BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 51, Number 1, January 2014, Pages 163–168
S 0273-0979(2013)01422-8
Article electronically published on July 1, 2013

Global aspects of ergodic group actions, by Alexander S. Kechris, AMS Mathematical Surveys and Monographs, vol. 160, American Mathematical Society, Providence, RI, 2010, xii+237 pp., ISBN 978-0-8218-4894-4

The study of dynamical systems has its origins in the classical mechanics of Newton and his successors. That theory concerns the behavior of solutions of certain differential equations on manifolds. The modern theory has flourished in many directions that are best described by focusing on different features of the classical systems. For example, keeping only the topological structure led to the development of topological dynamics as set out in the foundational monograph of by W. Gottschalk and G. Hedlund [GH], while focusing on the smooth structures gave rise to what is called smooth dynamics. The seminal ergodic theorems of J. von Neumann and G. D. Birkhoff involved only the measure structure and pointed the way to the development of ergodic theory which can be viewed as the study of transformations of measure spaces. In classical mechanics the acting group is the real line or the integers, and many of these theories were developed in that context. However, already in the papers in which Murray and von Neumann created the theory of what are now called *von Neumann algebras*, general groups were considered, and many of the recent developments in ergodic theory take place in the setting of quite general acting groups. Indeed many of the remarkable successes of ergodic theoretic methods in solving number theory problems involve actions of Lie groups and their discrete subgroups.

For the most part, the measure spaces studied in ergodic theory are isomorphic to the unit interval equipped with Lebesgue measure λ. These are called *standard measure spaces*. Recall that a measure space is a triple $(X, \mathcal{B}, \mu)$, where X is a space, $\mathcal{B}$ is a collection of subsets of X that is closed under countable unions and complements, and μ is a countably additive probability measure defined on $\mathcal{B}$. An isomorphism between such a measure space and the unit interval with Lebesgue measure is a measurable function θ from X to $[0, 1]$ that is invertible almost everywhere, maps $\mathcal{B}$ to the Lebesgue measurable sets, and for all such sets A, $\mu(\theta^{-1}(A)) = \lambda(A)$. The main characters are the measure preserving transformations. These are measurable mappings, T, of X that preserve the measure μ. More formally for $B \in \mathcal{B}$, we have that $T^{-1}(B) \in \mathcal{B}$ and $\mu(T^{-1}(B)) = \mu(B)$. Examples of these transformations are rotations of the circle, interval exchanges—in which the unit interval is partitioned into a finite number of intervals and these are permuted by translating each subinterval by a different number, toral automorphisms in which the measure space is the torus $\mathbb{T}^d$—which is easily seen to be a standard measure space.

Basic properties of such transformations are: (i) *ergodicity*, which means that invariant sets must have measure equal to zero or one; (ii) *strong mixing*, which means that successive images of any measurable set A are asymptotically independent of any fixed set B, i.e.,

$$\lim_{n\to\infty} \lambda(T^n(A) \cap B) = \lambda(A)\lambda(B);$$

2010 *Mathematics Subject Classification.* Primary 37A15, 37A20, 03E15.

and (iii) *weak mixing*, which replaces ordinary convergence in the above by Cesaro convergence. Irrational rotations of the circle are ergodic—but not weakly mixing—while a toral automorphism whose spectrum does not contain any root of unity is strongly mixing and possesses even stronger randomness properties.

While classical ergodic theory focused mainly on the asymptotic properties of individual measurable invertible transformations, already in the early years some attention was given to the group MPT, consisting of all measurable transformations T of the unit interval that preserves Lebesgue measure. This group has a natural topology induced by a metric which gives it the structure of a complete separable metric space (i.e., a Polish space) in which the group operation is continuous. Such topological groups are called *Polish groups*. In 1944 [H] P. Halmos proved that generically (in the topological sense, which means that the property in question holds for a dense G_δ) a transformation is weakly mixing and in particular it is ergodic. Several years later, V. Rokhlin [R] showed that generically a transformation is not strongly mixing. However, explicit examples of weakly mixing transformations that are not strongly mixing are not so easy to construct. These are the first results in what can be described as the global aspects of group actions. Here the group in question is of course $\mathbb{Z}$, the integers, and the action is identified with its generator, which is a single transformation.

Some of the more recent developments in this area involve some basic ideas from descriptive set theory, and we recall now some of these briefly. If we begin with a Polish space, we can start with the open sets and, by applying complements, unions, and intersections of countable families, build up the σ-algebra of Borel sets. These sets can be graded in terms of their complexity—countable intersections of open sets are the G_δ's, their complements are the F_σ's, countable unions G_δ's are the $G_{\delta\sigma}$'s etc. If B is a Borel set in some Polish space X and f is a continuous map from X to another Polish space Y, then $f(B)$ may not be a Borel set, contrary to a famous erroneous assertion of Lebesgue. Nonetheless, these sets, which are called *analytic sets*, do have some nice properties. For example they are measurable with respect to any regular Borel measure. The *co-analytic* sets are simply the complements of analytic sets.

A basic theorem of Suslin asserts that if a set is both analytic and co-analytic, then it is a Borel set. For both notions there is a complete object. For example, a complete analytic set C is an analytic subset of a Polish space X such that for any analytic set $A \subset Y$ there is a Borel measurable map $f : Y \to X$ such that $A = f^{-1}(C)$. These notions make it possible to ask more detailed questions about various collections of group actions.

Here is one of the first results in this circle of ideas. In topological dynamics, which studies continuous actions on compact spaces, there is a simple notion of *distality*, which is defined as follows. A homeomorphism T of a compact metric space (X, D) is said to be distal if for any two distinct points $x, y \in X$ there is a positive number c such that for all n the distance $d(T^n x, T^n y) \geq c$. There is a detailed structure theorem for these transformations which was discovered by H. Furstenberg [F] fifty years ago. A measure theoretic analogue of this notion called *measure distal* was first introduced by W. Parry, and a structure theorem for them was given by H. Furstenberg and R. Zimmer. The definition is more technical and we do not give it here. F. Beleznay and M. Foreman [BF] showed that the class of measure distal transformations in MPT is a complete co-analytic set and in particular is not Borel measurable.

In addition to studying classes of actions, one can also ask about the complexity of relations. In his pioneering paper [vN] J. von Neumann formulated the isomorphism problem for measure preserving transformations which can be also formulated as the conjugacy problem in MPT, namely when are two elements T and S conjugate, i.e., when is there a third $R \in$ MPT such that $S = RTR^{-1}$. B. O. Koopman introduced the unitary operator U_T associated to a $T \in$ MPT as the operator on L^2 defined by $U_T f(x) = f(Tx)$. Conjugate elements of MPT give rise to unitary operators that are unitarily equivalent, and this was the first invariant to be discovered for the conjugacy relation. For ergodic T, whose associated unitary operator has pure point spectrum, J. von Neumann showed that unitary equivalence implies conjugacy. This fails to be true for more general transformations and even for pure point spectrum in the nonergodic case. Work on the conjugacy problem motivated many of the developments in ergodic theory, such as entropy, but further discussion of this would lead us astray at this point. The descriptive set theory perspective sets the problem of how complex is the conjugacy relation itself as a subset of MPT $\times$ MPT. It is clearly an analytic set as the projection of the closed set of triples (R, S, T) such that $S = RTR^{-1}$.

G. Hjorth showed that it is a complete analytic set by using, in an essential way, nonergodic elements. Since the ergodic elements of MPT are easily seen to be a G_δ, they have the structure of a Polish space, and the question can be formulated for ergodic transformations. In joint work with M. Foreman and the late Dan Rudolph [FRW] it was shown that this too is a complete analytic set.

Ergodic theory for a countable group G studies the imbeddings of G into MPT. Each such imbedding is called an action of G, and naturally MPT acts by conjugation on the space of imbeddings which can be easily given a natural Polish space topology. Two such actions are isomorphic if they are conjugate as subgroups of MPT. In this way one can set the more general problem of the complexity of the isomorphism for general groups in the context of the complexity of the orbit equivalence relation of a Polish group action. A notion weaker than that of isomorphism arose in the theory of von Neumann algebras, and is called *orbit equivalence*. If $\phi(g)$ and $\psi(g)$ are two actions of G, they are said to be orbit equivalent if there is a element $R \in$ MPT such that for almost every $x \in [0, 1]$ the set $R(\phi(G)(x))$ equals $\psi(G)(R(x))$. Here what is essential is that the orbit equivalence relation, defined by the action and the transformation R, sends equivalences classes defined by the ϕ action to the equivalence classes defined by the ψ action. H. Dye proved the remarkable result that when G is abelian, any two ergodic actions are orbit equivalent, and in fact they are also orbit equivalent to an ergodic $\mathbb{Z}$ action.

Another remarkable result of Dye's is the following. A countable measurable equivalence relation on $(X, \mathcal{B}, \mu)$ is a measurable subset E of the product space which is an equivalence relation with countable classes such that measurable transformations of X whose graphs are subsets of E preserve the measure μ. Any measure preserving action of a countable group G defines such an E by its orbits. The *full group* of a countable measurable equivalence relation E is the subgroup of MPT of all transformations that map equivalent points to equivalent points almost everywhere. If E_1 and E_2 are two such measurable equivalence relations, then they are isomorphic if there is an element of MPT which sends the classes of E_1 to the classes of E_2. Dye proved that any abstract group isomorphism between the associated full groups of two countable measurable equivalence relations is induced by a measure preserving transformation.

Dye's theorem on orbit equivalence of ergodic $\mathbb{Z}$ actions was later extended to amenable groups by D. Ornstein and me. However, for nonamenable groups recent work has shown that there are many nonorbit equivalent ergodic actions. In fact, in some situations one finds the phenomenon of rigidity in which orbit equivalence implies isomorphism.

We turn now to the subject of our review *Global aspects of ergodic group actions* by Alexander Kechris. This is a very well written survey of many of the recent developments in the ergodic theory of countable groups, including many results of the author himself and his collaborators. The book consists of three chapters and nine appendices. The first chapter is devoted to the actions themselves. It gives a detailed, somewhat novel treatment of the results described above on orbit equivalence and much more. One of the sections of this chapter is devoted to showing that the conjugacy relation, when thought of as the action of MPT on the space of actions of $\mathbb{Z}$, is turbulent. This notion, which was introduced by G. Hjorth, is a property of the continuous action of a Polish group G on a Polish space Z and is defined as follows. The action is *turbulent* if all orbits of G are dense and meager, and, in addition for any two points x, y in Z and open sets $V \subset Z$ containing x and $U \subset G$ containing the identity, there are points x_i in V and g_i in U such that $x_0 = x$, $X_{i+1} = g_i x_i$, and for some subsequence i_n, the points x_{i_n} converge to a point in the orbit of y. Hjorth showed that if an action is turbulent, then countable structures cannot be complete invariants for the orbit relation. Thus no countable collection of invariants, such as the entropy of an action, which is a numerical conjugacy invariant introduced by A. Kolmogorov to distinguish between different kinds of Bernoulli shifts or the point spectrum of the associated unitary operator, can be complete.

Another topic discussed in this chapter is the cost of an equivalence relation, a numerical invariant which was introduced by G. Levitt and further developed by D. Gaboriau. Using it, Gaboriau showed that the equivalence relations induced by Bernoulli actions of free groups of unequal rank are not orbit equivalent. The full group of a countable measure preserving equivalence relation can be easily topologized so as to become a Polish group. A finite set topologically generates a topological group if the abstract group it generates is dense. Thus the minimum number of topological generators is an invariant of the group. Kechris discusses the relation between the cost of an equivalence relation and this minimal number of topological genreators. While writing this review, I was informed that Francois Le Maitre, a student of Gaboriau, has given optimal bounds relating these two invariants. In particular, the minimal number of topological generators for the full group defined by any free ergodic action of the free group of rank n is $n + 1$.

The second chapter, entitled "The space of actions", is devoted to a more global discussion of the space of measure preserving actions of a countable group Γ. Using the same topology that turned MPT into a Polish group, this becomes a Polish space, and the group MPT acts on it by conjugation and one can study its dynamical properties. These are the main objects of study in this chapter, and here is a small sampling of some of the results that are given a thorough treatment. First, a result that holds for all countable groups Γ, namely that the conjugacy action is topologically transitive, or in other words there is a dense conjugacy class. Certain classes of groups, such as groups with Kazhdan's property T and groups with the Haagerup Approximation property, can be characterized via properties of the space of actions. Turbulency of the action of MPT, which we mentioned earlier in connection with the classical situation of $\mathbb{Z}$ actions, is also given a detailed treatment here for more general groups. It is in this chapter that Kechris describes the latest results on the rich variety of nonorbit equivalent actions of nonamenable groups. Not only are there uncountably many orbit equivalence classes, but they cannot be classified by countable structures.

The main characters in the third chapter are cocycles and cohomology, which we have not yet introduced but which would take us quite far afield. Our review of this will be correspondingly quite brief but should serve at least to give a taste of this wonderland.

First a bit of history. The abstract study of cocycles and cohomology of measurable equivalence relations was first extensively developed by J. Feldman and C. Moore. Their motivation came from the theory of von Neumann algebras and ergodic theory.

The 1-cocycles arise naturally in several contexts in ergodic theory and are easily defined as follows. If ϕ is an action of Γ on a space X and G is any group, then a G-valued cocycle (here and in the sequel we drop the "1", since we will not be discussing the higher order cohomology) for this action is a measurable mapping u from $\Gamma \times X$ to G that satisfies the identity

$$u(\gamma\delta, x) = u(\gamma, \phi(\delta)(x))u(\delta, x).$$

If the cocycle is independent of x, then it is simply a homomorphism from the group Γ to the group G.

Two cocycles u and v are said to be cohomologous if there is a measurable function F from X to the group G such that

$$u(\gamma, x) = F(\phi(\gamma)(x))v(\gamma, x)F(x)^{-1}.$$

Cocycles that are cohomologous to the identity mapping are called coboundaries. Specifically, a cocycle u is a coboundary if there is a measurable function F such that $u(\gamma, x) = F(\phi(\gamma)(x))F(x)^{-1}$. The space of 1-cocycles modulo the cohomologous relation is called the 1-cohomolgy.

An orbit equivalence between two free Γ actions gives rise in a natural fashion to a cocycle with values in Γ, and this is one of the ways in which they arise. In fact the 1-comology depends only on the equivalence relation and not on the action, which is one of the reasons for their importance. They are also among the main characters in the generalizations of R. Zimmer of the famous superrigidity theory of G. Margulis. Here it is important that the acting group be far from amenable—in fact it should have property T.

Rather than trying to describe the wealth of results about this cohomology that are expertly surveyed in this part of the book, we will try to whet the readers appetite by explaining one striking consequence that has been proved using some of these ideas and much more by S. Popa. First of all we owe the reader who has come this far an example of a measure preserving action of a general group Γ. For this we start with an arbitrary probability space which, for simplicity, we take to be finite, thus we have a finite set A and a probability distribution $\mathbf{p} = \{p_a\}$ on A. Now let X be the space A^Γ with product measure $\mathbf{p}^\Gamma$. The group acts on this space of functions from Γ to A in a natural way by multiplication in the argument. In fact there are two actions, a left action and a right action. These actions are called Bernoulli since the coordinate functions are independent and Bernoulli was the first to prove a law of large numbers for independent random variables. In the classical situation of $\mathbb{Z}$, A. Kolmogorov showed that if two Bernoulli shifts were isomorphic, then the Shannon entropy of their underlying probability distribution $H(\mathbf{p})$ must be the same. D. Ornstein proved the converse, and in the course of doing so introduced many new ideas and techniques which led to a deep understanding of the ergodic theory of actions of amenable groups. One of his fundamental results was that a factor of a Bernoulli shift is isomorphic to a Bernoulli shift.

In a stark contrast to this, several years ago S. Popa [P] showed that for a large class of groups, including those with property T, a Bernoulli shift can have an uncountable family of factors that are mutually not orbit equivalent and none of them are orbit equivalent to a Bernoulli shift.

The appendices contain a concise presentation of some of the prerequisites for the book and include a discussion of Gaussian probability spaces, the Wiener chaos decomposition, some aspects of the theory of unitary representations of locally compact groups, and much more. Naturally the reader is expected to have a good background in ergodic theory, but it is to be regretted that some of the newer concepts, such as turbulence and cost, are not defined explicitly.

All in all, this is an important survey of many of the latest results in a newly developing branch of ergodic theory, well organized and clearly presented. It should serve as a basic reference for many years to come.

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