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SELECTED MATHEMATICAL REVIEWS

related to the paper in the previous section by
JEFFREY C. LAGARIAS

MR1968276 (2004k:11195) 11Y60; 11-01, 11M41, 11N05

Havil, Julian

Gamma. (English)

Princeton University Press, *Princeton, NJ*, 2003. xxiv+266 *pp.*. \$29.95.
ISBN 0-691-09983-9

This is a delightful book exploring Euler's constant γ. The author has a clear and engaging style and much of the book should be accessible to bright high school students and undergraduates. The author gives fascinating historical details of the story of γ and elaborates on some very interesting mathematics connected with it. He begins naturally with logarithms and the harmonic series and proceeds to the special values of $\zeta(s)$, Bernoulli numbers, summing the k-th powers of integers, the Euler-Maclaurin summation formula, the Γ-function, continued fractions, and ends with considering problems on primes and trying to explain the Riemann hypothesis! Some topics that I found particularly charming are $\int_0^1 dx/x^x = 1/1^1+1/2^2+1/3^3+\dots$ (on page 44, the author describes this as being "very easy on the eye", indeed!), a nice Putnam competition problem (page 131), the crossing the desert problem (page 127), and Benford's law (page 145). The author's enthusiasm for mathematics is evident throughout the book; for example, after a cute proof of Erdős for the infinitude of primes (page 29) he adds that "the proof leaves one breathless". There is much in this book to fire the imagination of eager young students of mathematics.

There are a few minor blemishes in the book. For example, the section "Why is the Riemann hypothesis important?" (page 205) doesn't really give a satisfactory reason. One of the given reasons is that RH implies every odd number is a sum of three primes; this is at best a half truth since we do know unconditionally that all large odd numbers are so represented, and the proof that all odd numbers are represented involves the Generalized Riemann Hypothesis. On page 115 there is a misprint: $\limsup_{n\to\infty}\varphi(n)=1$ should read $\limsup_{n\to\infty}\varphi(n)/n=1$. Earlier on the same page, the author mentions that it is not known whether $\liminf_{n\to\infty}(p_{n+1}-p_n)<\infty$. It would have been helpful to add that this is a weak version of the twin prime conjecture (which had been discussed earlier in the book), and that even

$$\liminf_{n\to\infty}(p_{n+1}-p_n)/\log p_n = 0$$

($\log p_n$ being the "usual" gap between primes) is unknown. On page 109 the formula $\sum_{n=2}^{\infty}(\Lambda(n)-1)/n=-2\gamma$ should be $\sum_{n=1}^{\infty}(\Lambda(n)-1)/n=-2\gamma$. Compared to the overall quality of the book, these small quibbles seem quite insignificant.

From MathSciNet, July 2013

K. Soundararajan

MR2321397 (2008h:01015) 01A50

Sandifer, C. Edward

How Euler did it. (English)

MAA Spectrum.
Mathematical Association of America, *Washington, DC*, 2007. xiv+237 *pp.*.
\$51.95. *ISBN* 978-0-88385-563-8

This book consists of forty monthly columns by Sandifer that appeared on MAA Online between November 2003 and February 2007. The range of topics is enormous, including results in number theory, geometry, combinatorics, analysis, and mechanics. Given Euler's genius, this range is certainly no surprise, but it does make clear the significance and scope of his contribution to the development of mathematics: nearly every topic includes some fundamental result in a branch of pure or applied mathematics.

The magnitude of Euler's achievements is highlighted by Sandifer's tongue-in-cheek "Official List of Euler's Top Ten Theorems" in the first chapter. These results include:

1. The Basel Problem: $\zeta(2) = \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$.
2. The Polyhedral Formula: $V - E + F = 2$.
3. The "Euler Identity": $e^{\pi i} = -1$.
4. The Königsberg bridge problem (cross seven bridges joining two islands and the shore exactly once) and the knight's tour (move a knight through all 64 squares of a chessboard landing exactly once on each).
5. The Euler Product formula: $\prod_{p \text{ prime}} \frac{1}{1-\frac{1}{p^n}} = \sum_{k=1}^{\infty} \frac{1}{k^n}$.
6. The Euler-Lagrange necessary condition: if a function f makes

$$J = \int_a^b f(x, y, y')dx$$

a maximum or a minimum, then

$$\frac{\partial f}{\partial y} - \frac{d}{dt}(\frac{\partial f}{\partial y}) = 0.$$

7. Density of primes: $\sum_{p \text{ prime}} \frac{1}{p}$ diverges.
8. Generating functions and partition numbers.
9. The Euler-Fermat Theorem: $a^{\varphi(n)} \equiv 1(\text{mod } n)$.
10. The gamma function.

The means of selecting these as the top ten are unlikely to satisfy the most stringent requirements of statistical methodology: participants at the Short Course on Euler at the 2007 Joint Mathematics Meeting in New Orleans were asked to select among thirty candidate theorems. Still, there can be little doubt that these are very important results, and the breadth of coverage is certainly remarkable.

The articles in the book are sorted into four sections: Geometry, Number Theory, Combinatorics, and Analysis. Each is relatively brief (averaging just over five pages, including references), but they offer an illuminating and stimulating introduction to Euler's work. As the title of the book indicates, the articles' emphasis is on showing how Euler approached a problem: identifying the means available to him, setting out the general outlines of his reasoning, and offering an assessment of the scope and significance of the result.

The book does an admirable job of introducing the reader to Euler's achievements, and it will be of interest to anyone curious about the development of eighteenth-century mathematics. The writing style is engaging, the mathematics is presented clearly, and the results discussed are of interest both historically and for their mathematical content. It is fascinating, for instance, to see the complexities involved in the proof of the polyhedral formula $V - E + F = 2$ (which Euler never quite proved in its full generality), and to learn that Descartes had wrestled with a very similar result involving sums of plane angles in a polyhedron, although Descartes' work on the problem was known only to Leibniz until the second half of the nineteenth century. Likewise, Sandifer's discussion of the question of who first proved the irrationality of e leads to an intriguing treatment of continued fractions along with some historical detective work that shows Euler to be correctly credited with the result, notwithstanding the fact that his initial approach to the problem falls short of a complete proof.

In short, Sandifer's articles can be read profitably by historians of mathematics and working mathematicians alike, and the collection is a welcome addition to the literature on the mathematician whom Laplace aptly termed "the master of us all".

From MathSciNet, July 2013

Douglas M. Jesseph

MR2219954 (2007h:01009) 01A70; 01A50, 11-03, 33-03, 40-03

Varadarajan, V. S.

Euler through time: a new look at old themes. (English)

American Mathematical Society, *Providence, RI*, 2006. viii+302 *pp.*.
ISBN 0-8218-3580-7

This book is a study in the mathematics of Leonhard Euler (1707–1783), with particular attention paid to those aspects of his voluminous work that connect to current mathematical research. As the author notes, he does not intend merely to describe what results Euler achieved and when, but rather to "pay great attention to the historical evolution of ideas and how they mesh with what we know and are interested in today" (Preface, vii). The result is very much a forward-looking book, which takes up themes from Euler and delineates their connection to current concerns in mathematics. On the whole, it is a satisfying achievement, although some historians of mathematics will find that it is insufficiently concerned with Euler's historical context and the details of his discoveries. The six chapters include a concise and accurate scientific biography of Euler (Chapter 1); a study of themes in the calculus and number theory (Chapter 2); a detailed look at the Eulerian theory of zeta values (Chapter 3); a treatment of the Euler-Maclaurin sum formula and its applications (Chapter 4); an extensive account of Euler's work on divergent series and its influence on later mathematics (Chapter 5); and a study of the Euler product formula, as well as subsequent research into Euler products and their role in number theory. A useful "portrait gallery" is appended, which gives thumbnail sketches of the careers and contributions of some of the mathematicians whose work is mentioned in the course of the book.

The presentation is clear and to the point, with technical material developed in a way that is generally accessible. Historians of mathematics with a somewhat rusty grasp of nineteenth and twentieth century developments in such subjects as the theory of infinite series and products may find parts of the book rather formidable. Nevertheless, the exposition and development of the key ideas is never inscrutable to someone with a sound undergraduate background in mathematics.

One of the most intriguing chapters of the book is its study of Euler's treatment of divergent series. Lacking our current understanding of such concepts as convergence and uniform convergence, Euler was left to navigate the murky waters of series expansions and infinite sums in ways that offend against current standards of rigor. He had no compunction about taking the assignment of a sum to a divergent series as a matter of more or less arbitrary convention, a willingness which opened up the possibility that different conventions might yield different sums for the same series. Guided by an almost infallible intuition, Euler nevertheless achieved results that withstood scrutiny. In the course of time, the inherent difficulties (indeed, incoherence) in the careless use of divergent series led to conflicting "results" and Abel's famous remark that "Divergent series are in general the work of the devil and it is shameful to base any demonstration whatever on them" (p. 145). Such scruples led to the modern theory of sequences, series, and convergence as developed by Abel and Cauchy, and became the foundation of analysis as we now know it. Varadarajan shows not only how Euler's main results (such as the functional equation of the zeta function) are recoverable in the modern approach to analysis, but that some of his ideas led into altogether new territory. In one case, Euler's work anticipated the theory of Borel summation in the solution of differential equations with asymptotic expansions, and in another his ideas can be traced to current work in summation of divergent Gaussian integrals that appear in quantum mechanics and quantum field theory.

Although not a traditional history of mathematics, Varadarajan's contribution is an interesting addition to the literature on eighteenth century mathematics. By taking some of Euler's most important insights, developing them, and showing their connection to contemporary research, this book offers a profound understanding of Euler's achievements and their role in the development of mathematics as we now know it.

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