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Semi-classical analysis, by Maciej Zworski, Graduate Studies in Mathematics, Vol. 138, American Mathematical Society, Providence, RI, xii+431 pp., ISBN 978-0-8218-8320-4, US $60.00

One way to think of semi-classical analysis is as an investigation of the mathematical implications of the Bohr correspondence principle: the assertion that classical mechanics is the limit, as $\hbar$ tends to zero, of quantum mechanics.[1] To illustrate how this principle works, consider a physical system consisting of a single point particle, p, of mass, m, in $\mathbb{R}^n$ acted on by a conservative force $F = -\nabla V$, $V \in \mathcal{C}^\infty(\mathbb{R}^n)$. The total energy of this system (kinetic plus potential) is given by $H(x,\xi) = \frac{1}{2m}|\xi|^2 + V(x)$, where x is the position and ξ the momentum of p, and the motion of this system in phase space is described by the Hamilton–Jacobi equations

$$\begin{aligned} \frac{dx}{dt} &= \frac{\partial H}{\partial \xi}(x,\xi), \\ \frac{d\xi}{dt} &= -\frac{\partial H}{\partial x}(x,\xi). \end{aligned} \tag{1}$$

The quantum mechanical description of this system on the other hand is given by the Schrödinger equation

$$ih\frac{\partial}{\partial t}\varphi = -\frac{h^2}{2m}\Delta\varphi + V\varphi, \tag{2}$$

whose L^2 normalized solution, $\int |\varphi|^2\, dx = 1$, gives one a probability measure $\mu_t = |\varphi(x,t)|^2\, dx$ that describes the "probable" position of p at time t.

Of particular interest are the steady state solutions of (2). If we assume for simplicity that the eigenvalues $\lambda_k(h)$ of the Schrödinger operator are discrete and the corresponding L^2 normalized eigenfunctions are $\varphi_k(x)$, then the functions, $e^{-i\frac{t\lambda_k}{\hbar}}\varphi_k(x)$, are steady state solutions of (2) in the sense that the measures $\mu_k = |\varphi_k(x,t)|^2\, dx$ are independent of t. The $\lambda_k(\hbar)$'s are, by definition, the energies of these steady state solutions, and the number of states with energies lying on the interval $a < \lambda < b$ is given by

$$N(a,b,\hbar) = \sharp\{a < \lambda_k(h) < b\}\,. \tag{3}$$

On the other hand a crude semi-classical method for computing this number of states is to invoke the Heisenberg uncertainty principle

$$|\delta x_i \delta \xi_i| \geq 2\pi\hbar \tag{4}$$

and the Pauli exclusion principle (which can be interpreted as saying that no two of these states can occupy the same position in phase space) to conclude that the

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[1]Mathematics are sometimes bothered by this formulation of the BCP since $\hbar$ is a fixed constant, i.e., is (approximately) 10^{-27} erg secs., not a parameter that one can vary at will. However, unlike e and π, it is a *physical* constant: in the world of classical physics, in which quantities are measured in ergs and secs, it is negligibly small, but in the world of subatomic physics it is not. Therefore the transition from quantum to semi-classical can legitimately be regarded as an "$\hbar$ tends to zero" limit.

maximum number of classical states with energies on the interval $a < H < b$ is approximately equal to the maximal number of disjoint rectangles lying in the region, $a < H(x,\xi) < b$, and satisfying the volume constraint imposed by (4). For $\hbar$ small the number of such rectangles is approximately

$$\left(\frac{1}{2\pi\hbar}\right)^n \mathrm{vol}\,(a < H(x,\xi) < b), \tag{5}$$

so as $\hbar$ tends to zero,

$$(2\pi\hbar)^n N(a,b,\hbar) \to \mathrm{vol}\,(a < H(x,\xi) < b)\,. \tag{6}$$

One of the more notable achievements of semi-classical analysis has been to make this heuristic derivation precise (and to show that the right-hand side of (6) can be replaced by $\mathrm{vol}(a < H < b) + \cdots$, where the dots represent an accurate estimate for the error term in this approximation).

Coming back to the assertion that the probability measure μ_t describes the probable position of the quantum state ϕ in equation (2) the tools that enable one to interpret this assertion semi-classically were developed in the 1960s by Kohn–Nirenberg and Hörmander in their seminal papers on the calculus of pseudo-differential operators. In particular this calculus enables one to replace the notion of "singular support" for distributional solutions ϕ of (2) by a more refined notion, that of "wave front set". To describe this notion, recall that a point x is **not** in the the singular support of ϕ if, for some neighborhood, U, of x, ρ times ϕ is a $\mathcal{C}^\infty$ function for all $\mathcal{C}^\infty$ functions ρ with support in U. The wave front set of ϕ is defined similarly: in its semi-classical form it asserts that a point (x,ξ) is **not** in the wave front set of ϕ if, for some neighborhood, U, of (x,ξ), $P\phi$ is a $\mathcal{C}^\infty$ function of order $O(h^\infty)$ for all pseudo-differential operators, P, with microsupport in U. (For semi-classical pseudo-differential operators this means that

$$P\phi(x,h) = \int p(x,y,h)\phi(y,h)\,dy \tag{7}$$

is a $\mathcal{C}^\infty$ function of order $O(h^\infty)$, where

$$p(x,y,h) = (2\pi h)^{-n} \int \mathfrak{p}(x,\xi,h) e^{\frac{i(x-y)\xi}{h}}\,d\xi \tag{8}$$

with supp $\mathfrak{p}(x,\xi,h)$ contained in U.)

Given this definition the semi-classical version of the assertion that the measure μ_t "describes the probable position of the quantum state ϕ at time t" is the assertion that if (x,ξ) is in the wave front set of ϕ, then the integral curve of the Hamiltonian system (1) passing through (x,ξ) at time $t = 0$ is in the wave front set of ϕ for all times, t. In other words, singularities of solutions of (2) propagate along trajectories of the associated classical system (1).

Two decades ago semi-classical results of this nature were accessible only in journal form, but fortunately there are now several excellent graduate level texts in which this material is well exposed. Among them are Dimassi and Sjöstrand's *Spectral asymptotics in the semi-classical limit*, Martinez's *An introduction to semiclassical and microlocal analysis*, Helffer's *Semi-classical analysis for the Schroedinger operator and applications*, and (in French) Robert's *Autour de l'approximation semi-classique*. (Also in French is a very nice set of notes by Colin de Verdiere, *Methodes semi-classique et theorie spectrale*.) The book reviewed here is a welcome addition to this list. For one thing it covers a lot of topics which are not covered

in most of the other texts cited above (such as the concept of "defect measures" in Chapter 5 and of "quasi-modes" in Chapter 7, both of which are striking illustrations of the "propagation of singularities" phenomenon that we alluded to above). Also, in comparison with these other texts, this book is more leisurely paced, and it assumes less in the way of prior expertise on the part of the reader. With some of the graduate level topics in the last few chapters omitted, it would be a close-to-perfect fit for a course in mathematical physics for bright undergraduates. (Full disclosure: The first two paragraphs of this review are taken verbatim from the introduction to *Semi-classical analysis* by the author and Shlomo Sternberg. (This exists at present as an e-book but will be issued shortly in printed form by International Press.) We would like to thank Maciej for citing it (favorably) in his preface.)

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