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COMMENTARY ON "LECTURES ON MORSE THEORY, OLD AND NEW"

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Immediately following the commentary below, this previously published article is reprinted in its entirety: Raoul Bott, *Lectures on Morse theory, old and new*, Bull. Amer. Math. Soc. (N.S.) **7** (1982) no. 2, 331–358.

1. Introduction

This masterly exposition of Raoul Bott, written in 1981, stands at several crossroads. The problem of closed geodesics on smooth Riemannian manifolds dates back to the late 19th century. Morse transformed the subject in the 1930s, and it is his basic critical point theory and its application to closed geodesics which Bott recalls in the first half of these lectures. This is *finite dimensional* Morse theory, and while it extends in this form to infinite dimensions via work of Palais and Smale, its more radical evolution to *infinite dimensions* in the hands of Floer and others lay in the future in 1981. The geodesic problem may be formulated on the infinite dimensional space of loops, but Bott approximates it by a finite space of polygons—piecewise geodesics—and so stays within the finite dimensional Morse theory. The first crossroads, then, is between finite and infinite dimensions. Another crossroads concerns the role of symmetry groups. Prior to 1980 Morse theory was profitably studied on the underlying manifold of a Lie group. The most prominent example is Bott's great discovery in the 1950s that the homotopy groups of classical Lie groups are periodic. In the last of these lectures Lie groups play an entirely new role. Namely, Bott considers the Morse theory of a function f on a manifold M with a Lie group G acting on M leaving f invariant. This is *equivariant* Morse theory. This Morse theory with symmetry was developed just at the time these lectures were written. The equivariant theory provides a beautiful coda to many results in the closed geodesic problem, as developed by Bott's student Hingston. Finally, these lectures lie at the beginning of a period in geometry of strong influence from physics, a current which is very much alive today. Indeed, a year or so before these lectures were written, Witten had already seen how supersymmetric gauge theory sheds light on Morse theory and he links it in a beautiful way with the de Rham complex. Bott's account of Witten's ideas is not in these lectures, but rather appeared in his next expository article on Morse theory eight years later [B].

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In the last section of the lectures at hand, physics appears in a different form. Taking inspiration from the Yang–Mills equations of four-dimensional quantum field theory, Atiyah and Bott use equivariant Morse theory to compute the cohomology of the space of holomorphic bundles on a Riemann surface. This links with results and techniques of Narasimhan, Seshadri, and Harder which arose in number theory. These ideas of Atiyah and Bott lie at the foundation of subsequent work in mathematical gauge theory. This eventually leads back to Morse theory, in the infinite dimensional versions of Floer, and then to new and unexpected applications in low dimensional topology. This is another crossroads: applications of Morse theory to higher dimensional topology versus applications to low dimensional topology. The latter also appears in the completely different guise of topological quantum field theory.

We will comment briefly on these and other developments over the past 30 years. The subject is vast and we can only touch lightly on a few significant lines of progress. The remarks are organized in two sections: outgrowths of the work of Atiyah and Bott described in the last lecture, and then other recent work in Morse theory proper and its applications. We mention only some of the mathematicians who have contributed, and encourage the reader to be mindful of the legions of others not identified here. The bibliography consists of a few expository articles about this material.

2. Beyond the work of Atiyah and Bott on Riemann surfaces

Let G be a compact Lie group and M a smooth manifold. A G-connection on M gives a notion of *parallel transport* along paths; indeed, it is determined by it. For example, for a two-dimensional sphere $M = S^2$, consider the family of great half-circles from the north pole to the south pole. The parallel transport of a connection along each geodesic half-circle can be identified as an element of G, and since the half-circles are parametrized by the equator, we obtain a map $S^1 \to G$, i.e., a loop in G. This provides a link between the loop space of G, a basic object in the closed geodesic problem, and connections on Riemann surfaces, the setting for Atiyah and Bott. The functional of Yang and Mills is an analog of the energy functional on loops:

$$YM(A) = \int_M \frac{1}{2}\|F_A\|^2 \, d\mathrm{vol}\,. \tag{1}$$

Here A is a connection and F_A its curvature. We assume that M is equipped with a Riemannian metric, used here both to measure the size of the curvature and to provide a measure against which to integrate. It is the equivariant Morse theory of (1) that Atiyah and Bott investigate. The first step is to determine the critical points of (1), that is, solutions to the Euler–Lagrange equation

$$d_A^* F_A = 0 \tag{2}$$

derived from (1) using the classical calculus of variations. This is the *Yang–Mills equation*. In the two-dimensional situation of Atiyah and Bott, the curvature is a single matrix-valued function on which (2) imposes (covariant) constancy. In particular, the *minima* of (1) are the flat connections with zero curvature, and a theorem of Narasimhan and Seshadri identifies the space of flat connections with the space of stable bundles, as defined in algebraic geometry from the complex structure on the Riemann surface M. This link to algebraic geometry mixes with

the topology of the space of connections, guided by the Morse theory of (1), to deduce the main results, as recounted in the lectures reprinted here.

Important analytic contributions to the study of (2) appeared in other works. Earlier, in 1978, Atiyah, Hitchin, and Singer applied the analytic techniques of Kodaira to construct the moduli space $\mathcal{M}$ of minima of (1) on a Riemannian four-dimensional manifold M whose metric is *self-dual*. These minima satisfy a first-order PDE called the *instanton equation*, which is special to four dimensions:

$$F_A = *F_A. \tag{3}$$

The Atiyah–Singer index theorem computes the dimension of $\mathcal{M}$ from the linearization of (3), which is an elliptic equation. (That computation for $M = S^4$ had direct application to four-dimensional quantum field theory and was one of the impetuses to the current interaction of geometry and quantum field theory.) The second analytic input came in the work of Uhlenbeck in 1982. Her estimates on the behavior of instantons, as well as her taming of the gauge invariance of the entire story, formed the basis of subsequent work. Taubes proved crucial existence theorems for instantons on Riemannian manifolds whose metric is not constrained to be self-dual. In their work Atiyah and Bott circumvent the hard analysis by using algebraic geometry, but subsequent work by Uhlenbeck's students Daskalopoulos and Råde shows that the Yang–Mills functional is well behaved, and the infinite dimensional Morse theory of Palais and Smale can be applied directly.

At the same time Atiyah and Bott were writing up their results in two dimensions, Donaldson embarked on a multifaceted study of the instanton equation (3) in four dimensions. The work of Atiyah, Hitchin, and Singer applies to construct moduli spaces $\mathcal{M}$ attached to a 4-manifold M, and the analytic results of Taubes and Uhlenbeck imply existence and compactness theorems about $\mathcal{M}$. In his first application to topology Donaldson used the nonlinear invariants $\mathcal{M}$ as the obstruction to certain 4-manifolds M. Combined with Freedman's solution of the topological Poincaré conjecture in four dimensions, there appeared exotic smooth structures on the standard four-dimensional vector space $\mathbb{R}^4$! In subsequent work Donaldson used the moduli spaces $\mathcal{M}$ to construct new invariants of smooth 4-manifolds. In another direction, inspired by Atiyah and Bott, he proves that on a complex surface—a special kind of 4-manifold—the moduli space $\mathcal{M}$ has an algebro-geometric interpretation in terms of stable bundles. This facilitates computation of the Donaldson invariants of complex surfaces, which eventually led to new results of Friedman, Morgan, and others about their topology and algebraic geometry.

Morse theory returns in the work of Floer, who investigated connections on a three-dimensional manifold N. However, the functional he studies is not the Yang–Mills functional (1), but rather the Chern–Simons functional. Its critical points are the flat connections on N, but now there is a crucial difference with the Atiyah–Bott situation. Namely, at a critical point the Hessian, or second derivative, of the Yang–Mills functional in two dimensions has a finite set of negative eigenvalues. This finiteness of the Morse index is a crucial reason for the applicability of classical Morse theory in this infinite dimensional setting of connections. But for the Chern–Simons functional there is an infinite dimensional space on which the Hessian is negative definite and an infinite dimensional space on which the Hessian is positive definite. Traditional Morse theory fails with gusto! But Floer realized that there is a finite *relative* index between two critical points, defined by specifying a curve which joins them. In this way he constructs a chain complex, analogous to that

in standard Morse theory, and its homology is naturally called the *Floer homology* of the 3-manifold. This links up with Donaldson's theory in four dimensions in a simple way: the Donaldson invariants of a 4-manifold M *with boundary* N take values in the Floer homology of N. Furthermore, if a 4-manifold M is cut into two pieces M', M'' along an embedded 3-manifold N, then the Donaldson invariants for M may be computed in terms of those for M' and M'' using the Floer homology of N. There are some technical restrictions which prevent this last statement from being true in general. Nonetheless, the Donaldson invariants and Floer homology groups come very close to satisfying this defining property of a *topological quantum field theory*, a notion introduced by Witten in the late 1980s. In fact, Witten wrote the Donaldson invariants as correlation functions in a supersymmetric gauge theory. Thus the Donaldson invariants, which were constructed at the beginning of the decade from the *classical* instanton equation, by the end of the decade found their natural home in *quantum* field theory. Floer also used his infinite dimensional Morse theory in the context of symplectic topology, which in the hands of Witten became part of a two-dimensional quantum field theory.

Mathematical gauge theory was transformed in 1994 when Seiberg and Witten deduced the long distance behavior of the supersymmetric gauge theory that computes Donaldson invariants. (The latter are seen directly in the short distance theory.) As the Donaldson invariants are topological, so independent of scale, they appear in the long distance approximation but in a new form. This intricate and highly nonformal physics argument leads to the predication that Donaldson's invariants can be computed from simpler equations, known as the Seiberg–Witten equations. These are simpler from a technical point of view—notably in the compactness arguments—and also conceptually: whereas the instanton moduli space $\mathcal{M}$ typically has positive dimension, the Seiberg–Witten equations usually have a finite set of solution. Invariants are then defined by a simple count. Many geometers ignored the physics and simply ran with the new equations. Old conjectures, such as one of René Thom about complex curves in the complex projective plane, were quickly solved by Kronheimer and Mrowka. Taubes turned his attention to symplectic manifolds and proved an important relation between gauge theory invariants from the Seiberg–Witten equations and the Gromov invariants in symplectic topology. Donaldson [D] wrote an excellent survey of these developments for this *Bulletin*. Morse theory continued to play an important role at this juncture. The Seiberg–Witten version of Floer theory for 3-manifolds is more tractable than the instanton version. In fact, gradually the focus shifted to three dimensions. New invariants of Osváth and Szabó have their origins in mathematical gauge theory, but can be defined and computed using combinatorial as opposed to analytic techniques. They have been used, sometimes in combination with gauge theory invariants, to settle old problems in low dimensional topology. This is a very active area of research, with close ties to symplectic topology. The article [H] by Hutchings, which also appeared in this *Bulletin*, contains an introduction to the various Floer theories.

As we have seen, the paper of Atiyah and Bott spawned the entire field of mathematical gauge theory and several close cousins. Now we survey two lines of development in Morse theory over the past 30 years which, at least initially, did not grow directly out of gauge theory.

3. Newer Morse theory

We have already mentioned that Witten, motivated by supersymmetric models in low dimensional quantum field theory, found a new approach to finite dimensional Morse theory in the early 1980s. Let M be a compact smooth manifold and $f\colon M \to \mathbb{R}$ a smooth function. Witten introduces a deformation

$$0 \longrightarrow \Omega^1(X) \xrightarrow{d+t\epsilon(df)} \Omega^2(X) \xrightarrow{d+t\epsilon(df)} \Omega^3(X) \xrightarrow{d+t\epsilon(df)} \cdots \tag{4}$$

of the standard de Rham complex, where $\epsilon(df)$ is exterior multiplication by the differential of f, and t is a real number. This specializes to the standard de Rham complex at $t = 0$. The novelty comes as $t \to \infty$. For then if f is a Morse function, and M is endowed with a Riemannian metric, there is a close relationship between the spectrum of the associated second-order Laplace operator Δ_t and the critical points of f. Namely, the small eigenvalues of Δ_t as $t \to \infty$ correspond to critical points, and the degree of the associated eigen-differential form equals the Morse index of the critical point. From the physics point of view these are the different vacuum states of the quantum system. Now quantum tunneling among these states can be encoded as a differential on the $\mathbb{Z}$-graded abelian group generated by the vacuum states, or critical points, where the grading is the Morse index. The complex then computes the homology of M. These ideas are implicit in earlier work of Smale, but the origins in physics of Witten's work opened up new possibilities. Namely, the relevant physical system for this finite dimensional Morse theory is supersymmetric quantum mechanics, which can be viewed as a one-dimensional quantum field theory. One can reasonably ask, then, if higher dimensional quantum field theories say something about Morse theory.

The answer is a resounding 'yes'. In fact, one can either increase the dimension or the amount of supersymmetry—for maximal effect, increase both. If we first double the amount of supersymmetry, then the manifold M is constrained to be a complex manifold with a Kähler metric. Furthermore, the function f is now complex valued and holomorphic. If it is in addition Morse, then the critical points all have middle dimensional index and there is no interesting flow (tunneling) among them. Moving up from quantum mechanics to a two-dimensional quantum field theory, called the Landau–Ginzburg model, there are interesting solitons connecting the critical points. The counting of these solitons then connects with the complex version of Morse theory, called Picard–Lefschetz theory. In another direction, returning to four-dimensional supersymmetric gauge theories, the study of vacua and tunneling on a 3-manifold leads to Floer homology.

Our account here slights the profound influence of Morse theory—including the complex Picard–Lefschetz theory—on symplectic topology. This includes both the finite dimensional theory as well as the infinite dimensional versions pioneered by Floer.

The connections between supersymmetric quantum field theory and Morse theory date from the 1980s and 1990s. Very recently, Witten [W] again took up these ideas in supersymmetric gauge theories with more supersymmetry and which exist in five and six dimensions. What emerges is a proposal for a new definition of the Jones polynomial and its "categorification", Khovanov homology. It is based on the Morse–Floer theory of a functional in four dimensions related to the Chern–Simons functional in three dimensions. The critical point equations are a specialization of the Kapustin–Witten equations introduced in the physics setting

of the Geometric Langlands Program. The flow equations are new equations in five-dimensional mathematical gauge theory, inspired by the Morse theory ideas connected to physics.

The second line of development since Bott's lectures has its origins around 1970 in the work of Cerf. As a simple example, consider the 2-sphere S^2. The simplest Morse function $f\colon S^2 \to \mathbb{R}$ is obtained by embedding the sphere in three-dimensional flat space and then letting $f(p)$ be the signed distance of $p \in S^2$ from a fixed equatorial two-dimensional plane. The function f has a single maximum at the north pole and a single minimum at the south pole. It is "perfect", as Bott explains in his lectures, as this is the minimal possible number of critical points. (It also satisfies Morse's lacunary principle.) Now imagine the embedded sphere to be made of rubber, and deform the embedding by pushing up in a small disk in the upper hemisphere not containing the north pole. The new height function g now has a new local maximum as well as a new critical point of index 1; a neighborhood of the latter is a saddle. These new critical points cancel, for example in the Morse complex introduced by Witten, as indeed they must since the cohomology of the sphere is immutable. It is easy to see that f and g cannot be connected in the space of Morse functions. Indeed, the critical point structure of a Morse function cannot jump along a path of Morse functions. This motivates Cerf to study a larger space of functions with a filtration in which the Morse functions form the lowest (dense, open) stratum. The next stratum lets in functions which fail to be Morse in one direction and whose third derivative in that direction is nonzero. A curve of Morse functions passing through such a non-Morse function experiences a birth of canceling critical points, as in the example above, or if time is reversed a death of such a pair of critical points. Cerf shows that allowing these birth-death singularities produces a connected space of functions. Morse functions decompose a smooth manifold into simple pieces, and the Cerf ideas can be used to compare different decompositions from different Morse functions. Cerf used this and a study of higher strata to investigate groups of exotic spheres in high dimensions. His ideas have long been used in low dimensional topology as well, for example as the foundation of Kirby calculus. Over the past 20 years they also lie at the heart of mathematical approaches to topological quantum field theories.

As might be expected, the physics approach to most topological quantum field theories depends on a functional integral which is not rigorously defined. Mathematical constructions typically begin with algebraic data which is then assigned to elementary bordisms between manifolds of a fixed dimension. The precise nature of that algebraic data depends on the dimension and type of topological structure (orientation, framing, etc.) imposed on the manifolds. Invariants of complicated manifolds and bordisms are obtained by decomposing into elementary bordisms using a Morse function. To see that one obtains an invariant, a comparison between different decompositions is needed, and here is where the Cerf theory enters to impose constraints on the topological data. A relatively simple instance of this argument occurs in two-dimensional field theory where the algebraic data is a commutative Frobenius ring. The argument is nicely explained by Moore and Segal in [MS, Appendix]. A much more elaborate and powerful result is the *cobordism hypothesis*. It was conjectured by Baez and Dolan and proved by Hopkins and Lurie in dimension two and Lurie in higher dimensions. The proof relies on a 1984 theorem of Igusa. He studies a refinement $\mathcal{F}$ of Cerf's space of generalized Morse

functions in which a certain framing is introduced. He then proves that not only is $\mathcal{F}$ connected, but it is highly connected in the sense that its homotopy groups vanish up to a degree which grows linearly with the dimension of the underlying manifold. That refinement is needed as the cobordism hypothesis applies to extended topological field theories and to families of field theories, so higher dimensional families of Morse functions are encountered. Finally, the renewed interest in Igusa's theorem from the cobordism hypothesis has led to a current sharper theorem: all homotopy groups of $\mathcal{F}$ vanish.

We leave the reader to envision the ample fruits not yet plucked from the tree of Morse theory and to enjoy in the following pages Bott's wonderful account of the fruits already harvested in 1981.

References

[B] Raoul Bott, *Morse theory indomitable*, Inst. Hautes Études Sci. Publ. Math. (1988), no. 68, 99–114 (1989). MR1001450 (90f:58027)

[D] S. K. Donaldson, *The Seiberg-Witten equations and* 4*-manifold topology*, Bull. Amer. Math. Soc. (N.S.) **33** (1996), no. 1, 45–70. MR1339810 (96k:57033)

[H] Michael Hutchings, *Taubes's proof of the Weinstein conjecture in dimension three*, Bull. Amer. Math. Soc. (N.S.) **47** (2010), no. 1, 73–125. MR2566446 (2011a:57049)

[MS] G. W. Moore and G. B. Segal, *D-branes and k-theory in* 2*d topological field theory*, Clay Mathematics Monographs, vol. 4, ch. 2, pp. x+681, American Mathematical Society, Providence, RI, 2009. MR2567952 (2011e:53148)

[W] Edward Witten, *Fivebranes and knots*, preprint.

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