mathfrak{u}$, the corresponding Levi subgroup $L$ is isomorphic to

$$\begin{aligned} L_0 &= SO_0(2k-1,1), \\ L_i &= SO(2)^i \times SO_0(2(k-i)-1,1), \quad 0 < i < k, \\ L_k &= SO(2)^{k-1} \times SO_0(1,1). \end{aligned}$$

**Proposition.** *Let $(\nu, F)$ be an irreducible finite-dimensional representation of $G = SO_0(n,1)$, $n = 2k-1$, with highest weight $\lambda$. Suppose that $\lambda$ satisfies the obvious symmetry condition $\lambda_{k-1} = \lambda_k$ with respect to a suitably chosen basis of the weight lattice in $\mathfrak{h}^*$. If $(\pi, H_\pi)$ is an irreducible unitary representation of $G$ with non-zero cohomology with respect to $F$, then the Harish-Chandra module of $(\pi, H_\pi)$ is one of the representations*

$$A_{\mathfrak{q}_i}(\lambda^i), \quad i = 0, \ldots, k-1,$$

*with $\lambda^i_{|\mathfrak{t}_{\mathbb{C}}} = \lambda$.*

*We have the following non-vanishing result:*

$$H^j(\mathfrak{g}, K, A_{\mathfrak{q}_i}(\lambda^i) \otimes F) = \begin{cases} \mathbb{C}, & \text{if } j = i, n-i, \\ 0, & \text{otherwise.} \end{cases}$$

In both cases, these representations can be described by their corresponding Langlands data $(P, \sigma, \nu)$ according to the classification of irreducible admissible representations of real reductive groups [83]. A given representation $A_{\mathfrak{q}}(\lambda)$ with data $(P, \sigma, \nu)$ is the unique irreducible unitary quotient of the principal series representation $\mathrm{Ind}_P^G(H_\sigma \otimes \mathbb{C}_{\rho_P + \nu})$, $P$ the standard minimal parabolic subgroup of $G$ with Langlands decomposition $P = MAN$, $\sigma$ a suitable irreducible representation of $M = SO(n)$, $\nu$ a uniquely determined character of $A$. We refer to [44] for a description of these Langlands data, following the general outline given in [154].

## 15. Geometric constructions versus automorphic forms

The geometric construction of (co)homology classes for arithmetically defined subgroups $\Gamma$ of a given reductive group over an algebraic number field has some important consequences for the automorphic spectrum of the group $\Gamma$. First of all, one can of think of various results concerning the existence of automorphic forms with specific properties. These will come up in the disguise of the related automorphic representation generated by all translates of a given automorphic form by elements of the group $G(\mathbb{R})$ of real points of $G$. In turn, in the case of a compact arithmetic quotient $X/\Gamma$, if there is a way to construct automorphic representations occurring with non-vanishing multiplicity in the spectrum which at the same time contribute to the cohomology of $X/\Gamma$ one is naturally led to questions about possible geometric objects corresponding to these "automorphic" classes. We substantiate these close relations between geometry and automorphic theory by discussing some examples. In view of the richness of this interplay and the various results which are beyond the scope of this survey, the account we give can only be very selective in its choices and can only touch upon the most salient features of this interaction. In particular, the global approach in the theory of automorphic forms via representations of the group $G$ over the adèles of $k$, the related interpretation of the arithmetic quotients and the arithmetic involved will not be discussed.

15.1. **Arithmetically defined locally symmetric spaces of real rank one.** We resume the discussion in Sections 8.3 and 10.1 on arithmetic quotients attached to real, complex and quaternionic hyperbolic $n$-space.

Let $k \neq \mathbb{Q}$ be a totally real algebraic number field of degree $d$. Let $\sigma_1, ..., \sigma_d$ denote the $d$ distinct field embeddings $k \to \mathbb{C}$. All of these factor through $k \to \mathbb{R}$. Consider a non-degenerate quadratic $k$-space $(E, f)$ of dimension $n+1$ where, in a suitable basis, the form $f$ is given by

$$f(x) = \sum_{1 \leq j \leq n+1} \mu_j x_j^2.$$

Let $(E^{\sigma_i}, f^{\sigma_i})$ denote the quadratic space whose underlying vector space is the real vector space $E^{\sigma_i}$ obtained as the tensor product $E \otimes_k \mathbb{R}$ via the embedding $\sigma_i : k \to \mathbb{R}$ and which is endowed with the quadratic form

$$f^{\sigma_i} = \sigma_i(\sum_{1 \leq j \leq n+1} \mu_j x_j^2).$$

Suppose that the signature of this form is $(n, 1)$ for $i = 1$, and that the form is positive definite otherwise. Let $G = SO(f)$ be the special orthogonal group of $f$ and set $G' = \mathrm{Res}_{k/\mathbb{Q}} G$; see Appendix C. A torsion-free arithmetic subgroup $\Delta$ of $SO(f)$ gives rise to a compact quotient $X/\Delta$ of hyperbolic $n$-space $X = H^n_{\mathbb{R}}$. As in the case discussed in Section 10.1, given an integer $m, 1 \leq m < n$, there is a rational involution $\sigma_m : G \longrightarrow G$ with its companion $\tau_m = \sigma_m \theta_x$, $x \in X(\langle \sigma_m \rangle)$ which give rise to special cycles $C(\langle \sigma_m \rangle, \Delta)$ and $C(\langle \tau_m \rangle, \Delta)$ of complementary dimension in $X/\Delta$. By passing over to a suitable subgroup $\Gamma$ of finite index in $\Delta$, their intersection number is positive. Thus, the cohomology classes corresponding to the fundamental classes $o_{C(\langle \sigma_m \rangle, \Gamma)}$ and $o_{C(\langle \tau_m \rangle, \Gamma)}$ by Poincaré duality are non-trivial. Moreover, since the compact dual of hyperbolic $n$-space is the $n$-sphere $S^n$, these classes cannot be represented by a $G'(\mathbb{R})$-invariant $\mathbb{R}$-valued $C^\infty$-form on $X$.

Interpreting the cohomology spaces $H^*(X/\Gamma, \mathbb{R})$ in terms of relative Lie algebra cohomology we have the isomorphism

$$H^*(X/\Gamma, \mathbb{R}) \tilde{\longrightarrow} \bigoplus_{\pi \in \hat{G}} m(\pi, \Gamma)\, H^*(\mathfrak{g}, K, H_{\pi,K}),$$

where the right-hand side is a finite direct algebraic sum. It ranges over the infinitesimal equivalence classes of irreducible unitary representations $(\pi, H_\pi)$ of $SO_0(n, 1)$ with non-vanishing cohomology $H^*(\mathfrak{g}, K, H_{\pi,K})$. In view of the enumeration of these representations in Section 14.4, given an integer $m, 1 \leq m < n$, there is exactly one representation $A_{\mathfrak{q}_m}(0) =: A_{\mathfrak{q}_m}$ with $H^m(\mathfrak{g}, K, A_{\mathfrak{q}_m}) \neq 0$, except in the even case $n = 2k$ where for $m = k$ we have two equivalence classes of such representations. These latter representations belong to the discrete series. Indeed we have in the notation of Section 14.4,

$$H^j(\mathfrak{g}, K, A_{\mathfrak{q}_m} \otimes \mathbb{C}) = \begin{cases} \mathbb{C}, & \text{if } j = m, n-m, \\ 0, & \text{otherwise;} \end{cases}$$

$$H^j(\mathfrak{g}, K, A_{\mathfrak{q}_k^\pm} \otimes \mathbb{C}) = \begin{cases} \mathbb{C}, & \text{if } j = k = m, \\ 0, & \text{otherwise.} \end{cases}$$

As a consequence, the non-vanishing of the cohomology classes corresponding to the special cycles $C(\langle\sigma_m\rangle,\Gamma)$, $1 \le m < n$, implies a non-vanishing of the multiplicities $m(A_{\mathfrak{q}_m},\Gamma)$ with which the irreducible representations $A_{\mathfrak{q}_m}$ occur in the spectrum $L^2(G'(\mathbb{R})/\Gamma)$ of $\Gamma$. Millson-Raghunathan [102] mention this implication though in a far less detailed form.

We can discuss in a similar way the case of arithmetic quotients of quaternionic hyperbolic $n$-space $H^n_{\mathbb{H}}$ dealt with in Section 10.1. We retain the notation used there. Given an integer $m$, $1 \le m < n$, there exists a $\langle\sigma_m,\tau_m\rangle$-stable subgroup $\Gamma$ of $G$ such that the Euler characteristics $\chi(F(\gamma))$, $\gamma \in \ker \mathrm{res}_{(\sigma_m,\tau_m)}$ of the connected components of the intersection of the special cycles $C(\langle\sigma_m\rangle,\Gamma)$ and $C(\langle\tau_m\rangle,\Gamma)$ are all positive. These cycles contribute non-trivially to cohomology. More precisely, they detect non-vanishing classes in $H^j(H^n_{\mathbb{H}}/\Gamma,\mathbb{R})$, $j = 4m, 4(n-m)$ which cannot be represented by a $G'(\mathbb{R})$-invariant $\mathbb{R}$-valued $C^\infty$-form on $H^n_{\mathbb{H}}$. Since the integer $m$ ranges over the set $\{1 \le m < n\}$ we get classes in degrees $i = 4, 8, ..., 4(n-1)$ in this way. However, in the case of the real Lie group $Sp(n,1)$, the enumeration of the infinitesimal equivalence classes of irreducible unitary representations $(\pi, H_\pi)$ of $Sp(n,1)$ with non-vanishing cohomology $H^*(\mathfrak{sp}(n,1),K,H_{\pi,K})$ shows the following result [86]: Given a degree $j$ with $0 \le j < n$ there is no irreducible unitary representation with non-zero cohomology in this degree if $j$ is odd. If $j$ is even, there exists (up to infinitesimal equivalence) exactly one representation with non-zero cohomology in degree $j$. In degrees $j$ with $n \le j \le 2n$ there are exactly one or two equivalence classes of irreducible unitary representations $(\pi, H_\pi)$ with $H^j(\mathfrak{sp}(n,1),K,H_{\pi,K}) \neq 0$ depending on whether $j$ is odd or even. Thus, the special cycles as constructed don't detect all possible degrees in which the arithmetic quotient can possibly have non-vanishing cohomology. Indeed, by using theta series, Li [86, Section 6] constructed "automorphic" cohomology classes in some of the missing degrees. The question arises if there is a way to have a geometric counterpart to these classes.

15.2. **Locally Hermitian symmetric spaces of type IV.** We consider a non-degenerate quadratic $k$-space $(E,f)$ of dimension $n+2$, $n > 1$, over a totally real algebraic number field $k \neq \mathbb{Q}$. Suppose that $f$ is of signature $(n,2)$ and all its conjugates $f^\sigma$, $\sigma \neq \mathrm{Id}$, are positive definite. Let $G = SO(f)$ be the special orthogonal group of $f$, and let $G' = \mathrm{Res}_{k/\mathbb{Q}}G$ be the algebraic $\mathbb{Q}$-group obtained by restriction of scalars. Then the symmetric space $X$ corresponding to the group of real points $G'(\mathbb{R})$ is of the form

$$X = (SO(n) \times SO(2))\backslash SO_0(n,2),$$

that is, the Hermitian symmetric space of type IV. A torsion-free arithmetically defined subgroup $\Delta \subset G$ gives rise to a compact locally symmetric space $X/\Delta$ of dimension $2n$. Passing over to a suitable subgroup $\Gamma$ of finite index in $\Delta$, for any integer $j$ strictly between 0 and $2n$ and divisible by either 2 or $n$, the cohomology $H^j(X/\Gamma,\mathbb{R})$ contains a non-trivial cohomology class which is not in the image of the homomorphism $\beta_\Gamma : H^*(X_u,\mathbb{R}) \to H^*(X/\Gamma,\mathbb{R})$. These cohomology classes are detected by geometric cycles of the form $X_H/\Gamma_H = C(\langle\sigma_j\rangle,\Gamma)$ where $H \subset G$ is a reductive subgroup with $H'^{der,0}(\mathbb{R}) \simeq SO_0(j,2)\times$ (compact orthogonal groups).

We intend to discuss the interpretation of such a class in terms of the automorphic cohomology. More precisely, if $n = 2m$ is even, we discuss the possible irreducible representations in the automorphic spectrum which correspond to the class determined by the geometric cycle $C(\langle\sigma_m\rangle, \Gamma)$ .

Due to the result of Matsushima, discussed in Section 13.1, we have an isomorphism

$$H^*(X/\Gamma, \mathbb{R}) \xrightarrow{\sim} \bigoplus_{\pi\in\hat{G}} m(\pi, \Gamma)\, H^*(\mathfrak{so}(n, 2), K, H_{\pi,K}),$$

where the right-hand side is a finite direct algebraic sum ranging over the infinitesimal equivalence classes of irreducible unitary representations $(\pi, H_\pi)$ of $SO_0(n, 2)$ with non-vanishing cohomology $H^*(\mathfrak{so}(n, 2), K, H_{\pi,K})$. Since $X$ is Hermitian symmetric the cohomology groups $H^*(X/\Gamma, \mathbb{R})$ as well as $H^*(\mathfrak{so}(n, 2), K, H_{\pi,K})$ acquire a bigrading, and the isomorphism is compatible with this structure on both sides.

Following [154] the infinitesimal equivalence classes of irreducible unitary representations $(\pi, H_\pi)$ of $SO_0(n, 2)$ with non-vanishing cohomology are represented by the representations $A_\mathfrak{q} := A_\mathfrak{q}(0)$, where $\mathfrak{q}$ ranges through the set of $\theta$-stable parabolic subalgebras (up to conjugation under $K$) of $\mathfrak{g}_\mathbb{C}$. In the case of the group $SO_0(n, 2)$ these $\theta$-stable parabolic subalgebras are given in [59, 1.5], resp. [89]. We write $\mathfrak{p} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$, where $\mathfrak{p}^+$ (resp. $\mathfrak{p}^-$) is the holomorphic (resp. antiholomorphic) tangent space of $X$ at the origin. Accordingly we write, given a $\theta$-stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$, $R^\pm = \dim(\mathfrak{u} \cap \mathfrak{p}^\pm)$. We put $k = [n/2]$ and $l = n - [n/2]$. The possible $\theta$-stable parabolic subalgebras $\mathfrak{q}$ fall into two classes:

[A] There are non-negative integers $r, s$ with $0 < r + s \le n/2$ such that

$$L \simeq T^{r+s} \times U(k - r - s, 1),$$
$$(R^+, R^-) = (r, l + s) \text{ or } (l + s, r).$$

The corresponding irreducible unitary representation $A_\mathfrak{q}$ has non-zero cohomology exactly in the bi-degrees

$$(p, q) = (R^+ + j, R^- + j), \quad 0 \le j \le k - r - s.$$

[B] There is a non-negative integer $r$ with $r \le n/2$ such that

$$L \simeq T^r \times SO_0(n - 2r, 2),$$
$$(R^+, R^-) = (r, r).$$

The corresponding irreducible unitary representation $A_{\mathfrak{q}_r}$ has non-zero cohomology exactly in the bi-degrees

$$(p, q) = (j, j), \quad r \le j \le n - r.$$

Consequently we obtain the following result: Given a cohomological bi-degree $(p, q)$ there exists an irreducible unitary representation $(\pi, H_\pi)$ with non-zero cohomology $H^{p,q}(\mathfrak{so}(n, 2), K, H_{\pi,K})$ in this bi-degree if and only if

[a] $\min(p, q) \le n/2 \le \max(p, q) \le n$
or
[b] $p = q \le n$.

Thus, we see that the usual Hodge diamond depicting the various Hodge types degenerates into a kind of butterfly.

Now we suppose that $n = 2m$ is even. We are interested in the cohomology in bi-degree $(m, m)$, that is, in the middle component in the total degree $2m = n = (1/2)\dim X$. In case [B], given a non-negative integer $r$ with $r \leq n/2$, the corresponding representation $A_{\mathfrak{q}_r}$ has non-zero cohomology in bi-degree $(m, m)$. In case [A], there are exactly two representations, say $A_{\mathfrak{q}_m^\pm}$, which meet this requirement, namely the ones given by $r = m, s = 0$. Together, there are (up to infinitesimal equivalence) exactly the irreducible unitary representations

$$A_{\mathfrak{q}_r}, \quad 0 \leq r < n/2 = m, \text{ and } A_{\mathfrak{q}_m^\pm},$$

which have non-trivial relative Lie algebra cohomology in the bi-degree $(m, m)$.

The fundamental class of the geometric cycle $X_H/\Gamma_H = C(\langle\sigma_m\rangle, \Gamma)$ as constructed above gives rise via Poincaré duality to a non-trivial class, to be denoted $[C(\langle\sigma_m\rangle, \Gamma)]$, in $H^{2m}(X/\Gamma, \mathbb{R})$. In the framework of deRham cohomology, this class cannot be represented by a $G'(\mathbb{R})$-invariant differential form on $X$. According to the decomposition

$$H^{2m}(X/\Gamma, \mathbb{R}) = \bigoplus_{p+q=2m} H^{p,q}(X/\Gamma, \mathbb{R})$$

we write

$$[C(\langle\sigma_m\rangle, \Gamma)] = \bigoplus_{p+q=2m} [C(\langle\sigma_m\rangle, \Gamma)]^{p,q}.$$

Interpreting these cohomology groups in terms of relative Lie algebra cohomology, each of the summands can be written as a finite direct algebraic sum

$$H^{p,q}(X/\Gamma, \mathbb{R}) = \bigoplus_{\pi \in \hat{G}} m(\pi, \Gamma)\ H^{p,q}(\mathfrak{so}(n, 2), K, H_{\pi,K})$$

ranging over the infinitesimal equivalence classes of irreducible unitary representations $(\pi, H_\pi)$ of $SO_0(n, 2)$ with non-vanishing cohomology $H^{p,q}(\mathfrak{so}(n, 2), K, H_{\pi,K})$. In the case $m = p = q$, these equivalence classes are represented by the irreducible unitary representations $A_{\mathfrak{q}_r}, 0 \leq r < n/2 = m,$ and $A_{\mathfrak{q}_m^\pm}$. Therefore we obtain

$$\begin{aligned} H^{m,m}(X/\Gamma, \mathbb{R}) = \bigoplus_{0 \leq r < m} & m(A_{\mathfrak{q}_r}, \Gamma)\ H^{m.m}(\mathfrak{so}(n, 2), K, A_{\mathfrak{q}_r}) \\ & \oplus\, m(A_{\mathfrak{q}_m^\pm}, \Gamma)\ H^{m.m}(\mathfrak{so}(n, 2), K, A_{\mathfrak{q}_m^\pm}). \end{aligned}$$

Kobayashi-Oda [72] proved that the $(m, m)$-component $[C(\langle\sigma_m\rangle, \Gamma)]^{m,m}$ is a suitable multiple of a generator of the group $H^{m.m}(\mathfrak{so}(n, 2), K, A_{\mathfrak{q}_0})$. Note that $\mathfrak{q}_0 = \mathfrak{g}_\mathbb{C}$; hence $A_{\mathfrak{q}_0}$ is the trivial representation of $SO_0(n, 2)$ and the corresponding Lie algebra cohomology is the image of $H^{m,m}(X_u, \mathbb{R}) = H^{2m}(X_u, \mathbb{R})$ in $H^{m,m}(X/\Gamma, \mathbb{R})$ under the injective homomorphism $\beta_\Gamma^{2m}$. However, since the class $[C(\langle\sigma_m\rangle, \Gamma)]$ is not invariant under the full group, there exists a non-trivial $(p, q)$-component $[C(\langle\sigma_m\rangle, \Gamma)]^{p,q}$ with $p \neq q,\ p + q = 2m$. As a consequence, one obtains a non-vanishing result for the corresponding summand $H^{p,q}(X/\Gamma, \mathbb{R})$. The method of proof in [72] relies on a detailed study of the restriction of a representation $A_\mathfrak{q}$ to the reductive subgroup $H$.

## 16. Concluding remarks

Finally I would like to draw attention to some topics left out of this paper.

16.1. **Arithmetic cycles.** In [61], Hirzebruch and Zagier discovered that the intersection numbers $a^N = [C_1] \cap [C_N]$ of certain arithmetically defined cycles $C_N$ on a Hilbert modular surface $X/\Gamma$ are the Fourier coefficients of a modular form of Nebentypus. Their results, as well as subsequent results of similar nature pertaining to arithmetic cycles on Picard modular surfaces [30], [74], were proven along purely geometric lines. Later on, Kudla and Millson started to examine such phenomena in the more conceptual framework of the theory of the Weil representation and theta functions for dual reductive pairs [76], [77], [78]. The case of arithmetic quotients attached to $O(n, 2)$, that is, Shimura varieties of orthogonal type, gained particular attention. There the generating function in question can be interpreted in terms of automorphic forms in a very precise way.

In recent work, Funke and Millson [40], [41], [42] treat the extension of the Kudla-Millson lift and the corresponding theory of cycles (even with local coefficient systems) to the case of non-compact hyperbolic manifolds.

16.2. **Lefschetz properties of subvarieties of Shimura varieties.** Let $S(G, X)$ be a Shimura variety, attached to a reductive algebraic $\mathbb{Q}$-group $G$ and a Hermitian symmetric domain $X$ on which the group $G(\mathbb{R})$ acts transitively. For the sake of simplicity we assume that $G$ is anisotropic over $\mathbb{Q}$; hence $S(G, X)$ is compact. Suppose that $S(H, Y)$ is a subvariety which itself is a Shimura variety attached to a reductive algebraic $\mathbb{Q}$-subgroup $H$ of $G$. In particular, the symmetric space $Y$ associated with $H$ is of Hermitian type. Then there is the natural restriction map $j^* : H^*(S(G, X), \mathbb{C}) \longrightarrow H^*(S(H, Y), \mathbb{C})$ induced by $j : S(H, Y) \longrightarrow S(G, X)$.

Motivated by Lefschetz's theorem on hyperplane sections, it is natural to ask whether or not these subvarieties satisfy a weak Lefschetz property in degrees smaller than the complex dimension, say $n$, that essentially is to investigate how much of the cohomology $H^i(S(G, X), \mathbb{C})$ in these degrees is captured by the cohomology of $S(H, Y)$ and its translates under Hecke correspondences. More formally, one may ask whether or not the map

$$H^i(S(G, X), \mathbb{C}) \longrightarrow \oplus H^i(S(H, Y), \mathbb{C})$$

is injective for $i < n$, where the sum ranges over the translates of $S(H, Y)$ under the natural action of the group $G(\mathbb{A}_f)$, $\mathbb{A}_f$ the ring of finite adèles of the field $\mathbb{Q}$. Oda's approach to this question in his study [107] of Picard modular surfaces has been very inspiring for subsequent works such as [57], [59], [29], [11] and [149], [150]. These rely on an extended arsenal of methods such as the classification of irreducible unitary representations with non-zero cohomology [154] or the work of Burger-Li-Sarnak on Ramanujan duals [27].

# Appendices

## Appendix A. Central simple algebras over algebraic number fields

A.1. **Central simple algebras.** Let $k$ be a field. A $k$-algebra $D$ is a division algebra if every non-zero element of $D$ is invertible. A central algebra over $k$ is a finite-dimensional $k$-algebra $A \neq (0)$ whose center is $k = k.1_A$. It is said to be simple if $A$ and $(0)$ are the only two-sided ideals in $A$.

Let $l/k$ be a field extension. If $A$ is a central simple algebra over $k$, then the $l$-algebra $A \otimes_k l =: A_l$, obtained from $A$ by extension of scalars from $k$ to $l$, is again central simple. We call $l$ a splitting field of $A$ (or we will say that $l$ splits $A$) if there exists an isomorphism $A_l \longrightarrow M_n(l)$ of $l$-algebras for some $n$. For a given central simple $k$-algebra $A$, such splitting fields always exist. For example, an algebraic closure $\overline{k}$ splits $A$. In fact, there even exists a finite extension $l/k$ so that $l$ splits $A$. The dimension of $A$ is a square, $\dim_k A = n^2$, if $A_l = M_n(l)$ for some field extension $l/k$. We call the integer $n$ the degree of $A$, to be denoted $\deg(A)$.

Let $l/k$ be a splitting field for a central simple algebra $A$ over $k$. We may therefore fix an $l$-algebra isomorphism $\phi : A_l \longrightarrow M_n(l)$. Given $a \in A$, the characteristic polynomial of $\phi(a \otimes 1)$ has coefficients in $k$ and is independent of the choice both of $l$ and $\phi$. Its constant term is given by $(-1)^n \det(\phi(a \otimes 1))$. The reduced norm of $a$, to be denoted $\mathrm{Nrd}(a)$, is defined by $\mathrm{Nrd}(a) = \det(\phi(a \otimes 1))$. The corresponding map $\mathrm{Nrd} : A \longrightarrow k$ is multiplicative and $\mathrm{Nrd}(a) \neq 0$ if and only if $a \in A^*$.

Given a central simple $k$-algebra one also has the opposite algebra $A^{op}$. The latter one is given as $\{a_{op} \mid a \in A\}$, endowed with the following operations: $a_{op} + b_{op} = (a+b)_{op}, a_{op} \cdot b_{op} = (ba)_{op}$, and $\lambda a_{op} = (\lambda a)_{op}$ for $a, b \in A$ and $\lambda \in k$. If $A$ is of degree $n$, then there is a natural isomorphism $A \otimes A^{op} \cong \mathrm{End}_k(A) \cong M_{n^2}(k)$ of $k$-algebras.

A.2. **The group $SL_1(A)$.** Let $A$ be a central simple algebra of degree $n$ over the field $k$. Notice that $\dim_k A = n^2$. Let $GL(A)$ be the algebraic group defined over $k$ whose rational points over an extension $k'/k$ equal the group of invertible elements in the $k'$-algebra $A \otimes_k k'$. The reduced norm defines a surjective homomorphism

$$\mathrm{Nrd} : GL(A) \to G_m$$

of $GL(A)$ into the multiplicative group $G_m$ over $k$. The kernel of the morphism Nrd is a semi-simple, simply connected algebraic group over $k$, to be denoted $SL_1(A)$. This group is a $k$-form of the group $SL_n/k$.

A.3. **Brauer groups.** Within the structure theory of central division algebras and, more generally, of central simple algebras over a field, the notion of the Brauer group plays a decisive role. Two finite-dimensional central simple algebras $A$ and $B$ over a field $F$ are said to be similar if there exist integers $l, m$ such that $A \otimes_F M_l(F) \cong B \otimes_F M_m(F)$. Every central simple algebra is similar to exactly one division algebra (up to isomorphism). This notion of similarity induces an equivalence relation on the set of isomorphism classes of finite-dimensional central simple algebras over $F$. The Brauer group of the field $F$, to be denoted $Br(F)$, is the corresponding set of equivalence classes $[A]$ of finite-dimensional central simple algebras over $F$, endowed with the tensor product as the group operation. The unit element in $Br(F)$ is the equivalence class $[F]$ of $F$. Note that $[M_r(F)] = [F]$ for every $r$. The inverse of the class $[A]$ is the class of the opposite algebra $A^{op}$. By use of crossed product algebras, one can show that $Br(F)$ is a torsion group (see, e.g., [65, 2.7]).

Let $A$ be a central simple $k$-algebra $A$ of degree $\deg(A)$ over an algebraic number field $k$. Suppose that the defining field $k$ contains a subfield $k'$ such that the degree $[k : k']$ of the extension $k/k'$ is 2. Such an extension is necessarily a Galois extension.

Let $\mathrm{Gal}(k/k') = \{\mathrm{Id}_k, c\}$ denote its Galois group.

Given a central simple $k$-algebra $A$, its conjugate algebra $A^c = \{a^c \mid a \in A\}$ is defined by the following operations:

$$a^c + b^c = (a+b)^c,\ a^c \cdot b^c = (ab)^c,\ (\lambda a)^c = c(\lambda)a^c$$

for all $a, b \in A$ and $\lambda \in k$. The map

$$s : A^c \otimes_k A \to A^c \otimes_k A$$

defined by $a^c \otimes b \mapsto b^c \otimes a$ is $c$-semilinear over $k$ and is a $k'$-algebra automorphism. The $k'$-subalgebra

$$\{z \in A^c \otimes_k A \mid s(z) = z\} =: N_{k/k'}(A)$$

of elements in $A^c \otimes_k A$ fixed under $s$ is called the norm of the central simple algebra $A$ over $k$. It is a central simple $k'$-algebra of degree deg $N_{k/k'}(A)$=$(\deg A)^2$.

This construction induces a group homomorphism

$$N_{k/k'} : Br(k) \longrightarrow Br(k'), \quad [A] \mapsto [N_{k/k'}(A)],$$

of the respective Brauer groups [71, 3.13].

A.4. **Quaternion algebras.** In our context, in order to discuss the notion of arithmeticity for discrete subgroups of $PGL_2(\mathbb{C})$, we need some basic facts in the theory of quaternion algebras over a number field $k$. In particular, for the convenience of the reader we recall the notion of ramification and some related results in the case of a quaternion algebra over a given local field $k_v$ for $v$ a place of $k$.

Let $Q$ be a quaternion algebra over $k$, that is, $Q$ is a central simple $k$-algebra of degree two. Viewed as a vector space over $k$, the algebra $Q$ has a basis $1, i, j, k$ subject to the relations $i^2 = a$, $j^2 = b$, $ij = -ji = k$ for some elements $a, b \in k^*$. Although the quaternion algebra $Q$ does not uniquely determine the elements $a, b \in k^*$, we may also use the notation $Q = Q(a, b \mid k)$ emphasizing the choice of $a$ and $b$. Given a place $v \in V$ there is the local analogue

$$Q_v = Q \otimes_k k_v,$$

defined as the tensor product over $k$ of $Q$ with $k_v$. The algebra $Q_v$ is a central simple algebra over $k_v$. If $v \in V_\infty$ is a complex place, that is, $k_v = \mathbb{C}$, this algebra is isomorphic to the matrix algebra $M_2(\mathbb{C})$. If $v \in V_\infty$ is a real place, the algebra $Q_v$ over $\mathbb{R}$ is isomorphic either to the matrix algebra $M_2(\mathbb{R})$ or to the division algebra $\mathbb{H} = Q(-1, -1|\mathbb{R})$ of Hamilton's quaternions. A similiar dichotomy exists in the case of a non-Archimedean place $v \in V_f$. For each local field $k_v, v \in V_f$, there is a unique quaternion division algebra $C_v$ over $k_v$. Using the unique unramified quadratic extension $k_v(\sqrt{a})$, where $a$ is a unit in the ring $\mathcal{O}_v$ of integers in $k_v$, it can be described as a cyclic algebra. Thus, the quaternion algebra $Q \otimes_k k_v$ is isomorphic either to the matrix algebra $M_2(k_v)$ or the unique division algebra $C_v$. This is a consequence of the fact that a quaternion algebra over an arbitrary field $L$ is isomorphic either to $M_2(L)$ or to a division algebra.

We say that $Q$ *ramifies at* $v \in V$, or that $v$ is ramified in $Q$, if $Q \otimes_k k_v$ is isomorphic to a division algebra; otherwise $Q$ *splits at* $v \in V$. Hilbert's law of reciprocity implies that a quaternion algebra $Q$ over $k$ splits at all but a finite number of places and that the set $\mathrm{Ram}(Q) = \{v \in V \mid Q \text{ ramifies at } v\}$ has even cardinality. Notice that the isomorphism class of $Q$ over $k$ is determined by its ramification set $\mathrm{Ram}(Q)$. Conversely, given a set $S \subset V \setminus \{v \in V_\infty \mid v$ complex

place} so that $S$ has even cardinality, there exists a quaternion algebra over $k$ with ramification set equal to $S$.

These results imply, for example, that for a given number field $k$ with exactly one complex place $w$ and an arbitrary non-empty set $T$ of real places there are infinitely many non-isomorphic quaternion algebras $D$ over $k$ such that $D$ ramifies at $v \in T$. This is achieved by supposing that the ramification set $\mathrm{Ram}(D) = \{v \in V \mid D$ ramifies at $v\}$ is any finite set of even cardinality so that $T \subset \mathrm{Ram}(D)$.

A.5. **$k$-forms of the algebraic group $PGL_2$.** We briefly describe all $k$-forms of the algebraic group $PGL_2$ (or $SL_2$) over an algebraic number field $k$. By definition, a linear algebraic group $G$ defined over $k$ is a $k$-form of the $k$-group $PGL_2$ (or $SL_2$) if there exists a field extension $k'/k$ such that $G$ is isomorphic as a $k'$-group to $PGL_2/k'$ (or $SL_2/k'$).

The $k$-forms in question can be described in the following way. Let $A$ be a quaternion algebra over the field $k$; that is, $A$ is a central simple algebra over $k$ of degree 2. The $k$-group $GL(A)$ attached in A.2 to $A$ has a one-dimensional center, and its derived group is $SL_1(A)$. Then the quotient $G$ of $GL(A)$ by its center is a $k$-form of $PGL_2/k$. This construction exhausts all possible $k$-forms of $PGL_2/k$.

## Appendix B. Groups of units of quadratic forms

In this appendix we describe the explicit construction of the algebraic groups $O(E, f)$ and their arithmetic subgroups that arise from unit groups of non-degenerate quadratic spaces $(E, f)$ over an algebraic number field $k$. Much of the material of this appendix can also be found in [144] or [15]. Siegel's work [139] contains a comprehensive discussion of the related reduction theory.

In the case of the algebraic $k$-groups $O(E, f)$, by using coordinates, we introduce parabolic $k$-subgroups and their subgroups in a quite elementary way. In particular, given a proper parabolic $k$-subgroup of some $O(E, f)$, we see that its Levi components are products of suitable isometry groups and general linear groups.

B.1. **Orthogonal groups.** Let $k$ be an algebraic number field, and let $(E, f)$ be a non-degenerate quadratic $k$-vector space of dimension $n$, that is, a $k$-vector space $E$ endowed with a non-degenerate quadratic form $f$. We denote the corresponding symmetric bilinear form on $E$ by $b_f$. If $f(x) = 0$ for $x \in E$, then $x$ is an isotropic vector, and a subspace $U$ of $E$ is an isotropic subspace if $b_f(x, x') = 0$ for all $x, x' \in U$. This is equivalent to the requirement that $f(x) = 0$ for all $x \in U$. If $x \in E$ is an isotropic vector, then there exists an isotropic vector $y \in E$ with $b_f(x, y) = 1$. In general, we call a two-dimensional subspace $H$ with a basis $x, y$, so that $f(x) = f(y) = 0$ and $b_f(x, y) = 1$ a hyperbolic plane.

If no non-zero vector in a subspace $U'$ of $E$ is isotropic, then $U'$ is called anisotropic. In other words, the quadratic form $f$ restricted to $U'$ does not represent zero rationally.

We denote the group of isometries of $(E, f)$ by $O(E, f)$; that is, by definition,

$$O(E, f) = \{\phi \in GL(E) \mid f(\phi(x)) = f(x) \text{ for all } x \in E\}.$$

Let $SO(E, f)$ denote the kernel of the determinant map $\det : O(E, f) \longrightarrow G_m$ into the multiplicative group over $k$, to be called the special orthogonal group of $f$. The corresponding algebraic group is connected.

Let $q$ denote the Witt index of $f$, that is, by definition, the dimension of the maximal isotropic subspaces in $E$. We suppose that $q > 0$. With respect to a suitable basis $\{e_i\}_{1\le i\le n}$ we can write

$$f(x) = x_1x_n + x_2x_{n-1} + \cdots + x_qx_{n-q+1} + f_0(x_{q+1}, \ldots, x_{n-q}),$$

where $f_0$ is a non-degenerate anisotropic quadratic form on the $k$-span $U$ of the set $\{e_{q+1}, ..., e_{n-q}\}$; that is, $(U, f_0)$ is an anisotropic space. The $k$-spans $U_q := \langle\{e_1, ..., e_q\}\rangle$, resp. $U'_q := \langle\{e_{n-q+1}, ..., e_n\}\rangle$, are both isotropic subspaces of the quadratic space $(E, f)$.

The special orthogonal group $SO(E, f)$ is split over $k$, that is, it contains a maximal $k$-split torus, if and only if $q = [n/2]$, equivalently, $n = 2q$ or $n = 2q + 1$. If $n = 2q + 2$, the group $SO(E, f)$ is quasi-split over $k$.

If $n$ is odd and $f$ has maximal Witt index $[n/2]$, then the $k$-split group $SO(E, f)$ is simple and adjoint. If $n$ is even, $q = n/2$, then $SO(E, f)$ is simple if $n > 5$. The corresponding adjoint group is the projective orthogonal group attached to $(E, f)$.

We now suppose that $k$ is a totally real algebraic number field of degree $d$. Let $\sigma_1, ..., \sigma_d$ denote the $d$ distinct field embeddings $k \to \mathbb{C}$. All of these factor through $k \to \mathbb{R}$. Let $\epsilon_i$, $i = 1, ..., d$, each be $+1$ or $-1$. As an application of the Weak Approximation Theorem in number fields, there exists $\alpha \in k^*$ such that $\mathrm{sign}(\sigma_i(\alpha)) = \epsilon_i$, $i = 1, ..., d$. Let $p_1, ..., p_d$ denote natural numbers with the property that $0 \le p_i \le n$, $i = 1, ..., d$. We fix an index $j, 1 \le j \le n$. By the remark above, we then may choose some $\alpha_j \in k^*$ such that

$$\mathrm{sign}(\sigma_i(\alpha_j)) = \begin{cases} +1, & \text{if } j \le p_i, \\ -1, & \text{if } j > p_i, \end{cases}$$

for $i = 1, ..., d$. Consider the non-degenerate quadratic $k$-space $(E, f)$, where, in a suitable basis, the form $f$ is given by

$$f(x) = \sum_{1\le j\le n} \alpha_j x_j^2.$$

Let $(E^{\sigma_i}, f^{\sigma_i})$ denote the quadratic space whose underlying vector space is the real vector space $E^{\sigma_i}$ obtained as the tensor product $E \otimes_k \mathbb{R}$ via the embedding $\sigma_i : k \to \mathbb{R}$ and which is endowed with the quadratic form

$$f^{\sigma_i} = \sigma_i(\sum_{1\le j\le n} \alpha_j x_j^2).$$

The signature of this form is $(p_i, n - p_i)$ for $i = 1, ..., d$.

Let $G$ be the special orthogonal group of $(E, f)$. By definition, the arithmetic subgroups of $G(k)$ are the subgroups that are commensurable with $G_{\mathcal{O}_k}$, the group of $\mathcal{O}_k$-units of $G$. The latter group is also called the *group of units of the quadratic form* $(E, f)$. Let $G' = \mathrm{Res}_{k/\mathbb{Q}}G$ be the algebraic $\mathbb{Q}$-group obtained by restriction of scalars.[13] Then the group of real points of $G'$ is of the form $\prod_{1\le i\le d} SO(p_i, n - p_i)$. Suppose that there exists $i_0$, $1 \le i_0 < d$ so that $p_i < n$ for $i \le i_0$ and $p_i = n$ for $i > i_0$. Then we have

$$G'(\mathbb{R}) = \prod_{1\le i\le i_0} SO(p_i, n - p_i) \quad \prod_{i>i_0} SO(n, 0). \tag{B.1}$$

[13] If $G$ is an algebraic group defined over some finite extension field $k$ of $\mathbb{Q}$, there is a canonical way to get from $G$ an algebraic group, to be denoted $\mathrm{Res}_{k/\mathbb{Q}}G$, defined over $\mathbb{Q}$. This process is usually called "restriction of scalars"; it is explained in some detail in Appendix C.

We write $(G')^{nc}$ for the non-compact real Lie group $\prod_{1\le i\le i_0} SO(p_i, n-p_i)$. Then $G'(\mathbb{R})$ is of the form $(G')^{nc}.C$ with $C$ the compact group $\prod_{i>i_0} SO(n,0)$. The associated symmetric space is then $X = K\backslash G'(\mathbb{R}) = K^{nc}\backslash(G')^{nc}$, where $K$, resp. $K^{nc}$, denotes a maximal compact subgroup of $G'(\mathbb{R})$, resp. $(G')^{nc}$. An arithmetic subgroup of $G(k)$ projects to $(G')^{nc}$ as a discrete subgroup. In the case that $i_0 = 1$ and $p_1 = n-1$ the corresponding symmetric space is hyperbolic $(n-1)$-space.

B.2. **Parabolic subgroups of $O(E,f)$.** Let $(E,f)$ be a non-degenerate quadratic $k$-vector space of dimension $n$, where $k$ is an arbitrary algebraic number field. We suppose that the Witt index $q$ of $f$ is non-zero. There exists a basis $\{e_i\}_{1\le i\le n}$ of $E$ so that $f$ is of the form

$$f(x) = x_1x_n + x_2x_{n-1} + \cdots + x_qx_{n-q+1} + f_0(x_{q+1},\ldots,x_{n-q}).$$

The form $f_0$ is a non-degenerate anisotropic quadratic form on $U = \langle\{e_{q+1},...,e_{n-q}\}\rangle$. The $k$-span of $\{e_1,...,e_q\}$, resp. $\{e_{n-q+1},...,e_n\}$, are both isotropic subspaces of the quadratic space $(E,f)$. The group

$$S = \{\mathrm{diag}(\lambda_1,...,\lambda_q,1,...,1,\lambda_q^{-1},...,\lambda_1^{-1})\}$$

of diagonal matrices is a maximal $k$-split torus in the orthogonal group $O(E,f)$. The centralizer of $S$ is $Z(S) = S\times O(U,f_0)$, where the group $O(U,f_0)$ is embedded into $O(E,f)$ by acting trivially on the subspaces $\langle\{e_1,...,e_q\}\rangle$, resp. $\langle\{e_{n-q+1},...,e_n\}\rangle$.

An isotropic flag $\mathcal{F}$ in $(E,f)$ is a chain

$$\mathcal{F}: \quad F_1\subset F_2\subset\cdots\subset F_r$$

of isotropic subspaces $F_j$ of $E$. We denote the dimension of $F_j$ over $k$ by $d_j$. The type of the flag is the ordered $r$-tuple $(d_1,...,d_r)$. By definition, a $k$-parabolic subgroup of $O(E,f)$ is the stabilizer $P_{\mathcal{F}} = P$ of an isotropic flag $\mathcal{F}$ in $(E,f)$. The type of $P = P_{\mathcal{F}}$ is defined to be the type of $\mathcal{F}$. A minimal parabolic $k$-subgroup of $O(E,f)$ is the stabilizer of an isotropic flag of type $(1,2,...,q-1,q)$. Note that two parabolic $k$-subgroups $P$ and $P'$ of $O(E,f)$ are conjugate over $k$ if and only if there exists an element in $O(E,f)$ mapping $\mathcal{F}$ onto $\mathcal{F}'$, equivalently, by the theorem of Witt on extensions of isometries, if and only if the defining flags $\mathcal{F}$ and $\mathcal{F}'$ have the same type.

Let $P$ be a parabolic $k$-subgroup with corresponding flag $\mathcal{F} = \{F_j\}_{j=1,\ldots,r}$. Then we can form the chain

$$F_1\subset F_2\subset\cdots\subset F_r\subset F_r^{\perp}\subset F_{r-1}^{\perp}\subset\cdots\subset F_1^{\perp}$$

in $E$. An element $p\in P$ stabilizes the flag $\mathcal{F}$; hence it induces maps on the quotients $F_i/F_{i-1}$ and $F_{i-1}^{\perp}/F_i^{\perp}$, $i = 2,...,r$, respectively. The subgroup of $O(E,f)$ formed by all elements $p\in P$ that act trivially on these quotients $F_i/F_{i-1}$ and $F_{i-1}^{\perp}/F_i^{\perp}$, $i = 2,...,r$, is called the unipotent radical of $P$, to be denoted $R_uP$. This group is a normal subgroup of $P$. Given a parabolic $k$-subgroup $P = P_{\mathcal{F}}$, let $\mathcal{F}'$ be another isotropic flag of the same type as $\mathcal{F}$, that is, $d_j = d_j'$ for all $j$. Suppose that $F_j + F_j'$ is a non-degenerate orthogonal sum of hyperbolic planes. We call the group $L := P_{\mathcal{F}}\cap P_{\mathcal{F}'}$ the Levi component of $P = P_{\mathcal{F}}$ attached to the flag $\mathcal{F}'$. Different choices of isotropic flags $\mathcal{F}'$ give different Levi components of $P$. However, the parabolic $k$-subgroup $P$ is the semi-direct product of its unipotent radical and any Levi component. Since the Levi component $L$ is defined as the intersection of the stabilizers of $\mathcal{F}$ and $\mathcal{F}'$, an element in $L$ stabilizes each of the subspaces $F_i + F_i'$ and the corresponding orthogonal complements $(F_i + F_i')^{\perp}$ as well.

We define the subspaces

$$V_1 = F_1,$$
$$V_i = F_i \cap F'^{\perp}_{i-1}, \quad i = 2, ..., r,$$

respectively,

$$V'_1 = F'_1,$$
$$V'_i = F'_i \cap F^{\perp}_{i-1}, \quad i = 2, ..., r,$$

and

$$W = F_r^{\perp} \cap F'^{\perp}_r.$$

Since the flags $\mathcal{F}$ and $\mathcal{F}'$ are of the same type, we have $\dim V_1 = d_1 = \dim V'_1$ and

$$\dim V_i = d_i - d_{i-1} = \dim V'_i.$$

Each of the spaces $V_j$, resp. $V'_j$, $j = 1, ..., r$ is orthogonal to $W$, and, furthermore, $V_s$, $s = 1, ..., r$, is orthogonal to $V_t$, resp. $V'_t$, $t = 1, ..., r$, if $t \neq s$. Thus, the Levi component in $P_{\mathcal{F}}$ attached to $\mathcal{F}'$ decomposes into the direct product of the isometry group of $(W, f_{|W})$ and the groups $GL_{d_i - d_{i-1}}(k)$, $i = 1, ..., r$, where we understand $d_0 = 0$ by convention.

With respect to the basis $\{e_i\}_{1 \leq i \leq n}$ of $E$ , let $\mathcal{F}_0$ be the standard isotropic flag

$$F_1 \subset F_2 \subset \cdots \subset F_{q-1} \subset F_q,$$

where $F_j = \langle \{e_1, ..., e_j\} \rangle$. It is of type $(1, 2, ..., q-1, q)$. Its stabilizer in $O(E, f)$ is the minimal parabolic $k$-subgroup $P_0 := P_{\mathcal{F}_0}$, to be called the standard minimal parabolic $k$-subgroup. Let $\mathcal{F}' = \{F'_j\}_{j=1,...,q}$ be the isotropic flag of type $(1, 2, ..., q-1, q)$ given as

$$F'_1 = \langle \{e_n\} \rangle \subset F'_2 = \langle \{e_n, e_{n-1}\} \rangle \subset \cdots \subset F'_q = \langle \{e_n, ..., e_{n-q+1}\} \rangle;$$

that is, $F'_j := \langle \{e_n, ..., e_{n-j+1}\} \rangle$ for $j = 1, ..., q$. Using the notation above, the spaces $F_j + F'_j$ form a non-degenerate orthogonal sum of hyperbolic planes and the subspace $W = F_q^{\perp} \cap F'^{\perp}_q$ coincides with $U = \langle \{e_{q+1}, ..., e_{n-q}\} \rangle$. Thus an element $p$ in $P_0$ (with respect to the chosen basis) has the form

$$\begin{pmatrix} M_1 & M_2 & M_3 \\ 0 & B & M_4 \\ 0 & 0 & M_5 \end{pmatrix},$$

where $M_1$ and $M_5$ are upper triangular matrices of size $q \times q$, and $B$ is an element of $O(W, f_{|W}) = O(U, f_0)$. There are further relations between the block determined by the requirement $p \in O(E, f)$.

If $n = 2q$ or $n = 2q + 1$, the centralizer $Z(S)$ equals $S$, the case where the group $O(E, f)$ splits. If $n = 2q + 2$, $Z(S)$ is a torus distinct from $S$, the quasi-split case.

The $k$-spaces $U_q := \langle \{e_1, ..., e_q\} \rangle$, resp. $U'_q := \langle \{e_{n-q+1}, ..., e_n\} \rangle$, are both maximal isotropic subspaces of the quadratic space $(E, f)$. The subspace $F_j = \langle \{e_1, ..., e_j\} \rangle$ of $U_q$ (resp. $F'_j = \langle \{e_{n-j+1}, ..., e_n\} \rangle$ of $U'_j$) of dimension $d_j = j$, $j = 1, ..., q$, gives rise to an isotropic flag $\mathcal{F}_j$ (resp. $\mathcal{F}'_j$) of type $(j)$. The parabolic $k$-subgroup $P_j := P_{\mathcal{F}_j}$ associated to $\mathcal{F}_j$ is a standard maximal parabolic $k$-subgroup of $O(E, f)$. With the two isotropic flags $\mathcal{F}_j$ and $\mathcal{F}'_j$ as defined, the Levi component $L_j$ of $P_j$ corresponding to this choice is, by definition, the group $P_{\mathcal{F}_j} \cap P_{\mathcal{F}'_j}$, isomorphic to $GL_d(k) \times O(W, f_{|W})$ with $W = F_j^{\perp} \cap F'^{\perp}_j$ of dimension $n - 2d_j$; the non-degenerate form $f_{|W}$ has Witt index $q - d_j$.

*Remarks.* (1) Suppose that $j = q$, that is, that the subspace $F_j$ is $U_q$. Then the Levi component $L_q$ is isomorphic to $GL_q(k) \times O(W, f_{|W})$ with $f_{|W}$ of Witt index 0; equivalently, the form $f_{|W}$ is anisotropic. Note that, if $n = 2q$, i.e., the group $O(E, f)$ splits, then the form $f_0$ is trivial. Hence, the Levi component $L_q$ is, up to isomorphism, the general linear group $GL_q(k)$.

(2) Suppose that $(E, f)$ is a non-degenerate quadratic $k$-vector space so that the Witt index of $f$ is 1. Up to conjugacy, there is only one parabolic $k$-subgroup of $O(E, f)$. An element $p$ in the standard parabolic $k$-subgroup $P_0$ is of the form

$$\begin{pmatrix} m & M_2 & M_3 \\ 0 & B & M_4 \\ 0 & 0 & m^{-1} \end{pmatrix},$$

where $m \in GL_1(k), B \in O(U, f_0)$. The quadratic form $f_0$ is anisotropic.

## Appendix C. Weil restriction of scalars

We consider an algebraic number field $k$ of degree $d = [k : \mathbb{Q}]$, and we denote by $\mathcal{O}_k$ the ring of integers in $k$. The ring $\mathcal{O}_k$ is a finitely generated torsion-free $\mathbb{Z}$-module of rank $d$; hence $\mathcal{O}_k$ has a $\mathbb{Z}$-basis $\omega_1, ..., \omega_d$. Moreover, as $\mathcal{O}_k$ spans $k$ over $\mathbb{Q}$, $(\omega_i)_{i=1,...,d}$ is also a basis of $k$ over $\mathbb{Q}$. We shall refer to such a basis as an integral basis of $k$.

Let $S$ be the set of distinct embeddings $\sigma_i : k \to \mathbb{C}$, $1 \leq i \leq d$. Among these embeddings some factor through $k \to \mathbb{R}$. Let $\sigma_1, ..., \sigma_s$ denote these real embeddings $k \to \mathbb{R}$. Given one of the remaining embeddings $\sigma : k \to \mathbb{C}, \sigma(k) \not\subset \mathbb{R}$, to be called imaginary, there is the conjugate one $\bar{\sigma} : k \to \mathbb{C}$, defined by $x \mapsto \overline{\sigma(x)}$, where $\bar{z}$ denotes the usual complex conjugation of the complex number $z$. Then the number of imaginary embeddings is an even number, which we denote by $2t$. We number the $d = s + 2t$ embeddings $\sigma_i : k \to \mathbb{C}$ , $i = 1, ..., d$, in such a way that, as above, $\sigma_i$ is real for $1 \leq i \leq s$, and $\overline{\sigma}_{s+i} = \sigma_{s+i+t}$ for $1 \leq i \leq t$.

Let $V_\infty$ be the set of Archimedean places of $k$. This set is naturally identified with the set of embeddings $\{\sigma_i\}_{1 \leq i \leq s+t}$; that is, we take all real embeddings and one representative for each pair of conjugate imaginary embeddings.[14]

Let $G$ be an algebraic subgroup of $GL_n(\mathbb{C})$ defined over $k$. If an integral basis $(\omega_i)_{i=1,...,d}$ of $k$ is chosen, following Weil [160], one can construct an algebraic subgroup $G'$ of $GL_{nd}(\mathbb{C})$, defined over $\mathbb{Q}$, and a rational homomorphism $\alpha : G' \longrightarrow G$ over $k$ such that

$$(\alpha^{\sigma_1}, ..., \alpha^{\sigma_d}) : G' \longrightarrow G^{\sigma_1} \times \cdots \times G^{\sigma_d}$$

is an isomorphism over $\mathbb{C}$. For a given $\sigma \in S$, $G^\sigma$ denotes the group obtained from $G$ by conjugation with $\sigma$. More precisely, if $I$ denotes the ideal of polynomials in $k[X_{11}, ..., X_{nn}]$ which vanish on $G$, let $I^\sigma$ be the ideal in $\sigma(k)[X_{11}, ..., X_{nn}]$ that is obtained by transforming the coefficients of the elements $h$ in $I$ with $\sigma$. Then $G^\sigma$ is the algebraic group over $\sigma(k)$ defined by $I^\sigma$. Up to isomorphism, the pair $(G', \alpha)$

[14]Recall that a prime $v$ (or a place) of an algebraic number field $K$ is a class of equivalent absolute values on $K$. We denote by $K_v$ the completion of $K$ with respect to the topology induced on $K$ by this class. There are two types of valuations possible. First, there is a finite number of so-called Archimedean completions $K_v = \mathbb{R}$ or $\mathbb{C}$ corresponding to the "infinite" places given by embeddings of $K$ into $\mathbb{R}$ or $\mathbb{C}$ (up to complex conjugation in the latter case). Second, there is an infinite number of non-Archimedean completions. These correspond to the "finite" places of $K$, one for each prime ideal in the ring of algebraic integers of $K$.

does not depend on the choice of the integral basis $(\omega_i)_{i=1,\dots,d}$ of $k$. The group $G'$ is the group, to be denoted $\mathrm{Res}_{k/\mathbb{Q}}\, G$, obtained from $G$ by restriction of scalars. The map $\alpha$ induces isomorphisms

$$(\mathrm{Res}_{k/\mathbb{Q}}\, G)(\mathbb{Q}) \longrightarrow G(k) \tag{C.1}$$

and

$$(\mathrm{Res}_{k/\mathbb{Q}}\, G)_{\mathbb{Z}} \longrightarrow G_{\mathcal{O}_k}. \tag{C.2}$$

That is, the group of $k$-points of $G$ is naturally identified with the group of $\mathbb{Q}$-points of the algebraic group $\mathrm{Res}_{k/\mathbb{Q}}\, G$.

For a given Archimedean place $v \in V_\infty$, corresponding to the embedding $\sigma : k \to \mathbb{C}$, we write $G_v = G^\sigma(k_v)$, where $k_v = \mathbb{R}$ if $\sigma$ is real and $k_v = \mathbb{C}$ if $\sigma$ is imaginary. Then the map $\alpha$ also induces an isomorphism

$$G'(\mathbb{R}) \longrightarrow \prod_{v \in V_\infty} G_v$$

for the group of real points.

The reader may wish to consult Appendix B.2 on groups of units of quadratic forms for a specific example.

## Appendix D. Cohomology

Mainly to fix notation and conventions we recall some basic definitions and results in the cohomology theory of manifolds which are needed in the text. In particular, we treat various products. Nevertheless we have to assume some familiarity with the basic sources as, for example, [24], [31] or [141].

### D.1. **Products in (co)homology.**

*Kronecker product.* Let $X$ be a topological space, $R$ a Dedekind ring and $E$ an $R$-module. As usual, $H_*(X, E)$ and $H^*(X, E)$ denote the singular homology and cohomology respectively of $X$ with coefficients in $E$. We denote the underlying chain, resp. cochain, complex by $C_*(X, E)$ and $C^*(X, E) = \mathrm{Hom}_R(C_*(X, R), E)$. If $\alpha \in C^*(X, E)$ is a cochain, we denote the value of $\alpha$ on a chain $\zeta \in C_*(X, R)$ by $\langle \alpha, \zeta \rangle$. This gives rise to the so-called Kronecker product

$$\langle \quad , \quad \rangle : H^j(X, E) \times H_j(X, R) \longrightarrow E, \tag{D.1}$$

defined by $(x, \sigma) \mapsto \langle \alpha_x, \zeta_\sigma \rangle$, where the cochain $\alpha_x$ represents $x$ and the chain $\zeta_\sigma$ represents $\sigma$. This definition does not depend on the choice of the representatives. If $E = R$ is a field and $H_j(X, R)$ is a finite-dimensional vector space, then the Kronecker product is a dual pairing.

More generally, there is a corresponding product in relative (co)homology for pairs $(X, A)$ and $R$-modules $E, F$, namely,

$$\langle \quad , \quad \rangle : H^j(X, A; E) \times H_j(X, A; F) \longrightarrow E \otimes F. \tag{D.2}$$

Suppose that $H_*(X, R)$ is a free $R$-module. Then the assignment $x \mapsto \langle x, \quad \rangle$ defines a canonical isomorphism $H^*(X, E) \to \mathrm{Hom}_R(H_*(X, R), E)$.

*Cup product.* Let $A, B$ be two subsets of a topological space $X$ so that $A, B$ are open in $A \cup B$, and let $E, E'$ be two $R$-modules. Then there is the cup-product

$$\begin{aligned} &\smile : H^i(X, A; E) \otimes_R H^j(X, B; E') \longrightarrow H^{i+j}(X, A \cup B; E \otimes_R E'), \\ &(x, y) \longmapsto x \smile y. \end{aligned}$$

This product is associative but not commutative. In fact, for $x \in H^*(X, A; E), y \in H^*(X, B; E')$, we have $x \smile y = (-1)^{\deg(x)\cdot\deg(y)} y \smile x$. For the class $1_X \in H^0(X, R)$ of the augmentation map $C_0(X) \to R$, we have $1_X \smile x = x = x \smile 1_X$, $x \in H^*(X, A; E)$. The cup-product is natural in $X$. Furthermore, the graded group $H^*(X, R)$ endowed with the cup-product structure is a (commutative) graded $R$-algebra, to be called the cohomology algebra of $X$ with coefficients in $R$. This algebra is functorial in $X$.

*Cap-product.* Given as before two subsets $A, B$ in $X$ that are relatively open as subsets of $A \cup B$, and $R$-modules $E, E'$, there is also the cap-product operation

$$\begin{aligned} &\frown : H^j(X, B; E') \otimes_R H_i(X, A \cup B; E) \longrightarrow H_{i-j}(X, A; E \otimes_R E'), \\ &(x, \sigma) \longmapsto x \frown \sigma \end{aligned}$$

induced by a bilinear pairing on the (co)chain level. In the case of a pair $(X, A)$ and coefficients $E' = R$, this product operation turns the homology $H_*(X, A; E)$ into a graded $H^*(X, R)$-module. If $f : X \to Y$ is a map of topological spaces, then

$$f_* : H_*(X, R) \to H_*(Y, R)$$

is a homomorphism of $H^*(Y, R)$-modules, where $H_*(X, R)$ is endowed with the $H^*(Y, R)$-module structure defined via the ring homomorphism $f^* : H^*(Y, R) \to H^*(X, R)$.

If $x \in H^*(X, A; E')$ and $\sigma \in H_*(X, A; E)$ are classes of the same degree, that is, $\deg(x) = \deg(\sigma)$, then

$$\langle x, \sigma \rangle \quad = \quad \epsilon_*(x \frown \sigma),$$

where $\epsilon$ denotes the augmentation map tensored with $\mathrm{Id}_{E \otimes E'}$. This equation relates the cap-product with the Kronecker product. In the case of trivial coefficients $E = E' = R$, this Kronecker product

$$\langle \quad , \quad \rangle : H^j(X, R) \times H_j(X, R) \longrightarrow R,$$

satisfies the identity

$$\langle x \smile y, \sigma \rangle \quad = \quad \langle x, y \frown \sigma \rangle$$

for $x, y \in H^*(X, R), \sigma \in H_*(X, R)$. Moreover, if $f : X \to Y$ is a map of topological spaces, we have

$$\langle f^*(y), \sigma \rangle \quad = \quad \langle y, f_*(\sigma) \rangle$$

with $y \in H^*(Y, R), \sigma \in H_*(X, R)$.

D.2. **Cohomology of manifolds.**

*Fundamental class.* Let $M$ be a manifold of dimension $n$, not necessarily compact. For each given point $x \in M$, the local homology group $H_i(M, M-x; \mathbb{Z}) = H_i(\mathbb{R}^n, \mathbb{R}^n - 0; \mathbb{Z})$ is zero for $i \neq n$ and infinite cyclic for $i = n$. We call a generator $o_x$ for $H_n(M, M-x; \mathbb{Z})$ a local orientation for $M$ at $x$. Notice that there are exactly two possible generators. An orientation for $M$ is a choice of a local orientation $o_x$ at each point $x \in M$ which varies continuously with $x$. More precisely, for each $x \in M$ there exist a compact neighborhood $U$ and a class $o_U \in H_n(M, M-U; \mathbb{Z})$ so that the natural homomorphism $j_u : H_n(M, M-U; \mathbb{Z}) \to H_n(M, M-u; \mathbb{Z})$ takes $o_U$ to $o_u$ for each $u \in U$. We say that $M$ is orientable if such an orientation exists. We call a manifold together with the choice of an orientation an oriented manifold.

Given a compact set $K$ in an oriented manifold of dimension $n$, there exists a uniquely determined element $o_K \in H_n(M, M-K; \mathbb{Z})$ so that $j_x(o_K) = o_x$ for each point $x \in K$. We call the class $o_K$ the fundamental class of $M$ around $K$. In particular, if $M$ itself is compact, we have the unique class $o_M \in H_n(M, \mathbb{Z})$, to be called the fundamental class of $M$.

In general, if $M$ is a connected manifold of dimension $n$, then $H_i(M, \mathbb{Z}) = 0$ for $i > n$, and $H_n(M, \mathbb{Z}) = \mathbb{Z}$ if $M$ is orientable and compact. If $M$ is not compact, then $H_n(M, \mathbb{Z}) = 0$. Via the universal coefficient theorem, if $M$ is compact and orientable, then $H^n(M, \mathbb{Z}) = \mathbb{Z}$, whereas in the non-compact case (even when $M$ is not orientable) $H^i(M, \mathbb{Z}) = 0$ for $i \geq n$.

In the case of an oriented compact manifold $M$ with boundary $\partial M$, there is an analogous construction of a uniquely determined fundamental class $o_M$ in $H_n(M, \partial M; \mathbb{Z})$. The boundary operator in homology takes $o_M$ to the fundamental class of $\partial M$ in $H_{n-1}(\partial M)$.

*Duality.* Suppose $M$ is a compact oriented manifold of dimension $n$. Then the assignment $x \mapsto x \frown o_M$ defines an isomorphism $H^i(M, R) \to H_{n-i}(M, R)$. This assertion is referred to as Poincaré duality. As cohomology can be interpreted in terms of homology, by means of the universal coefficient theorem, we obtain, if $R$ is a field, the sequence of isomorphisms

$$H_{n-i}(M, R) \cong H^i(M, R) \cong \mathrm{Hom}_R(H_i(M, R), R) = H_i(M, R)^*.$$

More generally, if $M$ is a compact oriented manifold with boundary, $R$ a field, and $o_M \in H_n(M, \partial M; R)$ is its fundamental class, then the map $x \mapsto x \frown o_M$ defines an isomorphism $H^i(M, \partial M; R) \tilde{\to} H_{n-i}(M, R)$. Furthermore, the pairing

$$H^i(M, \partial M; R) \times H^{n-i}(M, R) \longrightarrow H^n(M, \partial M; R) \longrightarrow H_0(M, R) = R,$$

defined by the assignment

$$(x, y) \mapsto x \smile y \mapsto \langle x \smile y, o_M \rangle = \langle x, y \frown o_M \rangle,$$

is a dual pairing.

There is a more general duality theorem which includes not necessarily compact manifolds as well. Let $M$ be a connected oriented manifold of dimension $n$, and let $H_c^*(M, R)$ denote its cohomology with compact support. This cohomology $H_c^*(M, R)$ can be defined as the direct limit $\varinjlim H^*(M, M-K; R)$, where $K$ ranges over the directed set of compact subsets of $M$. If $M$ is compact, then $H_c^*(M, R) = H^*(M, R)$. Given a compact subset $K \subset M$, we denote the fundamental class of

$M$ around $K$ by $o_K$. As the maps $\frown o_K : H^i(M, M-K; R) \longrightarrow H_{n-i}(M, R)$ are compatible with taking the limit over $K$, there is a well-defined homomorphism

$$\frown \{o_K\} : H^i_c(M, R) \tilde{\longrightarrow} H_{n-i}(M, R).$$

This homomorphism, which may be interpreted as the cap-product with the orientation class of $M$ (still to be defined), is indeed an isomorphism.

The homology group $H_0(M, R)$ has a canonical generator, the homology class of a point; thus there are identifications

$$H_0(M, R) \cong R \cong H^n_c(M, R). \tag{D.3}$$

We denote the homology with closed supports of $M$ with coefficients in $R$ by $H^{cl}_*(M, R)$. Analogously, Poincaré duality provides a canonical isomorphism

$$H^{cl}_i(M, R) \tilde{\longrightarrow} H^{n-i}(M, R), \quad i \in \mathbb{Z}. \tag{D.4}$$

Let $E$ be a finitely generated free $R$-module which is a module under the fundamental group $\pi_1(M)$ of $M$. This gives rise to a locally constant sheaf on $M$ with stalk isomorphic to $E$, to be denoted $\tilde{E}$. Let $E^*$ be the contragredient $\pi_1(M)$-module, and $\tilde{E}^*$ the associated locally constant sheaf. The canonical bilinear form on $E \times E^*$ induces a pairing of $\tilde{E}$ and $\tilde{E}^*$ to the constant sheaf $R_M$ on $M$ with stalk $R$. If we replace $R$ in the discussion above by $\tilde{E}$ or $\tilde{E}^*$, the Poincaré relations remain valid.

Suppose now that $R$ is a field, and for the sake of simplicity, that all (co)homology spaces to be considered are finite-dimensional vector spaces. Then there are the identifications

$$H^{cl}_i(M, \tilde{E}^*) = (H^i_c(M, \tilde{E}))^*, \quad H^i(M, \tilde{E}^*) = (H_i(M, \tilde{E}))^*. \tag{D.5}$$

Therefore Poincaré duality can be expressed by saying that there are canonical perfect pairings

$$H^i_c(M, \tilde{E}) \times H^{n-i}(M, \tilde{E}^*) \longrightarrow R, \quad i \in \mathbb{Z}, \tag{D.6}$$

$$H_i(M, \tilde{E}) \times H^{cl}_{n-i}(M, \tilde{E}^*) \longrightarrow R, \quad i \in \mathbb{Z}. \tag{D.7}$$

These pairings are given by the cup-product and the intersection product, respectively, followed by the pairing of $\tilde{E}$ and $\tilde{E}^*$ to the constant sheaf $R_M$ on $M$ with stalk $R$. In this generality, if $E = R$, endowed with the trivial $\pi_1(M)$-module structure, then there is a canonical isomorphism

$$H^{cl}_n(M, R) = R. \tag{D.8}$$

By definition, the homology class corresponding to the unit element 1 in $R$ is the fundamental class of $M$, to be denoted $o_M$.

## Appendix E. Lie algebra cohomology

### E.1. **Lie algebra cohomology: generalities.**

*$U(\mathfrak{g})$-modules.* Let $\mathfrak{g}$ be a Lie algebra over a field $F$, which we will think of as $\mathbb{R}$ or $\mathbb{C}$. Let $U$ be an associative $F$-algebra with unit. As usual such an algebra comes equipped with the natural Lie algebra structure defined by the bracket $[X,Y] = XY - YX$, $X, Y \in U$. Let $i : \mathfrak{g} \to U$ be a Lie algebra homomorphism. Then the pair $(U, i)$ is a universal enveloping algebra for $\mathfrak{g}$ if for a given associative $F$-algebra $A$ with unit and a Lie algebra homomorphism $\phi : \mathfrak{g} \to A$ there exists an $F$-algebra homomorphism $\Phi : U \to A$ such that $\phi = \Phi \circ i$.

For a given Lie algebra $\mathfrak{g}$, such a universal algebra exists. Due to the universal mapping property, it is uniquely determined up to isomorphism. Let $I$ denote the two-sided ideal in the tensor algebra $T(\mathfrak{g})$ of $\mathfrak{g}$ generated by the elements $XY - YX - [X,Y]$, $X, Y \in \mathfrak{g}$. Set $U(\mathfrak{g}) := T(\mathfrak{g})/I$, and let $i : \mathfrak{g} \to U(\mathfrak{g})$ be the canonical inclusion. Then the pair $(U(\mathfrak{g}), i)$ is a universal enveloping algebra for $\mathfrak{g}$. We note that for a given $\mathfrak{g}$-module $V$, the corresponding homomorphism $\mathfrak{g} \to \mathrm{End}_F(V)$ of Lie algebras extends to a homomorphism $U(\mathfrak{g}) \to \mathrm{End}_F(V)$ of $F$-algebras. In particular, $\mathfrak{g}$-modules are canonically $U(\mathfrak{g})$-modules and vice versa.

*Cohomology.* Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over $F$, and let $V$ be a $\mathfrak{g}$-module viewed as a $U(\mathfrak{g})$-module as well. In particular, the field $F$ carries a $U(\mathfrak{g})$-module structure by means of the trivial representation of $\mathfrak{g}$ on $F$. The Lie algebra cohomology $H^*(\mathfrak{g}, V)$ of $\mathfrak{g}$ with coefficients in $V$ is defined as the derived functor of $V \mapsto V^{\mathfrak{g}} = \{v \in V \mid Yv = 0, Y \in \mathfrak{g}\}$ in the category of $U(\mathfrak{g})$-modules, that is,

$$H^*(\mathfrak{g}, V) = \mathrm{Ext}_{U(\mathfrak{g})}(F, V).$$

A suitable projective resolution of $F$ as a $U(\mathfrak{g})$-module is given by the standard resolution $C_*$ with $C_p$ the free $\mathfrak{g}$-module $U(\mathfrak{g}) \otimes_F \Lambda^p \mathfrak{g}, p \geq 0$. Note that $\mathfrak{g}$ only acts on the first component. The complex $\mathrm{Hom}_{U(\mathfrak{g})}(C_*, V)$, endowed with the usual differential, is naturally isomorphic to the complex $(C^*(\mathfrak{g}, V), d^*)$ given by

$$C^*(\mathfrak{g}, V) := \mathrm{Hom}_F(\Lambda^* \mathfrak{g}, V),$$

and $d : C^q \longrightarrow C^{q+1}$ is defined by

$$\begin{aligned}(d\omega)(Y_0, ..., Y_q) : \quad &= \sum_{0 \leq i \leq q} (-1)^i Y_i \cdot \omega(Y_0, ..., \hat{Y}_i, ..., Y_q) \\ &\quad + \sum_{i<j} (-1)^{i+j} \omega([Y_i, Y_j], Y_0, ..., \hat{Y}_i, ..., \hat{Y}_j, ..., Y_q),\end{aligned}$$

where $\hat{Y}$ stands for omitting the argument underneath.

We may rewrite $C^*(\mathfrak{g}, V) = \mathrm{Hom}_F(\Lambda^* \mathfrak{g}, V)$ as $\Lambda^* \mathfrak{g}^* \otimes_F V$, and the differential $d$ has a corresponding interpretation. Let $\mathfrak{k}$ be a reductive subalgebra of $\mathfrak{g}$. Since duality reverses the role of subspaces and quotients, one has an inclusion

$$\Lambda(\mathfrak{g}/\mathfrak{k})^* \otimes_F V \longrightarrow \Lambda(\mathfrak{g})^* \otimes_F V. \tag{E.1}$$

By restricting the $\mathfrak{g}$-action, the subalgebra $\mathfrak{k}$ acts on $V$, and $\mathfrak{k}$ acts on $\Lambda(\mathfrak{g}/\mathfrak{k})^*$ via the adjoint action, and hence $\mathfrak{k}$ acts naturally on the tensor product $\Lambda(\mathfrak{g}/\mathfrak{k})^* \otimes_F V$. The space

$$C^*(\mathfrak{g}, \mathfrak{k}, V) := (\Lambda^*(\mathfrak{g}/\mathfrak{k})^* \otimes_F V)^{\mathfrak{k}} \tag{E.2}$$

of $\mathfrak{k}$-invariants in $\Lambda^*(\mathfrak{g}/\mathfrak{k})^* \otimes_F V$ is stable under the differential; thus it forms a subcomplex of the complex $C^*(\mathfrak{g}, V)$. The relative Lie algebra cohomology groups $H^*(\mathfrak{g}, \mathfrak{k}, V)$ of $\mathfrak{g}$ modulo $\mathfrak{k}$ with coefficients in $V$ are, by definition, the cohomology groups of this subcomplex. Note that we may identify $C^*(\mathfrak{g}, \mathfrak{k}, V)$ with the complex $\mathrm{Hom}_{\mathfrak{k}}(\Lambda(\mathfrak{g}/\mathfrak{k}), V)$.

*A slight twist.* Let $G$ denote a real Lie group with finitely many connected components, $\mathfrak{g}$ its Lie algebra, and let $K$ be a compact subgroup of $G$. Its Lie algebra $\mathfrak{k}$ is a reductive subalgebra in $\mathfrak{g}$. If $V$ is a $(\mathfrak{g}, K)$-module we define $C^*(\mathfrak{g}, K, V)$ to be $\mathrm{Hom}_K(\Lambda^*(\mathfrak{g}/\mathfrak{k}), V)$, where $K$ acts on $\Lambda^*(\mathfrak{g}/\mathfrak{k})$ via the adjoint action. Endowed with the differential

$$d : C^q(\mathfrak{g}, K, V) \longrightarrow C^{q+1}(\mathfrak{g}, K, V),$$

defined by

$$\begin{aligned}(d\omega)(Y_0, ..., Y_q) : \quad &= \sum_{0 \le i \le q} (-1)^i Y_i \cdot \omega(Y_0, ..., \hat{Y}_i, ..., Y_q) \\ &\quad + \sum_{i<j} (-1)^{i+j} \omega([Y_i, Y_j], Y_0, ..., \hat{Y}_i, ..., \hat{Y}_j, ..., Y_q),\end{aligned}$$

we have the complex $(C^*(\mathfrak{g}, K, V), d)$. We denote the cohomology of this complex by $H^*(\mathfrak{g}, K, V)$, to be called the relative Lie algebra cohomology of the $(\mathfrak{g}, K)$-module $V$.

Suppose that $K$ is a maximal compact subgroup of $G$, and let $K^0$ denote its identity component. Then the complex $(C^*(\mathfrak{g}, K, V), d)$ is a subcomplex of the complex $(C^*(\mathfrak{g}, K^0, V), d)$ and $H^*(\mathfrak{g}, K, V) = H^*(\mathfrak{g}, \mathfrak{k}, V)^{K/K^0}$; i.e., $H^*(\mathfrak{g}, K, V)$ is obtained by taking the $K/K^0$-invariants in $H^*(\mathfrak{g}, K^0, V)$ under the natural action of its component group $K/K^0$ on the complex $(C^*(\mathfrak{g}, K^0, V), d)$.

For a differentiable $G$-module $H$ we put $(C^*(\mathfrak{g}, K, H), d) := (C^*(\mathfrak{g}, K, H_K), d)$ and $H^*(\mathfrak{g}, K, H) = H^*(\mathfrak{g}, K, H_K)$, respectively.

E.2. **Lie algebra cohomology and invariant forms.** Let $G$ be a real Lie group with finitely many connected components. Recall that $F$ is a field we think of as $\mathbb{R}$ or $\mathbb{C}$. Suppose that $E/F$ is a finite extension of fields, $E \subset \mathbb{C}$. We let $l_g : G \to G$, resp. $r_g : G \to G$, denote the left, resp. right, translation by $g \in G$. We call a smooth $E$-valued differential form $\phi \in \Omega^*(G)$ on $G$ invariant if $l_g^*\phi = \phi$ and $r_g^*\phi = \phi$ for all $g \in G$. Analogously, we have the notion of a left-invariant, resp. right-invariant, form. The set of invariant forms on $G$ is a graded subalgebra of $\Omega^*(G)$, to be denoted $\Omega_I^*(G)$, which is stable under the exterior differential $\Omega^*(G) \longrightarrow \Omega^*(G)$. Note that the invariant forms on $G$ are closed.

Evaluating a given right-invariant form $\omega$ on $G$ at the origin of $G$, that is, the assignment $\omega \mapsto \omega(e)$, gives rise to an isomorphism

$$\Omega_R^*(G) \xrightarrow{\sim} C^*(\mathfrak{g}, E) = \Lambda^*\mathfrak{g}^* \otimes E \tag{E.3}$$

of the space of right-invariant $E$-valued smooth forms on $G$ onto the complex $C^*(\mathfrak{g}, E) = \mathrm{Hom}_F(\Lambda^*\mathfrak{g}, E)$ underlying the Lie algebra cohomology of the Lie algebra $\mathfrak{g}$ of $G$ with coefficients in the trivial $\mathfrak{g}$-module $E$. If $G$ is compact and connected, this isomorphism induces an isomorphism of $H^*(\mathfrak{g}, E)$ onto $H^*(G, E)$, and the latter space can be identified with the space $\Omega_I^*(G)$ of invariant forms on $G$ (e.g. [24], V,12).

Let $K$ be a closed connected subgroup of $G$, $\mathfrak{k}$ its Lie algebra. We denote the natural projection map $G \to K\backslash G$ by $\pi$. Then the triple $(G, \pi, K\backslash G)$ is a principal bundle with structure group $K$. The differential $d\pi_e$ induces an $\mathbb{R}$-linear isomorphism $T_{Ke}(K\backslash G)\tilde{\to}(\mathfrak{k}\backslash\mathfrak{g})$ of the tangent space at the point $Ke$ on the quotient $\mathfrak{k}\backslash\mathfrak{g}$. This isomorphism can be extended to an isomorphism of the tangent bundle of $K\backslash G$ and the bundle $(\mathfrak{k}\backslash\mathfrak{g}) \times_K G$. The evaluation map at the origin defines an isomorphism

$$\Omega_I^*(K\backslash G)\tilde{\longrightarrow}C^*(\mathfrak{g}, K, E) = \mathrm{Hom}_K(\Lambda^*(\mathfrak{k}\backslash\mathfrak{g}), E) \tag{E.4}$$

of the space of $G$-invariant $E$-valued smooth forms on $K\backslash G$ onto the complex $C^*(\mathfrak{g}, K, E)$. Therefore, the relative Lie algebra cohomology $H^*(\mathfrak{g}, K, E)$ is the cohomology of the space of $G$-invariant $E$-valued smooth forms on $K\backslash G$. If $G$ is compact and connected, the latter space can be identified with the cohomology space $H^*(K\backslash G, E)$.

Within the context of Riemannian symmetric spaces, some cases deserve particular notice. For example, suppose that $\mathfrak{g}$ is a non-compact semi-simple Lie algebra over $\mathbb{R}$, and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be any Cartan decomposition with corresponding involutive automorphism $\theta$. Then, by definition, the subalgebra $\mathfrak{k}$ is the set of fixed points of $\theta$, and $(\mathfrak{g}, \theta)$ is an (effective) orthogonal symmetric (or involutive) Lie algebra. We refer to [60, p. 213] for details. Similarly, let $\mathfrak{g}$ be a compact semi-simple Lie algebra endowed with an involutive automorphism $s$. Then the set $\mathfrak{k}$ of fixed points of $s$ is a compactly embedded subalgebra of $\mathfrak{g}$, and $(\mathfrak{g}, s)$ is an orthogonal symmetric Lie algebra. In such cases where $(\mathfrak{g}, s)$ is an orthogonal symmetric Lie algebra, the differential of the complex $C^*(\mathfrak{g}, \mathfrak{k}, \mathbb{R})$ vanishes identically [73, p. 101]. Thus we have the natural identification

$$H^*(\mathfrak{g}, \mathfrak{k}, \mathbb{R}) = C^*(\mathfrak{g}, \mathfrak{k}, \mathbb{R}). \tag{E.5}$$

## Appendix F. Some representation theory

We recall some basic definitions and results pertaining to the theory of Hilbert space representations of real Lie groups which are needed in the text. Nevertheless we have to assume some familiarity with the basic sources as, for example, [158], [159].

F.1. **$(\mathfrak{g}, K)$-modules.** Let $G$ be a real Lie group with finitely many connected components, let $K$ be a compact subgroup of $G$. The Lie algebras of $G$ and $K$ are denoted by $\mathfrak{g}$ and $\mathfrak{k}$, respectively. We denote by $U(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$. Let $V$ be a real of complex vector space which is a $\mathfrak{g}$-module endowed with a $K$-module structure. Then $V$ is called a $(\mathfrak{g}, K)$-module if the actions of $\mathfrak{g}$ and $K$ satisfy the following compatibility conditions:

- $f \in V$ is $K$-finite (i.e., the set $\{kf, k \in K\}$ of translates of $f$ spans a finite-dimensional space).
- $k.Y.v = \mathrm{Ad}k(Y).k.v$, $v \in V, k \in K, Y \in \mathfrak{g}$.
- If $v \in V$, then $Kv$ spans a finite-dimensional subspace $W_v$ such that the action of $K$ on $W_v$ is continuous; that is, $V$ is locally finite as a $K$-module.
- If $W \subset V$ is a $K$-stable finite-dimensional subspace of $V$, then the representation of $K$ on $W$ is differentiable and has the restriction of the $\mathfrak{g}$-action to $\mathfrak{k}$ as its differential.

By definition, a $(\mathfrak{g}, K)$-module is admissible if all the isotypic components for $K$ are finite dimensional. The module $V$ is said to be irreducible if $V$ and $(0)$ are the only subspaces invariant under $\mathfrak{g}$ and $K$. We denote by $C(\mathfrak{g}, K)$ the category of all $(\mathfrak{g}, K)$-modules and $(\mathfrak{g}, K)$-homomorphisms.

F.2. **Hilbert space representations.** Let $(\pi, H)$ be a Hilbert space representation of the given Lie group $G$. If $v \in H$ is such that the assignment $g \mapsto \pi(g).v$ defines a differentiable function $G \to H$, then we call $v$ a $C^\infty$-vector for $(\pi, H)$. The space of $C^\infty$-vectors for $(\pi, H)$, to be denoted $H^\infty$, is dense in $H$ and stable under $G$. Thus, $\pi$ defines a representation of $\mathfrak{g}$ on $H^\infty$, to be again denoted by $\pi$. The universal mapping property of the universal enveloping algebra $U(\mathfrak{g})$ makes it possible to extend $\pi$ to a representation of $U(\mathfrak{g})$.

A given Hilbert space representation $(\pi, H)$ is said to have an infinitesimal character $\chi$ if there is a homomorphism $\chi : Z_G(\mathfrak{g}) \longrightarrow \mathbb{C}$ such that $\pi(z)v = \chi(z)v$ for all $z \in Z_G(\mathfrak{g}) = \{Y \in U(\mathfrak{g}) \mid \mathrm{Ad}(g)Y = Y, g \in G\}$ and $v \in H^\infty$.

Let $K^e$ denote the set of all equivalence classes of irreducible unitary representations of the compact group $K$. With respect to the $K$-module structure, we may write the Hilbert space $H$ as a direct sum of its $K$-isotypic components $H(\tau)$, $\tau \in K^e$. For a given $\tau \in K^e$, the subspace $H(\tau) \cap H^\infty$ is dense in $H(\tau)$. Then the algebraic direct sum of these subspaces $H(\tau) \cap H^\infty$, to be denoted $H_K$, is a dense subspace of $H^\infty$. In particular, $H_K$ is $\mathfrak{g}$-invariant. Then $H_K$ is a $(\mathfrak{g}, K)$-module. One may characterize $H_K$ as the space of all $C^\infty$-vectors $v$ for $H$ such that $\pi(K).v$ spans a finite-dimensional subspace of $H$.

If $(\pi, H)$ and $(\pi', H')$are two Hilbert space representations of $G$, then, by definition, $\pi$ is infinitesimally equivalent to $\pi'$ if the associated $(\mathfrak{g}, K)$-modules $H_K$ and $H'_K$ are equivalent.

The following fundamental results are due to Harish Chandra. If $(\pi, H)$ is an irreducible unitary Hilbert space representation of $G$, then the associated $(\mathfrak{g}, K)$-module $H_K$ is admissible. Furthermore, a given unitary representation $(\pi, H)$ of $G$ is irreducible if and only if it is infinitesimally irreducible, that is, $H_K$ is irreducible as a $(\mathfrak{g}, K)$-module. If $(\pi, H)$ and $(\pi', H')$ are two irreducible unitary Hilbert space representations of $G$, then $\pi$ and $\pi'$ are unitarily equivalent if and only if they are infinitesimally equivalent. It is worth noting that every irreducible admissible $(\mathfrak{g}, K)$-module $V$ can be realized as the space of $K$-finite vectors in an irreducible admissible differentiable $G$-module. In such a case, $V$ has an infinitesimal character.

## About the author

Joachim Schwermer is professor of mathematics at the University of Vienna and scientific director at the Erwin-Schrödinger International Institute for Mathematical Physics, Vienna. His research focuses on questions arising in arithmetic algebraic geometry and the theory of automorphic forms. He takes a keen interest in the mathematical sciences in the nineteenth and twentieth centuries in their historical context.

## References

1. I. Agol, *Virtual Betti numbers of symmetric spaces*, arXiv:math/0611828v1.
2. A. A. Albert and N. Jacobson, *On reduced exceptional simple Jordan algebras*, Ann. of Math. (2) **66** (1957), 400–417. MR0088487 (19:527b)

3. R.C. Alperin, *An elementary account of Selberg's lemma*, L'Enseign. Math. **33** (1987), 269–273. MR925989 (89f:20051)
4. A. Ash, *Non-square-integrable cohomology of arithmetic groups*, Duke Math. J. **47** (1980), 435–449. MR575906 (82m:22013)
5. A. Ash, *A note on minimal modular symbols*, Proc. Amer. Math. Soc. **96** (1986), 394–396. MR822426 (87e:22024)
6. A. Ash, *Non-minimal modular symbols for GL(n)*, Invent. Math. **91** (1988), 483–491. MR928493 (89c:11089)
7. A. Ash, A. Borel, *Generalized modular symbols*, In: *Cohomology of Arithmetic Groups and Automorphic Forms*, ed. J.-P. Labesse , J. Schwermer, Lecture Notes in Math. 1447 (1990), 57–75. MR1082962 (92e:11058)
8. A. Ash, D. Ginzburg, S. Rallis, *Vanishing periods of cusp forms over modular symbols*, Math. Ann. **296** (1993), 709–723 MR1233493 (94f:11044)
9. A. Ash, D. Ginzburg, *Generalized modular symbols and relative Lie algebra cohomology*, Pacific J. Math. **175** (1996), 337–355. MR1432835 (98g:11057)
10. A. Ash and L. Rudolph, *The modular symbol and continued fractions in higher dimensions*, Invent. Math. **55** (1979), 241–250. MR553998 (82g:12011)
11. N. Bergeron, *Lefschetz properties for arithmetic real and complex hyperbolic manifolds*, Int. Math. Research Notices **20** (2003), 1089–1122. MR1963482 (2004a:11040)
12. N. Bergeron, *Restriction de la cohomologie d'une variété de Shimura à ses sous-variétés*, Transformation Groups **14** (2009), 41–86. MR2480852
13. L. Bianchi, *Sui gruppi di sostitutioni lineari con coefficienti a corpi quadratici imaginari*, Math. Ann. **40** (1892), 332–411. MR1510727
14. D. Birkes, *Orbits of linear algebraic groups*, Ann. of Math. (2) **93** (1971), 459–475. MR0296077 (45:5138)
15. A. Borel, *Introduction aux Groupes Arithmétiques*, Hermann, Paris, 1969. MR0244260 (39:5577)
16. A. Borel, *Stable real cohomology of arithmetic groups*, Ann. Sci. Ecole Norm. Sup. (4) **7** (1974), 235–272. MR0387496 (52:8338)
17. A. Borel, *Cohomologie de sous-groupes discrets et représentations de groupes semi-simples*, Astérisque **32-33** (1976), 73–112. MR0578913 (58:28282)
18. A. Borel, *Commensurability classes and volumes of hyperbolic* 3*-manifolds*. Ann. Scuola Norm. Sup. Pisa **8** (1981), 1–33. MR616899 (82j:22008)
19. A. Borel, Harish-Chandra, *Arithmetic subgroups of algebraic groups*, Ann. of Math. (2) **75** (1962), 485–535. MR0147566 (26:5081)
20. A. Borel, H. Jacquet, *Automorphic forms and automorphic representations*, In: *Automorphic Forms, Representations and L-functions*, Proc. Sympos. Pure Math. 33 I, Amer. Math. Soc., Providence, RI, 1979, pp. 189–202. MR546598 (81m:10055)
21. A. Borel, J.-P. Labesse, J. Schwermer, *On the cuspidal cohomology of S-arithmetic groups of reductive groups over number fields*, Compositio Math. **102** (1996), 1–40. MR1394519 (97j:11026)
22. A. Borel, J.-P. Serre, *Corners and arithmetic groups*, Comment. Math. Helvet. **48** (1973), 436–491. MR0387495 (52:8337)
23. A. Borel, N. Wallach, *Continuous Cohomology, Discrete Subgroups and Representations of Reductive Groups*, Ann. Math. Studies 94, Princeton: Princeton University Press, 1980. MR554917 (83c:22018)
24. G. Bredon, *Topology and Geometry*, Graduate Texts in Math. 139, New York-Berlin-Heidelberg: Springer-Verlag, 1993. MR1224675 (94d:55001)
25. G. Bredon, *Sheaf Theory*, 2nd edition, Graduate Texts in Math. 170, New York-Berlin-Heidelberg: Springer-Verlag, 1997. MR1481706 (98g:55005)
26. M.R. Bridson, A. Haefliger, *Metric Spaces of Non-Positive Curvature*, Grundlehren Math. Wiss. Bd. 319, Berlin-Heidelberg-New York: Springer-Verlag, 1999. MR1744486 (2000k:53038)
27. M. Burger, J.-S. Li, P. Sarnak, *Ramanujan duals and automorphic spectrum*, Bull. Amer. Math. Soc. **26** (1992), 253–257. MR1118700 (92h:22023)
28. L. Clozel, *On the cuspidal cohomology of arithmetic subgroups of* $SL(2n)$ *and the first Betti number of arithmetic* 3*-manifolds*, Duke Math. J. **55** (1987), 475–486. MR894591 (88m:22022)

29. L. Clozel and T.N. Venkataramana, *Restriction of the holomorphic cohomology of a Shimura variety to a smaller Shimura variety*, Duke Math, J. **95** (1998), 51–106. MR1646542 (99g:11077)
30. J. Cogdell, *Arithmetic cycles on Picard modular surfaces and modular forms of Nebentypus*, J. Reine Angew. Math. **357** (1985), 115–137. MR783537 (86m:11030)
31. A. Dold, *Lectures on Algebraic Topology*, Grundlehren Math. Wiss. Bd. 200, Berlin-Heidelberg-New York: Springer-Verlag, 1972. MR0415602 (54:3685)
32. N. Dunfield and W.P. Thurston, *The virtual Haken conjecture: Experiments and examples.* Geometric Topol. **7** (2003), 399–441. MR1988291 (2004i:57024)
33. J. Elstrodt, F. Grunewald, J. Mennicke, *Groups acting on hyperbolic space.* Berlin-Heidelberg-New York: Springer-Verlag, 1997. MR1483315 (98g:11058)
34. T. Enright, *Relative Lie algebra cohomology and unitary representations of complex Lie groups*, Duke Math. J. **46** (1979), 513–525. MR544243 (81i:22007)
35. F. Farell, P. Ontaneda, M.S. Raghunathan, *Non-univalent harmonic maps homotopic to diffeomorphisms*, J. Differential Geometry **54** (2000), 227–253. MR1818179 (2002e:58026)
36. J. Franke, *Harmonic analysis in weighted $L_2$-spaces*, Ann. Sci. Ecole Norm. Sup. (4) **31** (1998), 181–279. MR1603257 (2000f:11065)
37. J. Franke, *A topological model for some summand of the Eisenstein cohomology of congruence subgroups*, [preprint 1991]. In: Eisenstein Series and Applications, Progress Math. 258, pp. 27–85, Berlin-Heidelberg-New York: Birkhäuser-Verlag, 2008. MR2402680 (2009i:11069)
38. J. Franke and J. Schwermer, *A decomposition of spaces of automorphic forms, and the Eisenstein cohomology of arithmetic groups*, Math. Ann. **311** (1998), 765–790. MR1637980 (99k:11077)
39. W. Fulton, *Intersection Theory*, Ergebnisse der Mathematik (3), vol. 2, Berlin-Heidelberg-New York: Springer-Verlag, 1984. MR732620 (85k:14004)
40. J. Funke and J.J. Millson, *Cycles in hyperbolic manifolds of non-compact type and Fourier coefficients of Siegel modular forms*, Manuscripta Math. **107** (2002), 409–444. MR1906769 (2003d:11070)
41. J. Funke and J.J. Millson, *Cycles with local coefficients for orthogonal groups and vector-valued Siegel modular forms*, Amer. J. Math. **128** (2006), 899–948. MR2251589 (2007i:11068)
42. J. Funke and J.J. Millson, *Boundary behaviour of special cohomology classes arising from the Weil representation*, preprint 2007.
43. M. Goresky, *Compactifications and cohomology of modular varieties*, In: Harmonic Analysis, the Trace Formula, and Shimura Varieties, Clay Math. Proc. 4, pp. 551–582, Amer. Math. Soc., Providence, RI, 2005. MR2192016 (2006h:14033)
44. G. Gotsbacher and J. Schwermer, *Automorphic cohomology of arithmetically defined hyperbolic $n$-manifolds*, forthcoming.
45. M. Gromov and I. Piatetski-Shapiro, *Non-arithmetic groups in Lobachevsky spaces*, Publ. Math. Inst. Hautes Études Sci. **66** (1988), 93–103. MR932135 (89j:22019)
46. F. Grunewald and S. Kühnlein, *On the proof of Humbert's volume formula*, Manuscripta Math. **95** (1998), 431–436. MR1618190 (99h:11044)
47. F. Grunewald and J. Schwermer, *Arithmetic quotients of hyperbolic* 3*-space, cusp forms and link complements,* Duke Math. J. **48** (1981), 351–358. MR620254 (82j:10046)
48. F. Grunewald and J. Schwermer, *Free non-abelian quotients of $SL_2$ over orders of imaginary quadratic number fields,* J. Algebra **69** (1981), 298–304. MR617080 (82i:10027)
49. F. Grunewald and J. Schwermer, *A non-vanishing theorem for the cuspidal cohomology of $SL_2$ over imaginary quadratic integers,* Math. Ann. **258** (1981), 183–200. MR641824 (83c:10032)
50. P. E. Gunnells, *Modular symbols for $\mathbb{Q}$-rank one groups and Voronoĭ reduction,* J. Number Theory **75** (1999), 198–219. MR1681629 (2000c:11084)
51. P. E. Gunnells, *Symplectic modular symbols*, Duke Math. J. **102** (2000), 329–350. MR1749441 (2001i:11062)
52. G. Harder, *On the cohomology of $SL(2,\mathcal{O})$,* In: *Lie groups and their representations.* Proc. of the Summer School on Group Representations. pp. 139–150. London: Hilger, 1975. MR0425019 (54:12977)
53. G. Harder, *On the cohomology of discrete arithmetically defined groups*, In: Discrete subgroups of Lie groups and applications to moduli (Internat. Colloq., Bombay, 1973), pp. 129–160. Oxford Univ. Press, Bombay, 1975. MR0425018 (54:12976)

54. G. Harder, *General aspect in the theory of modular symbols,* Séminaire de théorie des nombres, Progress Math. 38, pp. 72–88. Boston: Birkhäuser, 1983. MR729161 (85h:11032)
55. G. Harder, *Eisenstein cohomology of arithmetic groups. The case $GL_2$*, Invent. Math. **89** (1987), no. 1, 37–118. MR892187 (89b:22018)
56. G. Harder, *A Gauss–Bonnet formula for discrete arithmetically defined groups,* Ann. Sci. École Norm. Sup.(4) **4** (1971), 409–455. MR0309145 (46:8255)
57. G. Harder, R.P. Langlands and M. Rapoport, *Algebraische Zyklen auf Hilbert-Blumenthal-flächen*, J. Reine Angew. Math. **366** (1986), 53–120. MR833013 (87k:11066)
58. Harish-Chandra, *Automorphic Forms on Semi-Simple Lie Groups*, Lecture Notes in Math. 62. Berlin-Heidelberg-New York: Springer-Verlag, 1968. MR0232893 (38:1216)
59. M. Harris and J. S. Li, *Lefschetz property for subvarieties of Shimura varieties*, J. Algebraic Geometry **7** (1998), 77–122. MR1620690 (99e:14027)
60. S. Helgason, *Differential Geometry, Lie Groups, and Symmetric Spaces*, New York and London: Academic Press, 1962. MR514561 (80k:53081)
61. F. Hirzebruch and D. Zagier, *Intersection numbers of curves on Hilbert modular surfaces and modular forms of Nebentypus*, Invent. Math. **36** (1976), 57–113. MR0453649 (56:11909)
62. N. Jacobson, *Composition algebras and their automorphisms*, Rend. Circ. Mat. Palermo (2) **7** (1958), 55–80. MR0101253 (21:66)
63. N. Jacobson, *Some groups of transformations defined by Jordan algebras I*, J. Reine Angew. Math. **201** (1959), 178–195. MR0106936 (21:5666)
64. N. Jacobson, *Some groups of transformations defined by Jordan algebras II, Groups of type $F_4$*, J. Reine Angew. Math. **204** (1960), 7–98. MR0159849 (28:3065)
65. N. Jacobson, *Finite-dimensional Division Algebras over Fields*, Berlin-Heidelberg: Springer-Verlag, 1996. MR1439248 (98a:16024)
66. H. Jacquet and R.P. Langlands, *Automorphic Forms on $GL(2)$*, Lecture Notes in Math., 114, Berlin-Heidelberg-New York: Springer-Verlag, 1970. MR0401654 (53:5481)
67. H. Jacquet, E. Lapid and J. Rogawski, *Periods of automorphic forms*, J. Amer. Math. Soc. **12** (1999), 173–240. MR1625060 (99c:11056)
68. H. Jacquet, I. Piatetski-Shapiro, J. Shalika, *Relèvement cubique non normal*, C.R. Acad. Sci. Paris **292** (1981), 567–571. MR615450 (82i:10035)
69. E. Kleinert, *Units of classical orders: A survey*, L'Enseignement Mathématique **40** (1994), 205–248. MR1309127 (95k:11151)
70. A. Knapp, D. Vogan, *Cohomological Induction and Unitary Representations*, Princeton: Princeton University Press, 1995. MR1330919 (96c:22023)
71. M.-A. Knus, A. Merkurjev, M. Rost and J.-P. Tignol, *The Book of Involutions*, Colloquium Publications 44. Amer. Math. Soc., 1998. MR1632779 (2000a:16031)
72. T. Kobayashi, T. Oda, *A vanishing theorem for modular symbols on locally symmetric spaces*, Comment. Math. Helv. **73** (1998), 45–70. MR1610583 (99b:11052)
73. J.-L. Koszul, *Homologie et cohomologie des algèbres de Lie*, Bull. Soc. Math. France **78** (1950), 65–127. MR0036511 (12:120g)
74. S. Kudla, *Intersection numbers for quotients of the complex 2-ball and Hilbert modular forms*, Invent. Math. **47** (1978), 189–208. MR501929 (80a:10044)
75. S. Kudla, *Algebraic cycles on Shimura varieties of orthogonal type*, Duke Math. J. **86** (1997), 39–78. MR1427845 (98e:11058)
76. S. Kudla, J. Millson, *Geodesic cycles and the Weil representation - I. Quotients of hyperbolic space and Siegel modular forms*, Compositio Math. **45** (1982), 207–271. MR651982 (83m:10037)
77. S. Kudla, J. Millson, *The theta correspondence and harmonic forms I*, Math. Ann. **274** (1986), 353–378; *II*, **277** (1987), 267–314. MR842618 (88b:11023)
78. S. Kudla, J. Millson, *Intersection numbers of cycles on locally symmetric spaces and Fourier coefficients of holomorphic modular forms in several complex variables*, Publ. Math. Inst. Hautes Études Sci., **71** (1990), 121–170. MR1079646 (92e:11035)
79. S. Kudla, M. Rapoport, T, Yang, *Modular Forms and Special Cycles on Shimura Curves*, Annals of Mathematics Studies 161, Princeton: Princeton University Press, 2006. MR2220359 (2007i:11084)
80. J.P. Labesse and J. Schwermer, *On liftings and cusp cohomology of arithmetic groups*, Invent. Math. **83** (1986), 383–401. MR818358 (87g:11060)

81. J.F. Lafont, B. Schmidt, *On submanifolds in locally symmetric spaces of noncompact type*, Algebraic & Geometric Topology **6** (2006), 2455–2472. MR2286032 (2008b:57041)
82. R.P. Langlands, *Base Change for $GL(2)$*, Ann. of Math. Studies, vol. 96, Princeton: Princeton University Press, 1980. MR574808 (82a:10032)
83. R.P. Langlands, *On the classification of irreducible representations of real algebraic groups*, In: Representation Theory and Harmonic Analysis in Semisimple Lie Groups, Math. Surveys Monogr. **31**, pp. 101–170, American Math. Soc., Providence, 1989. MR1011897 (91e:22017)
84. R. Lee and J. Schwermer, *The Lefschetz number of an involution on the space of cusp forms of $SL_3$*, Invent. Math. **73** (1983), 189–239. MR714089 (84k:22016)
85. R. Lee and J. Schwermer, *Geometry and arithmetic cycles attached to $SL_3(\mathbb{Z})$-I*, Topology **25** (1986), 159–174. MR837619 (87h:22015)
86. J.-S. Li, *Nonvanishing theorems for the cohomology of certain arithmetic quotients*, J. Reine Angew. Math. **428** (1992), 177–217. MR1166512 (93e:11067)
87. J.-S. Li and J.J. Millson, *On the first Betti number of a hyperbolic manifold with an arithmetic fundamental group,* Duke Math. J. **71** (1993), 365–401. MR1233441 (94e:11062)
88. J.-S. Li and J. Schwermer, *Constructions of automorphic forms and related cohomology classes for arithmetic subgroups of $G_2$*, Compositio Math. **87** (1993), 45–78. MR1219452 (96b:11071)
89. J.-S. Li and J. Schwermer, *Automorphic representations and cohomology of arithmetic groups,* In: Challenges for the 21st century (Singapore, 2000), 102–137, World Sci. Publishing, 2001. MR1875016 (2003c:11051)
90. J.-S. Li and J. Schwermer, *On the Eisenstein cohomology of arithmetic groups,* Duke Math. J. **123** (2004), 141–169. MR2060025 (2005h:11108)
91. J.-S. Li and J. Schwermer, *On the cuspidal cohomology of arithmetic groups,* Amer. J. Math. **131** (2009), 1431–1464.
92. A. Lubotzky, *Free quotients and the first Betti number of some hyperbolic manifolds*, Transformation Groups **1** (1996), 71–82. MR1390750 (97d:57016)
93. C. Maclachlan and A. Reid, *The Arithmetic of Hyperbolic* 3*-manifolds*, Graduate Texts in Math. 219, Berlin-Heidelberg-New York: Springer-Verlag, 2003. MR1937957 (2004i:57021)
94. G.A. Margulis and E. Vinberg, *Some linear groups virtually having a free quotient*, J. Lie Theory **10** (2000), 171–180. MR1748082 (2001h:22016)
95. Y. Matsushima, *On Betti numbers of compact, locally symmetric Riemannian manifolds*, Osaka Math. Journal **14** (1962), 1–20. MR0141138 (25:4549)
96. B. Mazur, *Courbes elliptiques et symboles modulaires.* Séminaire Boubaki 1971/72. exp. 414, In: Lecture Notes in Math., 317, pp. 277–294, Berlin-Heidelberg-New York: Springer-Verlag, 1973. MR0429921 (55:2930)
97. E. Mendoza, *Cohomology of $PGL_2$ over imaginary quadratic integers*, Bonner Math. Schriften 128. Math. Institut, Universität Bonn, Bonn, 1980. MR611515 (82g:22012)
98. A. Meyer, *Zur Theorie der indefiniten quadratischen Formen*, J. Reine Angew. Math. **108** (1891), 125–139.
99. J. Millson, *On the first Betti number of a constant negatively curved manifold*, Ann. of Math. (2) **104** (1976), 235–247. MR0422501 (54:10488)
100. J. Millson, *A remark on Raghunathan's vanishing theorem*, Topology **24** (1985), 495–498. MR816528 (87g:22010)
101. J. Millson, *Cycles and harmonic forms on locally symmetric spaces*, Canad. Math. Bull. **28** (1985), 3–38. MR778258 (87b:11038)
102. J. Millson and M.S. Raghunathan, *Geometric construction of cohomology for arithmetic groups* I. In: Geometry and Analysis (Papers dedicated to the memory of V. K. Patodi), pp. 103–123, Indian Academy of Sciences, Bangalore, 1980. MR592256 (81m:22020)
103. J. Milnor and J. Stasheff, *Characteristic classes*, Ann. of Math. Studies, vol.76, Princeton: Princeton University Press, 1974. MR0440554 (55:13428)
104. H. Minkowski, *Über den arithmetischen Begriff der Äquivalenz und über die endlichen Gruppen linearer ganzzahliger Substitutionen*, J. Reine Angew. Math. **100** (1887), 449–458.
105. H. Minkowski, *Zur Theorie der positiven quadratischen Formen*, J. Reine Angew. Math. **101** (1887), 196–202.
106. G. D. Mostow, *Strong Rigidity in Locally Symmetric Spaces*, Annals of Mathematics Studies 78, Princeton: Princeton University Press, 1973. MR0385004 (52:5874)

107. T. Oda, *A note on the Albanese variety of an arithmetic quotient of the complex hyperball*, J. Fac. Sci. Univ. Tokyo, Sect. 1A Math. **28** (1981), 481–486. MR656032 (83j:14035)
108. D. Quillen, *Elementary proofs of some results of cobordism theory using Steenrod operations*, Advances in Math., 7 (1971), 29–56. MR0290382 (44:7566)
109. M.S. Raghunathan, *A note on quotients of real algebraic groups by arithmetic subgroups*, Invent. Math. **4** (1968), 318–335. MR0230332 (37:5894)
110. M.S. Raghunathan, *Discrete Subgroups of Lie Groups*, Ergebnisse der Mathematik 68, Berlin-Heidelberg-New York: Springer-Verlag, 1972. MR0507234 (58:22394a)
111. M.S. Raghunathan, *The first Betti number of compact locally symmetric spaces.* In: Current Trends in Mathematics and Physics. 116–137, New Delhi: Narosa, 1995. MR1354176 (97a:11082)
112. M.S. Raghunathan, *Arithmetic lattices in semisimple groups*, Proc. Indian Acad. Sci. (Math. Sci), **91** (1982), 133–138. MR682519 (84a:22025)
113. M.S. Raghunathan and T.N. Venkataramana, *The first Betti number of arithmetic groups and the congruence subgroup problem.* In: Linear Algebraic Groups and their Representations, Contemporary Math. **153** (1993), 95–107. MR1247500 (94i:20086)
114. C.S. Rajan, *On the image and fibres of solvable base change*, Math. Res. Letters **9** (2002), 499–508. MR1928869 (2003g:11054)
115. C.S. Rajan, *On the non-vanishing of the first Betti number of hyperbolic three-manifolds*, Math. Ann. **330** (2004), 323–329. MR2089429 (2005j:11038)
116. J.G. Ratcliffe, *Foundations of Hyperbolic Manifolds.* Graduate Texts in Math. 149. New York-Berlin-Heidelberg: Springer-Verlag, 1994. MR1299730 (95j:57011)
117. I. Reiner, *Maximal orders.* London Math. Soc. Monographs, London-New York: Academic Press, 1975. MR0393100 (52:13910)
118. J. Rohlfs, *Arithmetisch definierte Gruppen mit Galoisoperation*, Invent. Math. **48** (1978), 185–205. MR507801 (80j:20043)
119. J. Rohlfs, *On the cuspidal cohomology of the Bianchi modular groups*, Math. Zeitschrift **188** (1985), 253–269. MR772354 (86e:11042)
120. J. Rohlfs, *The Lefschetz number of an involution on the space of classes of positive definite quadratic forms*, Comment. Math. Helvet. **56** (1981), 272–296. MR630954 (83a:10037)
121. J. Rohlfs, *Lefschetz numbers for arithmetic groups*, In: Cohomology of Arithmetic Groups and Automorphic Forms, (ed. J.-P. Labesse, J. Schwermer), Lecture Notes in Math., 1447, pp. 303–313, Berlin-Heidelberg-New York: Springer-Verlag, 1990. MR1082971 (92d:11055)
122. J. Rohlfs and J. Schwermer, *Intersection numbers of special cycles*, J. American Math. Soc. **6** (1993), 755–778. MR1186963 (94a:11075)
123. J. Rohlfs and J. Schwermer, *An arithmetic formula for a topological invariant of Siegel modular varieties*, Topology **37** (1998), 149–159. MR1480883 (98f:11044)
124. J. Schwermer, *A note on link complements and arithmetic groups,* Math. Ann. **249** (1980), 107–110. MR578717 (82e:22027)
125. J. Schwermer, *Kohomologie arithmetisch definierter Gruppen und Eisensteinreihen*, Lecture Notes in Math., 988, Berlin-Heidelberg-New York: Springer-Verlag, 1983. MR822473 (87i:22034)
126. J. Schwermer, *Cohomology of arithmetic groups, automorphic forms and L-functions* In: Cohomology of Arithmetic Groups and Automorphic Forms, (ed. J.-P. Labesse, J. Schwermer), Lecture Notes in Math., 1447, pp. 1–29, Berlin-Heidelberg-New York: Springer-Verlag, 1990. MR1082960 (92e:11059)
127. J. Schwermer, *Eisenstein series and cohomology of arithmetic groups: The generic case*, Invent. Math. **116** (1994), 481–511. MR1253202 (95h:11049)
128. J. Schwermer, *Arithmetic Groups—Geometric Aspects.* Lectures at ETH Zürich (Nachdiplomvorlesung) 1999/2000, in preparation.
129. J. Schwermer, *Special cycles and automorphic forms on arithmetically defined hyperbolic 3-manifolds*, Asian J. Mathematics **8** (2004), 837–860. MR2127951 (2005m:11073)
130. J. Schwermer, *Reduction theory of quadratic forms: Towards Räumliche Anschauung in Minkowski's early work*, In: The Shaping of Arithmetic after C. F. Gauss's Disquisitiones Arithmeticae (ed. C. Goldstein, N. Schappacher, J. Schwermer), pp. 483–504, Berlin-Heidelberg-New York: Springer-Verlag, 2007. MR2308294

131. J. Schwermer, *The cohomological approach to cuspidal automorphic representations*, In: Automorphic forms and $L$-functions I: Global Aspects—a volume in honor of Steve Gelbart, Contemporary Math. 488, pp. 257–284, American Math. Society, 2009.
132. J. Schwermer, *Minkowski in Königsberg 1884: a talk in Lindemann's colloquium*, Bull. Amer. Math. Soc. **47** (2010), no. 2, 355–362.
133. J. Schwermer, *Geometric cycles, Albert algebras and related cohomology classes for arithmetic groups* (preprint).
134. J. Schwermer and K. Vogtmann, *The integral homology of $SL_2$ and $PSL_2$ of Euclidean imaginary quadratic integers,* Comment. Math. Helvetici **58** (1983), 573–598. MR728453 (86d:11046)
135. J.-P. Serre, Corps Locaux. Paris: Hermann 1962. MR0354618 (50:7096)
136. J.-P. Serre, *Cohomologie Galoisienne*. Lecture Notes in Math. 5. Berlin-Heidelberg-New York: Springer, 1964.
137. J.-P. Serre, *Cohomologie des groupes discrets*, In: Prospects in Mathematics, Annals of Mathematics Studies 70, pp. 77–169, Princeton: Princeton University Press, 1971. MR0385006 (52:5876)
138. J.-P. Serre, *Le problème des groupes de congruence pour $SL_2$*, Ann. of Math. (2) **92** (1972), 489–527. MR0272790 (42:7671)
139. C. L. Siegel, *Einheiten quadratischer Formen*, Abh. Math. Seminar Hansische U. **13** (1940), 209–239. MR0003003 (2:148b)
140. C. L. Siegel, *Symplectic geometry*, Amer. J. Math. **65** (1943), 1–85. MR0008094 (4:242b)
141. E. H. Spanier, Algebraic Topology. New York: McGraw-Hill, 1966. MR0210112 (35:1007)
142. B. Speh, *Representation theory and the cohomology of arithmetic groups*, Proc. Int. Congress Math. Madrid, 2006, vol. II, European Math. Soc. 2006, 1327–1335. MR2275647 (2008b:11061)
143. B. Speh, T. N. Venkataramana, *Construction of some generalised modular symbols*, Pure Appl. Math. Q. **1** (2005), 737–754. MR2200998 (2006i:11052)
144. T.A. Springer, *Linear Algebraic Groups*, 2nd ed., Progr. in Math. 9, Boston: Birkhäuser, 1998. MR1642713 (99h:20075)
145. T.A. Springer, F.D. Veldkamp, Octonions, Jordan Algebras and Exceptional Groups. Berlin-Heidelberg-New York: Springer, 2000. MR1763974 (2001f:17006)
146. R. Swan, *Generators and relations for certain special linear groups.* Advances Math. **6** (1971), 1–77. MR0284516 (44:1741)
147. J. Tits, *Classification of algebraic semisimple groups*, In: Algebraic Groups and Discontinuous Subgroups, Proc. Sympos. Pure Math. **9** (1966), pp. 33–62. MR0224710 (37:309)
148. Y.L. Tong, S.P. Wang, *Theta functions defined by geodesic cycles in quotients of $SU(p,1)$*, Invent. math. **71** (1983), 467–499. MR695901 (85c:11046)
149. T.N. Venkataramana, *Cohomology of compact locally symmetric spaces*, Compositio Math. **125** (2001), 221–253. MR1815394 (2002c:11056)
150. T. N. Venkataramana, *On cycles on compact locally symmetric spaces*, Monatshefte Math. **135** (2002), 221–244. MR1897577 (2003a:14037)
151. M.-F. Vignéras, *Arithmétique des Algèbres de Quaternions.* Lecture Notes in Math. 800. Berlin-Heidelberg-New York: Springer-Verlag, 1980. MR580949 (82i:12016)
152. D. Vogan, *Unitarizability of certain series of representations*, Ann. of Math. (2) **120** (1984), 141–187. MR750719 (86h:22028)
153. D. Vogan, *Cohomology and group representations*, In: Proc. Sympos. Pure Math. **61**, Amer. Math. Soc., Providence, RI, 1997, pp. 219–243. MR1476500 (98k:22064)
154. D. Vogan, and G. Zuckerman, *Unitary Representations with Non-Zero Cohomology*, Compositio Math. **53** (1984), 51–90. MR762307 (86k:22040)
155. K. Vogtmann, *Rational homology of Bianchi groups,* Math. Ann. **272** (1985), 399–419. MR799670 (87a:22025)
156. F. Waldhausen, *The word problem in fundamental groups of sufficiently large irreducible 3-manifolds*, Ann. of Math. (2) **88** (1968), 272–280. MR0240822 (39:2167)
157. C. Waldner, *Geometric cycles and the cohomology of arithmetic subgroups of the exceptional group $G_2$*, thesis, Vienna, 2008; to appear in J. Topology.
158. N. R. Wallach, *Real Reductive Groups*, I. Pure Appl. Math. 132, Boston: Academic Press, 1988. MR929683 (89i:22029)

159. N. R. Wallach, *Real Reductive Groups*, II. Pure Appl. Math. 133, Boston: Academic Press, 1988. MR1170566 (93m:22018)
160. A Weil, *Adèles and Algebraic Groups*, Progress Math., 23, Boston: Birkhäuser, 1982. MR670072 (83m:10032)
161. X. Xue, *On the Betti numbers of a hyperbolic manifold,* Geom. Funct. Anal. **2** (1992), 126–136. MR1143667 (93b:57031)
162. J. Yang, *On the real cohomology of arithmetic groups and the rank conjecture for number fields*, Ann. Sci. Ecole Norm. Sup. (4) **25** (1992), 287–306. MR1169133 (93g:11056)
163. H. Zassenhaus, *On the units of orders,* J. of Algebra **20** (1972), 368–395. MR0289469 (44:6659)

Faculty of Mathematics, University of Vienna, Nordbergstrasse 15, A-1090 Vienna, Austria, and Erwin-Schrödinger International Institute for Mathematical Physics, Boltzmanngasse 9, A-1090 Vienna, Austria

*E-mail address*: `Joachim.Schwermer@univie.ac.at`