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COMMENTARY ON "AN ELEMENTARY INTRODUCTION TO THE LANGLANDS PROGRAM" BY STEPHEN GELBART

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Immediately following the commentary below, this previously published article is reprinted in its entirety: Stephen Gelbart, *An elementary introduction to the Langlands Program*, Bull. Amer. Math. Soc. (N.S.) **10** (1984) no. 2, 177–219.

When I was asked to suggest a *Bulletin* article worthy of reprinting, I knew the answer right away: *An Elementary Introduction to the Langlands Program* by Steven Gelbart, published by the *Bulletin* in 1984, remains one of the best reviews of the Langlands Program, even though this subject has expanded tremendously in the intervening years. This paper has greatly influenced my own research and is in fact one of my all-time favorites. I still keep my old xerox copy on roughly cut cheap yellow paper, made in a Moscow library in my student years, as a memento. It is precisely the kind of review article that the *Bulletin* strives to publish: describing a fascinating area of mathematics in a way that is accessible to nonspecialists.

Gelbart introduces in it the key concepts of the Langlands Program appealing to very little mathematical background. He starts with a brief recollection of basic number theory and a review of some "classical themes," such as the Local-Global Principle in number theory, modular forms, and Artin's L-functions. He then moves on to the key concept of *automorphic representations* of reductive groups over the adèles. This sets the stage for the conjectural Langlands' correspondence relating n-dimensional representations of the Galois group of a number field (or a function field) F and automorphic representations of the group GL_n over the ring of adèles of F. Introducing the Langlands dual group, Gelbart then presents Langlands' general *functoriality principle* (which Langlands himself considers as the central tenet of his Program). The last chapter reviews what was known about all this at the time when Gelbart's article went to press.

So much has happened since then! Though it was conceived initially [12] as a bridge between number theory and harmonic analysis, the Langlands Program has moved to other areas of mathematics, such as geometry, and even to quantum physics. It is tempting to think of it as a "grand unified theory" of mathematics, since it ties together so many different disciplines.

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André Weil has written (in a letter to his sister) about the deep analogy between number theory and the geometry of complex algebraic curves, with the theory of algebraic curves over finite fields appearing as the "go-between". Hence Weil talked about a "trilingual text", in which each of the above three subjects appears as one of the columns. Likewise, we can talk about three parallel tracks of the Langlands Program. (For more on this, see the recent Séminaire Bourbaki [6].) The two tracks involving algebraic curves (over finite fields and over the complex field) require a new geometric reformulation whose essential elements are already contained in Drinfeld's important paper [3] published around the time when Gelbart's review was submitted.

The geometric Langlands Program has been developing extensively for the last 20 years and a lot of progress has been made. In particular, the geometric Langlands correspondence has been proved for GL_n in [7, 9] (generalizing earlier work [3, 15]). For other groups a substantial part of the correspondence for curves over $\mathbb{C}$ has been proved by Beilinson and Drinfeld in [1]. This work brought ideas from two-dimensional conformal field theory into the subject (see [4, 5] for a review). Quantum field theories of a different kind, the four-dimensional supersymmetric gauge theories, also turned out to be closely related. Recent work by Witten and collaborators (starting with [10]) linking geometric Langlands correspondence to the so-called S-duality in these theories has already brought a host of exciting new ideas into both physics and mathematics (see [6] for an exposition and further references). Hence we can talk about the "fourth track" of the Langlands Program—in the realm of quantum physics.

In the meantime, many important results have also been obtained in the "classical" Langlands Program, described in Gelbart's article. The celebrated Shimura–Taniyama–Weil conjecture (which implies Fermat's Last Theorem), proved by Wiles, Taylor and others [19, 18, 2], may be viewed as a special case of the Langlands correspondence for the field of rational numbers: it relates two-dimensional Galois representations on the first étale cohomology of elliptic curves over $\mathbb{Q}$ to the automorphic representations of the group GL_2 over the adèles of $\mathbb{Q}$ encoded by certain modular forms on the upper half-plane. Similar methods and ideas have since been applied in greater generality.

The Langlands correspondence for GL_n has been proved in the case of curves over finite fields by L. Lafforgue [11]. The recent proof of the "fundamental lemma" by Ngô [16] was a breakthrough that has led to the stabilization of the Arthur–Selberg trace formula and many far-reaching consequences, as one can learn from the ambitious Book Project [17]. And there are some new ideas [13, 8, 14] on how to prove the functoriality principle in general.

This is just a small selection of recent results. But even from this incomplete list one can see clearly that this is a very exciting time to study the Langlands Program.

Like a well-aged bottle of Bordeaux, Gelbart's article still stands out as one of the best introductions to the subject. I am honored to present it to the readers of the *Bulletin*, and I hope that they will be seduced by it, and the Langlands Program, just like I was when I first read it.

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