BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 50, Number 2, April 2013, Pages 353–358
S 0273-0979(2012)01386-1
Article electronically published on July 24, 2012

Some problems of unlikely intersections in arithmetic and geometry, by Umberto Zannier, with appendixes by David Masser, Annals of Mathematics Studies, 181, Princeton University Press, Princeton, New Jersey, 2012, xiv+160 pp., ISBN 978-0-691-15371-1, US $75.00

What makes an intersection likely or unlikely? A simple dimension count indicates that two curves in the plane are likely to intersect, while two curves in three-space are unlikely to intersect. Indeed, in the complex projective plane, Bezout's theorem says that two algebraic curves always intersect. Moving down a dimension, it is unlikely that a curve C in the plane will intersect a given point P. More generally—and this is where the real interest begins—the intersection of C with a countable collection of "special" points $\{P_i\}_{i\geq 1}$ is unlikely to be large unless C itself is also special in some way.

A prototype for the subject of unlikely intersections is the following theorem in which the special points are those whose coordinates are roots of unity and the special curves are translates of subgroups of the torus $\mathbb{C}^{*2}$. This result was conjectured by Serge Lang and proved by Ihara, Serre, and Tate in the 1960s.

Theorem 1. *Let $f(X,Y) \in \mathbb{C}[X,Y]$ be a nonzero polynomial that has no factors of the form $X^d - cY^e$ or $X^dY^e - c$ with c a root of unity. Then the equation $f(X,Y) = 0$ has only finitely many solutions (ζ, η) in which ζ and η are roots of unity.*

A convenient way to describe the set of special points (ζ, η) in the theorem is as the set of points of finite order in the torus $\mathbb{C}^{*2}$, while as noted earlier, the special curves $X^d = cY^e$ and $X^dY^e = c$ are translates of algebraic subgroups of $\mathbb{C}^{*2}$ by points of finite order. A nontrivial generalization to $\mathbb{C}^{*n}$ was given by Laurent in 1984.

Replacing the torus by an abelian variety A leads to a famous problem raised by Manin and Mumford in the 1960s and solved by Raynaud in the 1980s. We suppose that $X \subset A$ is a subvariety of A that contains infinitely many torsion points of A. Raynaud proved that there is an abelian subvariety B of A of positive dimension and a torsion point $T \in A$ such that X contains the translate of B by T.

More generally, one can look at a semi-abelian variety, i.e., the extension of an abelian variety by a torus, and one can replace the torsion subgroup by the full divisible hull of a finitely generated subgroup.

Theorem 2. *Let G be a semi-abelian variety, let $\Gamma \subset G(\mathbb{C})$ be a finitely generated subgroup, let*

$$\Gamma^{\mathrm{div}} = \{g \in G(\mathbb{C}) : g^k \in \Gamma \text{ for some } k \geq 1\}$$

be the divisible hull of Γ, and let $X \subset G$ be a subvariety. Then $X \cap \Gamma^{\mathrm{div}}$ is finite unless X contains a translate of an algebraic subgroup of G of dimension at least 1.

2010 *Mathematics Subject Classification.* Primary 11G35; Secondary 11G50, 14G25, 14G35, 14K12.

Key words and phrases. unlikely intersection.

This deep result, which is a combined generalization of conjectures of Mordell–Lang and Manin–Mumford, was proven as the culmination of the work of a number of mathematicians, including in particular Faltings, Hindry, McQuillan, and Vojta. It may be summarized as saying that if a subvariety X of G contains infinitely many special points, i.e., points of Γ^{div}, then X contains a special subvariety, i.e., a translate of a positive dimensional subgroup of X.

Theorem 2 deals with unlikely intersections of subvarieties with a countable collection of special points. The situation becomes even more difficult if one replaces the set of special points with a countable collection of special subvarieties, as was done in a series of conjectures by Bombieri, Masser and Zannier [4], Pink [14], and Zilber [16]. For simplicity, we restrict our attention for the moment to the case of an n-dimensional torus $G = \mathbb{C}^{*n}$. For a given subvariety $X \subset G$, we define

$$X_{(d)} = \bigcup_{H \subset G,\ \dim H \leq d} (X \cap H),$$

where H runs over all algebraic subgroups of G of dimension at most d. For example, the set $X_{(0)}$ is the set considered earlier, namely the intersection of X with the torsion points of G. Algebraic subgroups of G are special, so we would expect $X_{(d)}$ to be small unless X itself is also special. Illustrating this intuition is a recent theorem of Maurin [11], which says that if X is an irreducible curve defined over $\bar{\mathbb{Q}}$ that is not contained in a proper algebraic subgroup of G, then $X_{(n-2)}$ is finite. This is the best that one can hope, since each intersection $X \cap H$ with $\dim H = n-1$ is likely to be nonempty, leading to $X_{(n-1)}$ being infinite.

When X has dimension greater than 1, the situation is considerably more complicated, as illustrated by the following example of Bombieri, Masser and Zannier [4]. Let

$$X = \bigl\{(x_1, x_2, x_3, x_4) \in \mathbb{C}^{*4} : x_1 + x_2 = 1,\ x_3 + x_4 = 2\bigr\},$$

and consider the translated subgroup $gH = \{x_1 = x_2 = \frac{1}{2}\} \subset \mathbb{C}^{*4}$. Then X and gH have dimension 2, and one can check that X contains no translated subgroups. So one would expect $X \cap gH$ to have dimension 0, but in fact $\dim(X \cap gH) = 1$.

Generalizing this construction, Bombieri, Masser and Zannier gave the right formulation for unlikely intersections. For an irreducible subvariety $X \subset G = \mathbb{C}^{*n}$ and a subgroup $H \subset G$, the expected dimension $\mathrm{ED}(G; X, H)$ of the intersection $X \cap H$ is

$$\mathrm{ED}(G; X, H) = \dim G - \operatorname{codim}_G X - \operatorname{codim}_G H.$$

For $\delta \geq 0$ and $g \in G$, they write $(X \cap gH)^\delta$ for the union of the components of $X \cap gH$ of dimension δ, and they define the "nonanomalous" part of X to be

$$X^{\circ a} = X \setminus \bigcup_{H \subset G} \bigcup_{g \in G} \bigcup_{\delta > \mathrm{ED}(G;X,H)} (X \cap gH)^\delta.$$

They prove that $X^{\circ a}$ is a (possibly empty) Zariski open subset of X, and they make a number of unlikely intersection conjectures for $X^{\circ a}$. Their formulation was spectacularly justified by recent work of Habegger [6, 7] in which he shows that $X^{\circ a} \cap X_{(n - \dim X - 1)}$ is a finite set. In a subsequent work, Habegger proves an analogous result for products of abelian varieties, and work is continuing by Habegger and others to extend these results to more general (commutative) algebraic groups. However, despite this great progress, we still do not know the best possible finiteness result for surfaces sitting in a five-dimensional torus.

Up to this point we have looked at algebraic groups, whose special subsets are translates of subgroups. A rather different type of special subset arises in the theory of moduli spaces and, more generally, Shimura varieties. It would require too many technical definitions to describe the full theory in this short review, so we consider the special case of elliptic curves. An elliptic curve E is a compact algebraic curve (Riemann surface) that has the structure of an algebraic group. The isomorphism class of E is determined by a single number $j(E)$, called the j-invariant of E. The set of algebraic maps from E to itself, denoted $\mathrm{End}(E)$, has the natural structure of a ring via $(f+g)(P) = f(P) + g(P)$ and $(fg)(P) = f(g(P))$. For most elliptic curves, the ring $\mathrm{End}(E)$ is just $\mathbb{Z}$, but for some special curves it is larger, isomorphic to an order in an imaginary quadratic field. Such curves are said to have complex multiplication, or CM for short. As an example, the elliptic curve $E : y^2 = x^3 + x$ admits the endomorphism $\varphi(x, y) = (-x, iy)$ whose square $\varphi^2(x, y) = (x, -y)$ is the negation map on E. In this way, one can show that $\mathrm{End}(E) = \mathbb{Z}[i]$, the ring of Gaussian integers. There are a countably infinite number of CM elliptic curves.

We now view the complex plane $\mathbb{C}^2$ as parameterizing pairs of elliptic curves (E, E') via the identification $(E, E') \leftrightarrow \big(j(E), j(E')\big)$. Among those pairs, the special points will be those (E, E') such that both elliptic curves E and E' have CM. We will call them CM points. Now given an irreducible algebraic curve $X \subset \mathbb{C}^2$, we might ask under what circumstances X contains infinitely many CM points. André [2] answered this question: the curve X is either a horizontal line, a vertical line, or a modular curve $Y_0(N)$, where $Y_0(N)$ is the set of pairs (E, E') such that there is a homomorphism $E \to E'$ whose kernel is a cyclic group of order N. In brief, if X contains infinitely many special points, then X itself has a special form.

Various conjectural generalizations of this result were formulated independently by André [1] and Oort [12]. Later Pink [14] described a version that combines unlikely intersection problems on Shimura varieties with the unlikely intersection problems we discussed earlier on algebraic groups, including the case in which the special points are replaced by special subvarieties of higher dimension. Without getting too much into technicalities, a Shimura variety is roughly a quotient of a symmetric Hermitian domain by an arithmetic group. The special subvarieties of Shimura varieties are themselves Shimura varieties; for example, the zero-dimensional Shimura subvarieties are CM points. Then one version of the André–Oort conjecture says that if an irreducible subvariety X of a Shimura variety V contains a Zariski dense set of CM points, then X is itself a Shimura subvariety of V. Recently, Pila [13] has proved certain cases of the combined André–Oort/Lang–Manin–Mumford conjecture using, among other tools, the theory of o-minimality from logic and model theory.

Returning to the case of algebraic groups, we reconsider the earlier theorem that a nonspecial subvariety X of an abelian variety A has small intersection with the set of special points A_{tors}. In this statement, we may view the abelian variety A and its special points as fixed and the subvariety X as varying. Suppose now that we allow the abelian variety A to vary in a family. In this setting, Masser and Zannier [9] proved a deep, but very particular, result whose generalizations are an active area of current research. They consider the family of elliptic curves

$$E_\lambda : y^2 = x(x-1)(x-\lambda)$$

parameterized by λ and the two families of points

$$P_\lambda = (2, \sqrt{4-2\lambda}) \quad \text{and} \quad Q_\lambda = (3, \sqrt{18-6\lambda})$$

on E_λ. By setting $nP_\lambda = 0$ for various values of n, one can easily find values of $\lambda \in \mathbb{C}$ such that P_λ is a torsion point, i.e., such that P_λ is special; and similarly for Q_λ. However, it seems "unlikely" that there are many λ for which both P_λ and Q_λ are torsion points. Masser and Zannier prove that the set of such λ is finite.

There are also analogues of unlikely intersection problems in the theory of dynamical systems. Rather than listing these analogues case by case, we instead describe one recent beautiful result due to Baker and DeMarco [3] that is a dynamical analogue of Masser and Zannier's result on unlikely intersections in varying families. For $c \in \mathbb{C}^*$, let $f_c(z) = z^2 + c$, and write $f_c^{\circ n}$ for the nth iterate of f_c. The forward orbit of a point $a \in \mathbb{C}$ is the set $\{f_c^{\circ n}(a) : n \geq 1\}$. The point a is said to be preperiodic for f_c if its forward orbit is finite. In this dynamical setting, preperiodic points are special points. The Baker–DeMarco theorem says that if $a, b \in \mathbb{C}$ have the property that there are infinitely many values of c such that both a and b are preperiodic for f_c, then $a^2 = b^2$.

Before turning to the book under review, we mention briefly a situation in which intersections are "likely, but not too likely." To describe these results, we need to find a way of measuring the arithmetic complexity of a point P whose coordinates are algebraic numbers. For this purpose we use the height $h(P)$, which is roughly the number of bits it takes to store the coordinates of the point P on a computer. For example, if we consider points having rational coordinates, then for any bound B there are only finitely many points $P \in \mathbb{Q}^n$ whose height satisfies $h(P) \leq B$.

Returning to the family of elliptic curves E_λ and the family of points P_λ considered by Masser and Zannier, they show that there are infinitely many algebraic numbers λ such that P_λ is a torsion point. So it would not be appropriate to say that P_λ is unlikely to be a torsion point. However, it turns out that the set

$$\{\lambda \in \bar{\mathbb{Q}} : P_\lambda \text{ is a torsion point}\}$$

is a set of bounded height, so for example, we immediately see that there are only finitely many $\lambda \in \mathbb{Q}$ for which P_λ is a torsion point. This is a very special case of a theorem of the reviewer [15], which generalized earlier work of Dem'janenko [5] and Manin [8] in the split case and deals with general one-dimensional families of abelian varieties and points. In a related vein, Masser [10] gave a height density result for higher dimensional families, and Habegger [6, 7] has proven deep bounded height theorems for certain unlikely intersections. Aside from the intrinsic interest of such results, they often form an important part of finiteness proofs for unlikely intersection, but we do not have space here to comment further.

We turn now to the monograph under review, which is an expanded version of a series of four lectures given by the author at the IAS in May 2010. The first lecture forms an introductory chapter in which the author gives an overview of the theory of unlikely intersections, including most of the relevant definitions and statements of many of the main theorems and conjectures. Three of the remaining four chapters (Chapters 1, 3, and 4) follow the remaining three lecures and cover unlikely intersections in multiplicative groups, unlikely intersections on families of elliptic curves (elliptic surfaces), and unlikely intersections on moduli spaces and Shimura varieties (the André–Oort conjecture). We have described these topics already. An additional chapter (Chapter 2) discusses unlikely intersections in number fields, which was not covered in the lectures. Each chapter ends with notes on related topics, directions, and guides to the literature.

As an example of the sort of theorems discussed in Chapter 2, we mention the following special case of a result of Corvaja and the author: Let $a, b \geq 2$ be integers such that $a^n \neq b^m$ for all $m, n \geq 1$. Then for every $\epsilon > 0$ there is a constant $C = C(a, b, \epsilon)$ such that

$$\gcd(a^n - 1, b^m - 1) \leq C \max\{a^{n\epsilon}, b^{m\epsilon}\} \quad \text{for all } m, n \geq 1.$$

Although deceptively simple to state, the proof is ineffective and uses Schmidt's subspace theorem.

Zannier's book is well written and a pleasure to read, although since it is a monograph based on a lecture series, the pace is at times uneven, and the proofs, of course, are generally only sketched. But the author always makes an effort to point out key ideas and key steps, so a reader who wants to read and understand the complete proofs in this technically demanding field will find this monograph to be an extremely helpful entree into the subject. Seven appendices round out the volume. In the first appendix, the author sketches Pila's theorem describing the distribution of rational points on subanalytic varieties, which plays a crucial role in many unlikely intersection proofs. The remaining six appendices are short notes by David Masser on a variety of related subjects and tools used in the study of unlikely intersection. Since any book review must find at least one item to criticize, the reviewer warns the reader that some sort of indexing glitch has caused many index entries to be off by up to four pages. But aside from this minor caveat, the reviewer highly recommends Zannier's book as an excellent survey of and introduction to the important and hot topic of unlikely intersections in arithmetic geometry.

References

[1] Y. André. *G-functions and geometry.* Aspects of Mathematics, E13. Friedr. Vieweg & Sohn, Braunschweig, 1989. MR990016

[2] Y. André. Finitude des couples d'invariants modulaires singuliers sur une courbe algébrique plane non modulaire. *J. Reine Angew. Math.*, 505:203–208, 1998. MR1662256

[3] M. Baker and L. DeMarco. Preperiodic points and unlikely intersections. *Duke Math. J.*, 159(1):1–29, 2011. MR2817647

[4] E. Bombieri, D. Masser, and U. Zannier. Anomalous subvarieties—structure theorems and applications. *Int. Math. Res. Not. IMRN*, (19):Art. ID rnm057, 33, 2007. MR2359537

[5] V. A. Dem'janenko. Rational points of a class of algebraic curves. *Izv. Akad. Nauk SSSR Ser. Mat.*, 30:1373–1396, 1966. MR0205991

[6] P. Habegger. Intersecting subvarieties of abelian varieties with algebraic subgroups of complementary dimension. *Invent. Math.*, 176(2):405–447, 2009. MR2495768

[7] P. Habegger. On the bounded height conjecture. *Int. Math. Res. Not. IMRN*, (5):860–886, 2009. MR2482128

[8] J. Manin. The p-torsion of elliptic curves is uniformly bounded. *Izv. Akad. Nauk SSSR Ser. Mat.*, 33:450–465, 1969. MR0272786

[9] D. Masser and U. Zannier. Torsion anomalous points and families of elliptic curves. *Amer. J. Math.*, 132(6):1677–1691, 2010. MR2766181

[10] D. W. Masser. Specializations of finitely generated subgroups of abelian varieties. *Trans. Amer. Math. Soc.*, 311(1):413–424, 1989. MR974783

[11] G. Maurin. Courbes algébriques et équations multiplicatives. *Math. Ann.*, 341(4):789–824, 2008. MR2407327

[12] F. Oort. Canonical liftings and dense sets of CM-points. In *Arithmetic geometry (Cortona, 1994)*, Sympos. Math., XXXVII, pages 228–234. Cambridge Univ. Press, Cambridge, 1997. MR1472499

[13] J. Pila. *O*-minimality and the André–Oort conjecture for $\mathbb{C}^n$. *Ann. of Math. (2)*, 173:1779–1840, 2011. MR2800724

[14] R. Pink. A common generalization of the conjectures of André–Oort, Manin–Mumford, and Mordell–Lang. unpublished manuscript dated 17th April 2005.

[15] J. H. Silverman. Heights and the specialization map for families of abelian varieties. *J. Reine Angew. Math.*, 342:197–211, 1983. MR703488

[16] B. Zilber. Exponential sums equations and the Schanuel conjecture. *J. London Math. Soc. (2)*, 65(1):27–44, 2002. MR1875133

Joseph H. Silverman
Mathematics Department
Brown University
Providence, Rhode Island 02912