

L^p THEORY FOR OUTER MEASURES AND TWO THEMES OF LENNART CARLESON UNITED

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Dedicated to Lennart Carleson

ABSTRACT. We develop a theory of L^p spaces based on outer measures generated through coverings by distinguished sets. The theory includes as a special case the classical L^p theory on Euclidean spaces as well as some previously considered generalizations. The theory is a framework to describe aspects of singular integral theory, such as Carleson embedding theorems, paraproduct estimates, and $T(1)$ theorems. It is particularly useful for generalizations of singular integral theory in time-frequency analysis, the latter originating in Carleson's investigation of convergence of Fourier series. We formulate and prove a generalized Carleson embedding theorem and give a relatively short reduction of the most basic L^p estimates for the bilinear Hilbert transform to this new Carleson embedding theorem.

1. INTRODUCTION

Two seminal papers of Lennart Carleson of the 1960s each introduced a new tool into analysis that had profound influence. In his paper *Interpolation by bounded analytic functions and the corona problem* [2], he introduced what later became known as Carleson measures. Carleson measures revolutionized singular integral theory, where they are, for example, related to the space BMO and related areas in real and complex analysis. In his celebrated paper *On convergence and growth of partial sums of Fourier series* [3], Carleson introduced what we now call time-frequency analysis. Time-frequency analysis has remained until now an indispensable tool for its original application of controlling Fourier series pointwise as well as a number of other applications, including L^p estimates for the bilinear Hilbert transform. Our present paper shows that a natural L^p theory for outer measures offers a unifying language for both Carleson measures and time-frequency analysis. The fundamental nature of our L^p theory for outer measures might in hindsight be an explanation for the important role of Carleson measures.

This paper is divided into three parts. In the first part, Sections 2 and 3, we carefully develop the basic L^p theory for outer measure spaces. This part is open ended in nature and will hopefully lead to further investigations of outer measure spaces. We have focused only on those aspects of the theory that are directly relevant for the applications that we have in mind in the other parts of this paper.

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Outer measures are subadditive set functions. In contrast to measures, outer measures do not necessarily satisfy additivity for disjoint finite or countable collections of sets. Some outer measures give rise to interesting measures by restriction to Carathéodory measurable sets, the most prominent example being classical Lebesgue theory. However, general outer measures need not give rise to interesting measures, and one is led to studying outer measure spaces for their own sake. Lacking additivity for disjoint sets, one cannot expect a useful linear theory of integrals with respect to outer measure. A good replacement is a sublinear or quasi-sublinear theory, which leads directly to norms or quasi-norms rather than integrals. Naturally, L^p norms are among the most basic norms to be considered in the context of outer measures.

There is a rich literature on outer measures, for example on capacity theory. In contrast to previously developed theories based on the Choquet integral, we do not in general base our L^p theory on the outer measure of super level sets $\{x : f(x) > \lambda\}$ for a function f . Instead, we use a more subtly defined quantity (Definition 2.5) to replace the outer measure of a super level set. This new quantity, which we call *super level measure*, involves predefined averages over the generating sets of the outer measure. If the predefined averages are of L^∞ type, the super level measure specializes to the outer measure of the super level set, but in general the two quantities are quite different. Once we have introduced the super level measure, the L^p theory develops in standard fashion, and we develop it to the extent that we need for subsequent parts of the paper.

In the second part of this paper, Section 4, we describe how outer measures can be used in the context of Carleson measures. It is our first example of an outer measure space in which our refined definition of super level measure does not coincide with the classical case of the outer measure of super level set. The outer measure space in question is the upper half-plane and the outer measure is generated by tents. The essentially bounded functions with respect to the outer measure in this upper half-plane are Carleson measures. Moreover, the identification of a function on the boundary with the harmonic extension in the interior of the upper half-plane, that is the Carleson embedding map, turns out to be a basic example of a bounded map from a classical L^p space to an outer L^p space. We describe in Section 4 how classical estimates for paraproducts and $T(1)$ theorems can be proved by an outer Hölder inequality together with such embedding theorems. In this setting, the use of outer L^p spaces is very much in the spirit of the use of tent spaces introduced in [4]. It is an artifact in this particular situation that our notion of outer measure may be replaced with more classical concepts.

The full power of the new outer L^p spaces becomes evident in its applications in time-frequency analysis, which we discuss in the third part of this paper (Sections 5 and 6). The underlying space for the outer measure becomes the Cartesian product of the upper half-plane with a real line. In this setting there are no evident analogues of the tent spaces of [4] that one could use in place of outer L^p spaces. We formulate and prove a novel generalized Carleson embedding theorem, Theorem 5.1, in Section 5. It is a compressed and elegant way to state an essential part of time-frequency analysis. In Section 6 we then use the generalized Carleson embedding theorem to reprove bounds for the bilinear Hilbert transform.

The generalized Carleson embedding theorem can also be used as an ingredient to prove almost everywhere convergence of partial Fourier integrals of L^p functions

with $2 < p < \infty$. One would need an additional Carleson embedding theorem, either analogous to the interplay between energy and mass in [12], or analogous to some vector valued version of the Carleson embedding theorem as in [7]. We also envision the generalized Carleson embedding theorem and variants thereof to be useful in further advances in time-frequency analysis. We were led to the theory of outer L^p spaces while working on variation norm estimates as in [16] in the setting of biest type operators as in [14]. For brevity of the present paper and because of the various possible routes toward Carleson’s theorem, we decided to restrict this exposition to a discussion of the bilinear Hilbert transform. This already captures many essential parts of Carleson’s time-frequency analysis.

Gaining a streamlined view on time-frequency analysis was the original motivation for the present paper, which is the outcome of a long evolution process. In traditional time-frequency analysis, one proves bounds of multilinear forms passing through model sums

$$\Lambda = \sum_{P \in \mathbf{P}} c_P \prod_{j=1}^n a_j(P),$$

where the summation index runs through a discrete set, typically a collection of rectangles (tiles) in the phase plane. The coefficients c_P are inherent to the multilinear form, while the sequences a_j each depend on one of the input functions for the multilinear form in question. There is a multitude of examples in the literature for the tile sequences a_j , the most basic example being normalized wave packet coefficients

$$(1.1) \quad a_j(P) = \langle f, \phi_P \rangle$$

for the L^1 normalized wave packets

$$\phi_P(x) = 2^{-k} \phi(2^{-k}x - n) e^{2\pi i 2^{-k}xl},$$

where k, n, l are integers and parameterize the space \mathbf{P} , and ϕ is a suitably chosen Schwartz class function. These coefficients are much in the spirit of the embedding maps considered in Sections 5 and 6 of the present paper. Use of such wavepackets in the study of the bilinear Hilbert transform appears in [10]. In the dyadic model as in [18] one defines wave packets with respect to abstract Fourier analysis on the group $\mathbb{Z}/2\mathbb{Z}$. More generally one can have tile seminorms

$$(1.2) \quad a_j(P) = \sup_{\phi \in \Phi} |\langle f, \phi_P \rangle|,$$

where one maximizes and possibly also averages over a suitably chosen set Φ of generating functions. This approach has been useful in [19] and more explicitly in [15]. To prove bounds on Carleson’s operator, [12] uses modified wave packets

$$(1.3) \quad a_j(P) = \langle f, \phi_P \mathbf{1}_{\{(x, N(x)) \in P\}} \rangle$$

for the linearizing function N of the linearized Carleson operator. In some instances, such as in [14], the definition of $a_j(P)$ may involve itself a multilinear operator whose analysis requires another level of time-frequency analysis. For variational estimates of the Carleson operator, as in [16], one has variational wave packets

$$(1.4) \quad a_j(P) = \langle f, \phi_P \sum_k v_k \mathbf{1}_{\{(x, N_k) \in P_2, (x, N_{k-1}) \notin P\}} \rangle$$

for a sequence of linearizing functions $N_0(x) < N_1(x) < \dots$ and a sequence of dualizing functions $v_1(x), v_2(x), \dots$, such that for some $r > 2$ we have the uniform bound

$$\sum_k |v_k(x)|^{r'} = O(1).$$

A point of the present paper is that in many of these examples the bound on Λ is a Hölder inequality with respect to an outer measure on the space P :

$$|\Lambda| \leq C \sup_{P \in \mathcal{P}} |c_P| \prod_{j=1}^n \|a_j\|_{L_j^p(\mathbf{P}, \dots)},$$

where the dots stand for specifications of the outer measure structures in each example. The rest of the proof of boundedness of Λ then becomes modular in that one has to prove bounds for each j separately on the outer L^p norms of the sequences a_j , estimates which, for example, take the form

$$\|a_j\|_{L_j^p(\mathbf{P}, \dots)} \leq \|f_j\|_p,$$

where f_j may be the corresponding input function to the original multilinear form as for example in (1.1), and the L^p norm is in the classical sense.

A novelty in the present paper is that we do not have to pass through a discrete model form, but rather work with an outer measure space on a continuum. This avoids both the cumbersome introduction of the discrete spaces as well as the usual technicalities in the discretization process.

The factorization of the multilinear form in time-frequency analysis into embedding theorems on the one hand and an outer Hölder's inequality on the other hand is a clear modularization of the matter, and it promises to be useful in other applications of time-frequency analysis. Indeed, we were explicitly studying the modularization process because with Camil Muscalu we were considering a program outlined in [8] of estimating multilinear forms with nested levels of time-frequency analysis.

2. OUTER MEASURE SPACES

2.1. Outer measures. An outer measure or exterior measure on a set X is a monotone and subadditive function on the collection of subsets of X with values in the extended nonnegative real numbers, and with the value 0 attained by the empty set.

Definition 2.1 (Outer measure). Let X be a set. An outer measure on X is a function μ from the collection of all subsets of X to $[0, \infty]$ that satisfies the following properties:

- (1) If $E \subset E'$ for two subsets of X , then $\mu(E) \leq \mu(E')$.
- (2) $\mu(\emptyset) = 0$.
- (3) If E_1, E_2, \dots is a countable collection of sets in X , then

$$(2.1) \quad \mu\left(\bigcup_{j=1}^{\infty} E_j\right) \leq \sum_{j=1}^{\infty} \mu(E_j).$$

In the examples we have in mind, the space X is an infinite complete metric space and thus uncountable. The set of all subsets of X then has even larger cardinality than the continuum, and it can only be organized in abstract ways.

The description of an outer measure then typically comes in two steps: First, one specifies concretely a quantity that we may call a premeasure on a small collection of subsets. Second, one passes abstractly from the premeasure to the outer measure by means of covering an arbitrary subset by sets in the small collection. This covering process is the intuition behind the adjective *outer* in the term *outer measure*.

Proposition 2.1 (Abstract generation of outer measure by a concrete premeasure). *Let X be a set, and let \mathbf{E} be a collection of subsets of X . Let σ be a function from \mathbf{E} to $[0, \infty)$. Define for an arbitrary subset E of X*

$$\mu(E) := \inf_{\mathbf{E}'} \sum_{E' \in \mathbf{E}'} \sigma(E'),$$

where the infimum is taken over all countable subcollections \mathbf{E}' of \mathbf{E} which cover the set E , that is whose union contains E . Here we understand that an empty sum is 0. Then μ is an outer measure.

The concrete premeasure requires the data \mathbf{E} and σ . For simplicity we will often omit explicit mention of \mathbf{E} , since \mathbf{E} is implicitly determined as the domain of σ . The proof of the proposition is basic and standard, we reproduce it here for emphasis.

Proof. We need to prove the three defining properties of outer measures.

The empty collection of subsets covers the empty set, which shows $\mu(\emptyset) = 0$ since the empty sum of nonnegative numbers is 0.

If $F \subset F'$ for two subsets of X , then every cover of F' is a cover of F and hence $\mu(F) \leq \mu(F')$. Let F_1, F_2, \dots be a countable collection of subsets of X , and pick $\epsilon > 0$. Find for each i a countable subcollection \mathbf{E}_i of \mathbf{E} which covers F_i and satisfies

$$\sum_{E \in \mathbf{E}_i} \sigma(E) \leq \mu(F_i) + \epsilon 2^{-i}.$$

Then the union \mathbf{E}' of the collections \mathbf{E}_i covers the union of the sets F_i and satisfies

$$\sum_{E \in \mathbf{E}'} \sigma(E) \leq \left(\sum_i \mu(F_i) \right) + \epsilon.$$

Since ϵ was arbitrary, we conclude that $\mu(\bigcup F_i) \leq \sum_i \mu(F_i)$. □

It is in general not true that for $E \in \mathbf{E}$ we have $\sigma(E) = \mu(E)$; however, this identity can be established in many examples in practice. Clearly, this identity holds precisely if for every set $E \in \mathbf{E}$ and every cover of E by a countable subcollection \mathbf{E}' of \mathbf{E} , we have

$$(2.2) \quad \sigma(E) \leq \sum_{E' \in \mathbf{E}'} \sigma(E').$$

Then the most efficient cover of E is by the trivial collection $\{E\}$, which establishes $\sigma = \mu|_{\mathbf{E}}$.

We did not allow σ to take value ∞ . This is not a restriction, since if we had $\sigma(E') = \infty$ for some $E' \in \mathbf{E}$, then using the set E' in any cover of E will make the sum $\sum_{E' \in \mathbf{E}'} \sigma(E')$ equal to ∞ , a value that is as already the default even if no cover of E exists at all.

If the collection \mathbf{E} is countable, the contribution of sets $E \in \mathbf{E}$ with $\sigma(E) = 0$ could be ignored. Namely, we may consider the union E_0 of the countably many generating sets with premeasure 0. Then E_0 has outer measure zero, and we can

construct an outer measure on $X \setminus E_0$ which reflects the structure of the outer measure on X but does not contain any generating set with premeasure 0.

2.2. Examples for outer measures.

Example 1 (Lebesgue measure via dyadic cubes). Let X be the Euclidean space \mathbb{R}^m for some $m \geq 1$, and let \mathbf{E} be the set of all dyadic cubes, that is all cubes of the form

$$Q = [2^k n_1, 2^k(n_1 + 1)) \times \cdots \times [2^k n_m, 2^k(n_m + 1))$$

with integers k, n_1, \dots, n_m . For each dyadic cube Q we set

$$\sigma(Q) = 2^{mk}.$$

Then σ generates an outer measure which is the classical Lebesgue outer measure on \mathbb{R}^m . We have $\sigma(Q) = \mu(Q)$ for every dyadic cube. This latter fact requires a bit of work; in fact it is one of the more laborious items in the standard introduction of Lebesgue measure.

Example 2 (Lebesgue measure via balls). Let $X = \mathbb{R}^m$ as above, and let \mathbf{E} be the set of all open balls $B_r(x)$ with radius r and center $x \in \mathbb{R}^m$. Let $\sigma(B_r(x)) = r^m$ for each such ball. Then σ generates a multiple of the Lebesgue outer measure, and again we have $\sigma = \mu|_{\mathbf{E}}$.

If one desires a countable generating set, one may restrict the collection of generating sets to the collection of balls which have rational radius and rational center. This choice will result in the same outer measure.

Example 3 (Outer measure generated by tents). Let $X = \mathbb{R} \times (0, \infty)$ be the open upper half-plane, and let \mathbf{E} be the set of tents that is open isosceles triangles of the form

$$T(x, s) = \{(y, t) \in \mathbb{R} \times (0, \infty) : t < s, |x - y| < s - t\}$$

(see Figure 1 in Section 4) for some pair $(x, s) \in \mathbb{R} \times (0, \infty)$ which describes the tip of the tent. Note that the constraint $t < s$ is implied by the constraint $|x - y| < s - t$, but it is kept for emphasis. Define $\sigma(E) = s$ for any such tent, and note that $\sigma(E)$ is equal to $\frac{1}{2}\sigma_L(\pi(E))$, where $\pi(E)$ is the projection of E onto the first coordinate and thus is an open ball in \mathbb{R} , and σ_L is the generator of Lebesgue outer measure on \mathbb{R} described in Example 2.

By projection onto the first coordinate it easily follows from Example 2 that μ satisfies (2.2). Again one obtains the same outer measure restricting the collection of generating sets to the tents with rational tip.

Example 4 (Capacity). We restrict our attention to a particular example of capacity; more examples can be found in the survey [1]. Let $X = \mathbb{R}^n$ with $n \geq 3$, and let \mathbf{E} be the collection of open sets in X . Define the kernel $K(x) := |x|^{2-n}$, which is a multiple of the classical Newtonian kernel. Let σ assign to each open set its capacity with respect to K , that is the least upper bound for the total mass $\|\nu\|$ of a positive Borel measure ν which has compact support in E and satisfies $\|\nu * K\|_\infty \leq 1$. Note that $\sigma(E) > 0$ for every nonempty open set E . This can be seen by testing with a measure ν associated with a smooth nonnegative density supported in a small compact ball contained in E .

To see property (2.2), assume E is some open set covered by a countable collection \mathbf{E}' of open sets. Let ν be a measure supported on a compact set $F \subset E$ such that $\|\nu * K\|_\infty \leq 1$. Then

$$\begin{aligned} \|\nu\| &\leq \sum_{E' \in \mathbf{E}'} \|\nu 1_{E'}\| \\ &\leq \sum_{E' \in \mathbf{E}'} \sup_{F \subset E'} \|\nu 1_F\| \\ &\leq \sum_{E' \in \mathbf{E}'} \sigma(E') \sup_{F \subset E'} \|(\nu 1_F) * K\|_\infty \\ &\leq \sum_{E' \in \mathbf{E}'} \sigma(E') \|\nu * K\|_\infty \leq \sum_{E' \in \mathbf{E}'} \sigma(E'). \end{aligned}$$

Since \mathbf{E}' was arbitrary, this proves (2.2).

2.3. Remarks on measurable sets. Outer measures are used in classical textbooks such as [20] as a stepping stone toward the introduction of measures. In measure theory, one is interested in equality in (2.1) under the additional assumption that the sets E_i are pairwise disjoint. Such equality does not follow in general from the properties of outer measure. A sufficient additional criterion is that each of the sets E_i is measurable, as in the following definition.

Definition 2.2 (Measurability). Let μ be an outer measure on a set X generated by a premeasure on a collection \mathbf{E} . An arbitrary subset F of X is called measurable if for every generating set $E \in \mathbf{E}$ we have

$$\mu(F \cap E) + \mu(F^c \cap E) = \mu(E).$$

We note that if F is measurable, then it also satisfies the Carathéodory criterion that for arbitrary subset G of X we have

$$\mu(F \cap G) + \mu(F^c \cap G) = \mu(G).$$

We briefly sketch the argument. If $\mu(G)$ is infinite, then it is easy to see that one of the outer measures on the left-hand side has to be infinite as well. If $\mu(G)$ is finite, pick $\epsilon > 0$ and a cover \mathbf{E}' of G by generating sets such that

$$\sum_{E \in \mathbf{E}'} \sigma(E) \leq \mu(G) + \epsilon.$$

Then we have

$$\begin{aligned} \mu(G) &\leq \mu(F \cap G) + \mu(F^c \cap G) \leq \sum_{E \in \mathbf{E}'} \mu(F \cap E) + \sum_{E \in \mathbf{E}'} \mu(F^c \cap E) \\ &\leq \sum_{E \in \mathbf{E}'} \mu(F \cap E) + \mu(F^c \cap E) = \sum_{E \in \mathbf{E}'} \mu(E) \leq \sum_{E \in \mathbf{E}'} \sigma(E) \leq \mu(G) + \epsilon. \end{aligned}$$

Since ϵ was arbitrary, it follows that the first inequality in this line of reasoning is indeed an equality.

In Example 1 above the measurable sets are called Lebesgue measurable. To see existence of many Lebesgue measurable sets, one observes that dyadic cubes are Lebesgue measurable. This follows from two observations: First, one may estimate the outer measure of F by coverings with cubes of side length at most that of the given cube E . Second, each such small cube is either contained in E or disjoint

from E allowing us to split the covering into two disjoint collections, of which one covers $F \cap E$ and the other covers $F \cap E^c$.

One can show in general that the collection of measurable sets is closed under countable union and countable intersection. Thus from Lebesgue measurability of dyadic cubes one can conclude Lebesgue measurability of all Borel sets in \mathbb{R}^m .

In contrast, no set other than \emptyset and X is measurable in Example 3. Assume we are given a nontrivial subset E of X , let (x_0, s_0) be a point in the boundary of E , and consider a tent $T(x, s)$ which contains (x_0, s_0) and satisfies $s < 2s_0$. Then we find points $(y, t) \in E \cap T(x, s)$ and $(y', t') \in E^c \cap T(x, s)$ in the vicinity of (x_0, s_0) such that $s < t + t'$. Then we have

$$\mu(T(x, s)) = \sigma(T(x, s)) = s < t + t' \leq \mu(T(x, s) \cap E) + \mu(T(x, s) \cap E^c),$$

where we used that if a set F contains a point (y, t) , then $\mu(F) > t$ because any cover of F needs to contain a tent with height at least t . The last display shows that the set E is not measurable.

In Example 4, it is well known that no bounded open set E is measurable. Namely, let E_1 and E_2 be disjoint bounded open sets such that $\text{dist}(E_1, E_2) > 0$ and set $E := E_1 \cup E_2$. Let ν be a positive Borel measure on a compact subset of E with $\|\nu * K\|_\infty \leq 1$. Since E is bounded, for some finite constant M that depends on the diameter of E and n , it holds that

$$\|\nu\| \leq M \inf_{x \in E} (\nu * K)(x) \leq M \|\nu * K\|_\infty \leq M.$$

In particular, it follows that $\sigma(E), \sigma(E_1), \sigma(E_2) < \infty$ (they are positive from a previous discussion). Then, by inner regularity of Borel measures, we obtain

$$\|\nu\| = \sum_{j=1}^2 \|\nu 1_{E_j}\| \leq \sum_{j=1}^2 \sigma(E_j) \|(\nu 1_{E_j}) * K\|_\infty.$$

Using the fact that σ satisfies the countably subadditive property (2.2), we obtain $\mu(E_j) = \sigma(E_j)$. Using harmonicity of $\nu 1_{E_j} * K$ in the interior of E_j^c , it follows that

$$\begin{aligned} \|\nu\| &\leq \sum_{j=1}^2 \mu(E_j) \|(\nu 1_{E_j}) * K\|_\infty \\ &\leq \sum_{j=1}^2 \mu(E_j) \|(\nu 1_{E_j}) * K\|_{L^\infty(\overline{E_j})} \\ &\leq \sum_{j=1}^2 \mu(E_j) (\|\nu * K\|_\infty - \inf_{x \in \overline{E_j}} ((\nu 1_{E_{3-j}}) * K)(x)) \\ &\leq \sum_{j=1}^2 \mu(E_j) - \sum_{j=1}^2 \mu(E_j) \inf_{x \in \overline{E_j}} ((\nu 1_{E_{3-j}}) * K)(x). \end{aligned}$$

Since E is bounded, it follows that for some finite positive constant M that depends on the diameter of E and n we have

$$\inf_{x \in \overline{E_j}} ((\nu 1_{E_{3-j}}) * K)(x) \geq \frac{1}{M} \|\nu 1_{E_{3-j}}\|.$$

It follows that

$$\begin{aligned} \|\nu\| &\leq \sum_{j=1}^2 \mu(E_j) - \sum_{j=1}^2 \mu(E_j) \frac{1}{M} \|\nu 1_{E_{3-j}}\| \\ &\leq \sum_{j=1}^2 \mu(E_j) - \frac{1}{M} \min(\mu(E_1), \mu(E_2)) \sum_{j=1}^2 \|\nu 1_{E_{3-j}}\| \\ &\leq \sum_{j=1}^2 \mu(E_j) - \frac{1}{M} \min(\mu(E_1), \mu(E_2)) \|\nu\|. \end{aligned}$$

Since $\mu(E_1) > 0$ and $\mu(E_2) > 0$, it follows that for some constant $c > 0$ that depends only on E_1, E_2, n it holds that

$$\|\nu\| \leq \frac{1}{1+c} \sum_{j=1}^2 \mu(E_j).$$

Taking the supremum over all such ν , it follows that $\mu(E) = \sigma(E) < \mu(E_1) + \mu(E_2)$, thus neither E_1 nor E_2 is measurable. While in Example 3 the lack of measurable sets is intuitively caused by the scarceness of the collection of generating sets, the collection \mathbf{E} in this example is very rich and can hardly be blamed for the shortage of measurable sets.

2.4. Functions and sizes. We propose an L^p theory for functions on outer measure spaces. One possible way of introducing an L^p norm of a nonnegative function f and $1 \leq p < \infty$ is via the following definition:

$$(2.3) \quad \left(\int_0^\infty p\lambda^{p-1} \mu(\{x \in X : f(x) > \lambda\}) d\lambda \right)^{1/p}.$$

In many instances, this is the correct definition. However, we propose a different formula, which in many examples such as Lebesgue theory coincides with the above but differs in full generality. The motivation for our definition is that it appears more useful in the applications that we have in mind.

Our different approach already finds a motivation in the efficiency of encoding of functions in classical Lebesgue theory. Classical coding describes functions as assignment of a value to every point in the space X . For an L^p function this assignment has to be consistent with the measurability structure. The set of such assignments has a very large cardinality, which is only reduced after consideration of equivalence classes of L^p functions. This detour over sets of large cardinality can be avoided by coding functions via their averages over dyadic cubes. There are only countably many such averages, and by the Lebesgue differentiation theorem these averages contain the complete information of the equivalence class of the L^p function.

Unlike in the above definition of the L^p norm, which regards the function f as a pointwise assignment, we propose to build the L^p theory on outer measure spaces via averages over generating sets. The theory then splits again into a concrete and an abstract part, parallel to the construction of outer measures by generating sets. There will be a concrete procedure to assign averages over generating sets to functions, and further on there will be an abstract procedure to define the L^p norms of functions from such averages. The concrete averaging procedure itself is based on some other measure theory (which by itself might be an outer measure

theory, but in the current paper we will not delve into such higher level iteration of the theory). We will consider this other measure theory as concrete external input into the outer measure theory, while the genuine part of the outer measure theory is the abstract passage from the concrete averages to outer L^p norms.

The class of functions from which we will be able to take L^p norms will depend on the concrete averaging procedure we choose. To avoid too abstract a setup we shall assume that X is a metric space and that every set of the collection \mathbf{E} is Borel. We shall assume the concrete averaging procedure will allow to average positive functions in the class $\mathcal{B}(X)$, the set of Borel measurable functions on X . If the set X is countable, a case that exhibits many of the essential ideas of the theory, the space $\mathcal{B}(X)$ is the space of all functions on X .

As linearity is closely related with measurability, in the absence of measurability we will not require averages to be linear but merely sublinear or even quasi-sublinear. We will call these averages “sizes”.

Definition 2.3 (Size). Let X be a metric space. Let σ be a function on a collection \mathbf{E} of Borel subsets of X , and let μ be the outer measure generated by σ . A size is a map

$$S : \mathcal{B}(X) \rightarrow [0, \infty]^{\mathbf{E}}$$

satisfying the following properties for every $f, g \in \mathcal{B}(X)$ and every $E \in \mathbf{E}$.

- (1) Monotonicity: if $|f| \leq |g|$, then $S(f)(E) \leq S(g)(E)$.
- (2) Scaling: $S(\lambda f)(E) = |\lambda|S(f)(E)$ for every $\lambda \in \mathbb{C}$.
- (3) Quasi-subadditivity:

$$(2.4) \quad S(f + g)(E) \leq C[S(f)(E) + S(g)(E)]$$

for some constant C depending only on S but not on f, g, E .

Note that (1) above implies $S(f)(E) = S(|f|)(E)$ for all f and E . Hence, our theory is essentially one of nonnegative functions, and the size needs initially be only defined for nonnegative Borel functions and can then be extended via the above identity to all functions.

We discuss sizes for Examples 1 through 4, and we give a number of forward looking remarks on particular aspects of the outer L^p theory to be developed.

In Lebesgue theory in Example 1, we define for every Borel function $f \in \mathcal{B}(X)$ and every cube Q

$$S(f)(Q) = \mu(Q)^{-1} \int_Q |f(x)| dx.$$

The integral is in the Lebesgue sense. Note the coincidence that the measure theory used to define the size is the same as the measure theory associated with the outer measure (X, μ) . This coincidence is a particular feature of Example 2 (and 2 below). The circularity of this setup does not invalidate our theory; certainly, Lebesgue measure can be introduced without reference to the outer integration theory that we develop in this paper.

Note that $S(f)(Q)$ is finite for every locally integrable function on \mathbb{R}^m . For such a function we may define the *martingale*

$$M(f)(Q) := \mu(Q)^{-1} \int_Q f(x) dx.$$

A consistency condition applies for $M(f)$, namely, the value of $M(f)$ on a dyadic cube is equal to the average of the values on the dyadic subcubes of half the side-length. By the dyadic Lebesgue differentiation theorem, the martingale uniquely determines the value of the function f at every Lebesgue point, and this uniquely determines the equivalence class of the measurable function f in the Lebesgue sense. As noted before, the martingale is a very efficient way of encoding the function f . The space $L^\infty(\mathbb{R}^m)$ can be described as all bounded maps from \mathbf{E} to \mathbb{C} which satisfy the consistency condition. This example is a strong indication that a useful general theory of outer measure may be built out of assigning values to elements $E \in \mathbf{E}$. Indeed, it would be possible in this example to build the theory entirely out of maps $M : \mathbf{E} \rightarrow \mathbb{C}$ satisfying the consistency condition, without reference to any Borel function f .

Turning to Example 2, we may similarly define

$$S(f)(B) = \mu(B)^{-1} \int_B |f(x)| dx$$

for every ball B . Again, these averages determine f by the Lebesgue differentiation theorem. In this case there does not exist an easy algebraic consistency condition that identifies maps from \mathbf{E} to \mathbb{C} that arise from locally integrable functions f as the average

$$M(f)(B) := \mu(B)^{-1} \int_B f(x) dx.$$

This provides the evidence that it is impracticable to build a theory of functions on outer measure space entirely out of maps from \mathbf{E} to \mathbb{C} and without reference to a function f .

In Example 3 we assign a value to each tent by averaging a Borel measurable function on the tent:

$$(2.5) \quad S(F)(T(x, s)) := s^{-1} \int_{T(x, s)} |F(y, t)| dy \frac{dt}{t}.$$

This averaging is based on weighted Lebesgue measure on X , which however is not the outer measure (X, μ) in Example 3. In the literature, one often works with the class of Borel measures ν on X rather than the class of Borel measurable functions, and one defines

$$S(\nu)(T(x, s)) := |s|^{-1} |\nu|(T(x, s)).$$

If the function $S(\nu)$ is bounded, the measure ν is called a Carleson measure in the literature, the concept of which dates back to the seminal paper [2]. The space of Carleson measures may be considered the space L^∞ on the outer measure space, as will be discussed more thoroughly further below.

A specific Carleson measure of interest is the following. For some function $f \in L^\infty(\mathbb{R})$ consider the function F on X defined by

$$F(y, t) = \int f(z) t^{-1} \phi(t^{-1}(y - z)) dz,$$

where ϕ is some smooth and rapidly decaying function of integral zero. Then $|F(y, t)|^2 dy \frac{dt}{t}$ turns out to be a Carleson measure.¹ The quadratic nature of this

¹For details see the special case $p = \infty$ of (4.3).

example suggests we define a size

$$S(F)(T(x, s)) = \left(s^{-1} \int_{T(x,s)} |F(y, t)|^2 dy \frac{dt}{t} \right)^{1/2}.$$

This example provides evidence for why we do not try to base a theory of outer measure on linear averaging, as could have been done in the example of martingales or the linear averaging over balls.

In Example 4, the most commonly (implicitly) used size is

$$S(f)(E) = \sup_{x \in E} |f(x)|.$$

Rather than the L^1 or L^2 based averages from the previous examples, this is an L^∞ based average. Such an L^∞ average has the effect that the more generally defined outer L^p norms we will introduce below specialize to the case of the integral (2.3), which is frequently referred to as the Choquet integral in the context of capacity theory. We conclude this very brief discussion of Example 4 with the remark that it may be interesting to compare the capacity strong type inequalities [1], whose intensive study goes back to the work of Maz'ya with the embedding theorems that we discuss further below.

2.5. A note on subadditivity. We have chosen to only demand quasi-subadditivity in the definition of size. Many sizes will be subadditive, which means that the constant in (2.4) can be chosen to be 1. The general constant in (2.4) allows for certain more general examples, for example L^p type sizes with $p < 1$. It also sets the stage for quasi-subadditivity throughout our discussion, which will simplify some of the arguments.

Note that L^p type sizes occur naturally in factorizations. Generalizing the classical factorization $|f| = |f|^\alpha |f|^{1-\alpha}$ for a Borel measurable function f and $0 < \alpha < 1$, one may consider modified sizes $S^{[\alpha]}$ defined, for every nonnegative function f , by

$$(2.6) \quad S^{[\alpha]}(f)(E) := \left[S(f^{\frac{1}{\alpha}})(E) \right]^\alpha.$$

One then has the factorization

$$S(f) = S^{[\alpha]}(f^\alpha) \times S^{[1-\alpha]}(f^{1-\alpha}).$$

Even if S is subadditive, the fractional size $S^{[\alpha]}$ with $0 < \alpha < 1$ might only be quasi-subadditive.

2.6. Essential supremum and super level measure. This section contains the most subtle points in the development of our L^p theory on outer measure spaces, with definitions carefully adjusted to the precise setup and the applications we have in mind. To develop an L^p theory, we need a space X , which we assume to be a metric space. We need a premeasure σ on a collection \mathbf{E} of Borel subsets, generating an outer measure μ on X . Finally, we need a size S . So as to not overburden the notation, we collect this data into a triple (X, σ, S) , because σ determines the generating collection and the outer measure. We use the letters \mathbf{E} and μ for these as standing convention. We call the triple (X, σ, S) an outer measure space.

Definition 2.4 (Outer essential supremum). Assume (X, σ, S) is an outer measure space. Given a Borel subset F of X , we define the outer essential supremum of $f \in \mathcal{B}(X)$ on F to be

$$\text{out sup}_F S(f) := \sup_{E \in \mathbf{E}} S(f1_F)(E).$$

We emphasize that the values $S(f)(E)$ for fixed f and all $E \in \mathbf{E}$ are, in general, not enough information to determine the essential supremum of f on a Borel set F other than X or \emptyset . It is important to refer back to the function f and to truncate it according to the set F .

We also emphasize that, unlike in Examples 1 and 2, the outer essential supremum in general does not coincide with the essential supremum of f on F in the Borel sense. In Example 3 with size given by (2.5), we note that every Lebesgue integrable Borel function supported above a line $t = t_0 > 0$ in the space X has finite outer essential supremum. Namely, the size of such a function with respect to some tent vanishes if the tent is small and is bounded above by t_0^{-2} times the Lebesgue integral of the function for arbitrary tent. On the other hand, if we define the size $S(f)(E)$ to be the supremum of f on the set E , then the outer essential supremum defined above coincides with the classical supremum on the set F , under the mild assumption that F can be covered by generating sets E .

The following properties of the outer essential supremum are inherited from the corresponding properties for the size. We have the following for every $f, g \in \mathcal{B}(X)$ and every Borel set $F \subset X$.

- (1) Monotonicity: if $|f| \leq |g|$, then $\text{out sup}_F S(f) \leq \text{out sup}_F S(g)$.
- (2) Scaling: for $\lambda \in \mathbb{C}$ we have $\text{out sup}_F S(\lambda f) = |\lambda| \text{out sup}_F S(f)$.
- (3) Quasi-subadditivity: for some constant $C < \infty$ independent of f, g, F , we have

$$\text{out sup}_F S(f + g) \leq C(\text{out sup}_F S(f) + \text{out sup}_F S(g)).$$

The use of the outer essential supremum is the main subtle point in the following definition.

Definition 2.5 (Super level measure). Let (X, σ, S) be an outer measure space. Let $f \in \mathcal{B}(X)$ and $\lambda > 0$. We define

$$(2.7) \quad \mu(S(f) > \lambda)$$

to be the infimum of all values $\mu(F)$, where F runs through all Borel subset of X which satisfy

$$\text{out sup}_{X \setminus F} S(f) \leq \lambda.$$

We emphasize once more that in general $\mu(S(f) > \lambda)$ is not the outer measure of the Borel set where $|f|$ is larger than λ , even though it is precisely that in many special examples such as the case of Lebesgue outer measure or in cases where the outer essential supremum above coincides with the classical supremum.

We obtain the following properties of super level measure.

- (1) Monotonicity: if $|f| \leq |g|$, then

$$\mu(S(f) > \lambda) \leq \mu(S(g) > \lambda).$$

- (2) Scaling: for a complex number λ' we have

$$\mu(S(\lambda' f) > |\lambda'| \lambda) = \mu(S(f) > \lambda).$$

(3) Quasi-subadditivity: for some constant $C < \infty$ independent of f, g, F ,

$$\mu(S(f + g) > C\lambda) \leq \mu(S(f) > \lambda) + \mu(S(g) > \lambda).$$

Note that a constant $C = 2$ would be necessary in general in the last inequality even if S was subadditive.

3. OUTER L^p SPACES

The definition of an outer L^p space and subsequent development of the theory of outer L^p spaces follows classical lines of reasoning, once the crucial definitions of the outer essential supremum and the super level measure from the previous section have replaced their classical counterparts. The only minor deviation comes in the proof of the triangle inequality, since we do not have a satisfactory theory of duality in outer L^p spaces. This manifests itself in a loss of a factor 2 in the triangle inequality.

Definition 3.1 (Outer L^∞). Let (X, σ, S) be an outer measure space. Let $f \in \mathcal{B}(X)$, then we define

$$\|f\|_{L^\infty(X, \sigma, S)} := \text{out sup}_X S(f) = \sup_{E \in \mathbf{E}} S(f)(E)$$

and $L^\infty(X, \sigma, S)$ to be the space of elements $f \in \mathcal{B}(X)$ for which $\sup_{E \in \mathbf{E}} S(f)(E)$ is finite. For notational convenience we define

$$L^{\infty, \infty}(X, \sigma, S) := L^\infty(X, \sigma, S).$$

As Example 3 of Carleson measures shows, $f \in L^\infty(X, \sigma, S)$ need not be an essentially bounded function on X in the Borel sense.

Definition 3.2 (Outer L^p). Let $0 < p < \infty$. Let (X, σ, S) be an outer measure space. We define for $f \in \mathcal{B}(X)$:

$$\|f\|_{L^p(X, \sigma, S)} := \left(\int_0^\infty p\lambda^{p-1} \mu(S(f) > \lambda) d\lambda \right)^{1/p},$$

$$\|f\|_{L^{p, \infty}(X, \sigma, S)} := \left(\sup_{\lambda > 0} \lambda^p \mu(S(f) > \lambda) \right)^{1/p}.$$

Moreover, we define $L^p(X, \sigma, S)$ and $L^{p, \infty}(X, \sigma, S)$ to be the spaces of elements in $\mathcal{B}(X)$ such that the respective quantities are finite.

Clearly $\mu(S(f) > \lambda)$ is monotone in λ , so that the integral in the definition of $\|f\|_{L^p(X, \sigma, S)}$ is well defined and a number in $[0, \infty]$. As in the classical case we trivially have

$$\|f\|_{L^{p, \infty}(X, \sigma, S)} \leq \|f\|_{L^p(X, \sigma, S)}.$$

The following properties hold, with elementary proofs that follow in most cases from the corresponding statements for super level measure.

Proposition 3.1 (Basic properties of outer L^p). *Let (X, σ, S) be an outer measure space, and let f, g be in $\mathcal{B}(X)$. Then we have the following for $0 < p \leq \infty$.*

- (1) *Monotonicity: if $|f| \leq |g|$, then $\|f\|_{L^p(X, \sigma, S)} \leq \|g\|_{L^p(X, \sigma, S)}$.*
- (2) *Scaling: $\|\lambda f\|_{L^p(X, \sigma, S)} = |\lambda| \|f\|_{L^p(X, \sigma, S)}$ for any $\lambda \in \mathbb{C}$.*
- (3) *Quasi-subadditivity: there is a constant C independent of f, g such that*

$$\|f + g\|_{L^p(X, \sigma, S)} \leq C(\|f\|_{L^p(X, \sigma, S)} + \|g\|_{L^p(X, \sigma, S)}).$$

Moreover, we have for $\lambda > 0$

$$\|f\|_{L^p(X, \lambda\sigma, S)} = \lambda^{1/p} \|f\|_{L^p(X, \sigma, S)}.$$

Corresponding statements hold for the spaces $L^{p,\infty}(X, \sigma, S)$.

Note that the proof of quasi-subadditivity for L^p with $p < \infty$ is based on quasi-subadditivity of super level measure, which yields a constant C different from 1 even if the size S is subadditive. It might be interesting to study conditions under which one may have subadditivity for L^p .

We turn to the behavior of outer L^p spaces under mappings between outer measure spaces. Note that Borel measurable functions as well as classical L^p functions are typically pulled back under a continuous map, while in contrast Borel measures are pushed forward under such maps. This is one of the motivations for us to use the class of Borel measurable functions to develop the theory of outer L^p functions, even though much of the theory can be developed for Borel measures as well.

Let X_1 and X_2 be two metric spaces, and let $\Phi : X_1 \rightarrow X_2$ be a continuous map. For $j = 1, 2$, let \mathbf{E}_j be a collection of Borel sets covering X_j , and let $\sigma_j : \mathbf{E}_j \rightarrow [0, \infty]$ be a function generating an outer measure μ_j on X_j . Let S_1 and S_2 be sizes turning (X_1, σ_1, S_1) and (X_2, σ_2, S_2) into outer measure spaces.

Proposition 3.2 (Pullback). *Assume that for every $E_2 \in \mathbf{E}_2$, we have*

$$(3.1) \quad \mu_1(\Phi^{-1}E_2) \leq A\mu_2(E_2).$$

Further, assume that for each $E_1 \in \mathbf{E}_1$, there exists $E_2 \in \mathbf{E}_2$ such that for every $f \in \mathcal{B}(X_2)$ we have

$$(3.2) \quad S_1(f \circ \Phi)(E_1) \leq BS_2(f)(E_2).$$

Then we have for every $f \in \mathcal{B}(X_2)$ and $0 < p \leq \infty$ and some universal constant C :

$$\begin{aligned} \|f \circ \Phi\|_{L^p(X_1, \sigma_1, S_1)} &\leq A^{1/p} BC \|f\|_{L^p(X_2, \sigma_2, S_2)}, \\ \|f \circ \Phi\|_{L^{p,\infty}(X_1, \sigma_1, S_1)} &\leq A^{1/p} BC \|f\|_{L^{p,\infty}(X_2, \sigma_2, S_2)}. \end{aligned}$$

Proof. First note that by scaling properties it is not a restriction to prove the proposition with constants $A = B = 1$ in (3.1) and (3.2).

For every Borel set $F_2 \subset X_2$ we have

$$\mu_1(\Phi^{-1}F_2) \leq \mu_2(F_2).$$

Namely, given F_2 , without loss of generality, we may assume that $\mu_2(F_2) < \infty$. Let $\mathbf{E}'_2 \subset \mathbf{E}_2$ be a cover of F_2 which attains, up to a factor $(1 + \epsilon)$ with small $\epsilon > 0$, the outer measure of F_2 :

$$\sum_{E_2 \in \mathbf{E}'_2} \mu_2(E_2) \leq \sum_{E_2 \in \mathbf{E}'_2} \sigma_2(E_2) \leq (1 + \epsilon)\mu_2(F_2).$$

Then we obtain

$$\mu_1(\Phi^{-1}(F_2)) \leq \sum_{E_2 \in \mathbf{E}'_2} \mu_1(\Phi^{-1}(E_2)) \leq \sum_{E_2 \in \mathbf{E}'_2} \mu_2(E_2) \leq (1 + \epsilon)\mu_2(F_2).$$

This proves the claim, since $\epsilon > 0$ can be chosen arbitrarily.

Assume $F \subset X_2$ is a Borel set such that

$$\mu_2(F) \leq (1 + \epsilon)\mu_2(S_2(f) > \lambda),$$

and for every $E_2 \in \mathbf{E}_2$ we have $S_2(f1_{F^c})(E_2) \leq \lambda$. Pick $E_1 \in \mathbf{E}_1$, then there exists $E_2 \in \mathbf{E}_2$ such that we have

$$S_1((f \circ \Phi)(1_{\Phi^{-1}(F^c)}))(E_1) = S_1((f1_{F^c}) \circ \Phi)(E_1) \leq S_2(f1_{F^c})(E_2) \leq \lambda,$$

and hence

$$\begin{aligned} \mu_1(S_1(f \circ \Phi) \geq \lambda) &\leq \mu_1((\Phi^{-1}(F^c))^c) \\ &\leq \mu_1(\Phi^{-1}F) \leq \mu_2(F) \leq (1 + \epsilon)\mu_2(S_2(f) > \lambda). \end{aligned}$$

This proves the desired inequalities for $p < \infty$. The case $p = \infty$ follows immediately from the assumption on sizes. \square

Proposition 3.3 (Logarithmic convexity). *Let (X, σ, S) be an outer measure space, and let $f \in \mathcal{B}(X)$. Assume $\alpha_1 + \alpha_2 = 1$, $0 < \alpha_1, \alpha_2 < 1$, and*

$$1/p = \alpha_1/p_1 + \alpha_2/p_2$$

for $p_1, p_2 \in (0, \infty]$ with $p_1 \neq p_2$. Then

$$(3.3) \quad \|f\|_{L^p(X, \sigma, S)} \leq C_{p, p_1, p_2} \left(\|f\|_{L^{p_1, \infty}(X, \sigma, S)} \right)^{\alpha_1} \left(\|f\|_{L^{p_2, \infty}(X, \sigma, S)} \right)^{\alpha_2}.$$

Proof. Assume without loss of generality $p_1 < p_2$. We first consider the case $p_2 < \infty$. If either of the norms on the right-hand side of (3.3) vanishes, then $\mu(S(f) > \lambda)$ vanishes for all $\lambda > 0$ and then the left-hand side of (3.3) vanishes as well. By scaling, we may then assume

$$A := \|f\|_{L^{p_1, \infty}(X, \sigma, S)}^{p_1} = \|f\|_{L^{p_2, \infty}(X, \sigma, S)}^{p_2}.$$

Optimizing the use of these two identities, we have with $p_1 < p < p_2$

$$\begin{aligned} \mu(S(f) > \lambda) &\leq A \min(\lambda^{-p_2}, \lambda^{-p_1}), \\ \|f\|_p &\leq (Ap \left(\int_0^1 \lambda^{p-p_1-1} d\lambda + \int_1^\infty \lambda^{p-p_2-1} d\lambda \right))^{\frac{1}{p}} \\ &\leq C_{p, p_1, p_2} A^{1/p} = C_{p, p_1, p_2} \left(\|f\|_{L^{p_1, \infty}(X, \sigma, S)}^{p_1} \right)^{\frac{\alpha_1}{p_1}} \left(\|f\|_{L^{p_2, \infty}(X, \sigma, S)}^{p_2} \right)^{\frac{\alpha_2}{p_2}}. \end{aligned}$$

This completes the proof in case $p_2 < \infty$. If $p_2 = \infty$, we may assume by scaling that $\|f\|_{L^\infty(X, \sigma, S)} = 1$. Then for $\lambda > 1$ we have $\mu(S(f) > \lambda) = 0$. Consequently,

$$\|f\|_{L^p(X, \sigma, S)} \leq (p \|f\|_{L^{p_1, \infty}(X, \sigma, S)}^{p_1} \int_0^1 \lambda^{p-p_1-1} d\lambda)^{\frac{1}{p}} \leq C_{p, p_1, p_2} \|f\|_{L^{p_1, \infty}(X, \sigma, S)}^{\alpha_1}. \quad \square$$

Proposition 3.4 (Hölder’s inequality). *Assume we have a metric space X , three collections $\mathbf{E}, \mathbf{E}_1, \mathbf{E}_2$ of Borel subsets, and three functions $\sigma, \sigma_1, \sigma_2$ on these collections generating outer measures μ, μ_1, μ_2 on X . Assume $\mu \leq \mu_j$ for $j = 1, 2$. Assume S, S_1, S_2 are three respective sizes such that for any $E \in \mathbf{E}$ there exist $E_1 \in \mathbf{E}_1$ and $E_2 \in \mathbf{E}_2$ such that for all $f_1, f_2 \in \mathcal{B}(X)$ we have*

$$(3.4) \quad S(f_1 f_2)(E) \leq S_1(f_1)(E_1) S_2(f_2)(E_2).$$

Let $p, p_1, p_2 \in (0, \infty]$ such that $1/p = 1/p_1 + 1/p_2$. Then

$$(3.5) \quad \|f_1 f_2\|_{L^p(X, \sigma, S)} \leq 2 \|f_1\|_{L^{p_1}(X, \sigma_1, S_1)} \|f_2\|_{L^{p_2}(X, \sigma_2, S_2)}.$$

Proof. We assume $0 < p_1, p_2 < \infty$. The case $\max(p_1, p_2) = \infty$ can be argued similarly. Without loss of generality assume that the factors on the right-hand side of (3.5) are finite. For $j = 1, 2$, pick Borel sets $F_j \subset X$ such that for every $E_j \in \mathbf{E}_j$ we have

$$S_j(f_j 1_{F_j^c})(E_j) \leq \lambda^{p/p_j}$$

and

$$\mu_j(F_j) \leq \mu_j(S_j(f_j) > \lambda^{p/p_j}) + \epsilon.$$

Define $F = F_1 \cup F_2$. Let $E \in \mathbf{E}$ be arbitrary. Then by (3.4) there exists $E_1 \in \mathbf{E}_1$ and $E_2 \in \mathbf{E}_2$ such that

$$\begin{aligned} S(f_1 f_2 1_{F^c})(E) &\leq S_1(f_1 1_{F_1^c})(E_1) S_2(f_2 1_{F_2^c})(E_2) \\ &\leq S_1(f_1 1_{F_1^c})(E_1) S_2(f_2 1_{F_2^c})(E_2) \leq \lambda^{p/p_1} \lambda^{p/p_2} = \lambda, \end{aligned}$$

with passage from the first to second line by monotonicity of the sizes.

It follows from subadditivity of μ and domination of μ by μ_1 and μ_2 that for all $\lambda > 0$

$$\begin{aligned} \mu(S(f_1 f_2) > \lambda) &\leq \mu(F) \leq \mu(F_1) + \mu(F_2) \\ (3.6) \qquad &\leq \mu_1(F_1) + \mu_2(F_2) \leq 2\epsilon + \sum_{i=1}^2 \mu(S_i(f_i) > \lambda^{p/p_i}). \end{aligned}$$

To prove (3.5) we may assume via scaling that

$$\|f_1\|_{L^{p_1}(X, \sigma_1, S_1)} = \|f_2\|_{L^{p_2}(X, \sigma_2, S_2)} = 1.$$

Then (3.5) follows from (3.6), using that $\epsilon > 0$ is arbitrarily small,

$$\begin{aligned} \int p \lambda^{p-1} \mu(S(f_1 f_2) > \lambda) d\lambda &\leq \int p \lambda^{p-1} \sum_{i=1}^2 \mu(S_i(f_i) > \lambda^{p/p_i}) d\lambda \\ &= \sum_{i=1}^2 \int p_i \lambda^{p_i-1} \mu(S_i(f_i) > \lambda) d\lambda = 2. \qquad \square \end{aligned}$$

In the following proposition, let $L^p(Y, \nu)$ denote the classical space of complex valued functions on a measure space (Y, ν) such that $\|f\|_{L^p(Y, \nu)} := (\int_Y |f(x)|^p d\nu)^{1/p}$ is finite.

The following proposition is an outer measure version of classical Marcinkiewicz interpolation, which in practice is used to obtain strong bounds in a range of exponents p from weak bounds at the endpoints of the range.

Proposition 3.5 (Marcinkiewicz interpolation). *Let (X, σ, S) be an outer measure space. Assume $1 \leq p_1 < p_2 \leq \infty$. Let T be an operator that maps $L^{p_1}(Y, \nu)$ and $L^{p_2}(Y, \nu)$ to the space of Borel functions on X , such that for any $f, g \in L^{p_1}(Y, \nu) + L^{p_2}(Y, \nu)$ and $\lambda \geq 0$ we have the following.*

- (1) *Scaling:* $|T(\lambda f)| = |\lambda T(f)|$.
- (2) *Quasi-subadditivity:* $|T(f + g)| \leq C(|T(f)| + |T(g)|)$.
- (3) *Boundedness properties:*

$$\|T(f)\|_{L^{p_1, \infty}(X, \sigma, S)} \leq A_1 \|f\|_{L^{p_1}(Y, \nu)},$$

$$\|T(f)\|_{L^{p_2, \infty}(X, \sigma, S)} \leq A_2 \|f\|_{L^{p_2}(Y, \nu)}.$$

Then we also have

$$\|T(f)\|_{L^p(X,\sigma,S)} \leq A_1^{\theta_1} A_2^{\theta_2} C_{p_1,p_2,p} \|f\|_{L^p(Y,\nu)},$$

where $p_1 < p < p_2$ and θ_1, θ_2 are such that

$$\theta_1 + \theta_2 = 1,$$

$$\frac{1}{p} = \frac{\theta_1}{p_1} + \frac{\theta_2}{p_2}.$$

Proof. We may normalize ν to become $\tilde{\nu} = \lambda^{-1}\nu$, with λ chosen so that

$$A_1\lambda^{1/p_1} = A_2\lambda^{1/p_2} := A.$$

Then

$$A_1^{\theta_1} A_2^{\theta_2} \lambda^{1/p} = A\lambda^{-\theta_1/p_1 - \theta_2/p_2} \lambda^{1/p} = A.$$

Thus it suffices to prove the theorem with $A_1 = A_2 = A$. Further normalizing T to become $\tilde{T} = A^{-1}T$, we observe that it suffices to prove the theorem with $A_1 = A_2 = 1$.

If $f_1 \in L^{p_1}(Y, \nu)$ and $f_2 \in L^{p_2}(Y, \nu)$, then we have for every $E \in \mathbf{E}$

$$S(T(f_1 + f_2))(E) \leq C \left(S(Tf_1)(E) + S(Tf_2)(E) \right).$$

Then we also have for some possibly different constant C

$$(3.7) \quad \mu(S(T(f_1 + f_2)) > C\lambda) \leq \mu(S(Tf_1) > \lambda) + \mu(S(Tf_2) > \lambda).$$

We first assume $0 < p_1 < p_2 < \infty$. Let $f \in L^p(Y, \nu)$. We decompose $f = f_{1,\lambda} + f_{2,\lambda}$ with $f_{1,\lambda} = f1_{|f|>\lambda}$. It is clear that $f_{j,\lambda} \in L^{p_j}(Y, \nu)$. Using (3.7) we obtain

$$\begin{aligned} \mu(S(Tf) > C\lambda) &\leq C \sum_{j=1}^2 \lambda^{-p_j} \|f_{j,\lambda}\|_{L^{p_j}}^{p_j} \\ &= C\lambda^{-p_1} \int_Y |f|^{p_1} 1_{|f|>\lambda} d\nu(y) + C\lambda^{-p_2} \int_Y |f|^{p_2} 1_{|f|\leq\lambda} d\nu(y), \end{aligned}$$

and therefore

$$\begin{aligned} \|Tf\|_{L^p(X,\sigma,S)} &= \left(p \int_0^\infty \lambda^{p-1} \mu(S(Tf) > \lambda) d\lambda \right)^{1/p} \\ &\leq C \left(\int_Y |f|^{p_1} \left(\int_0^{|f|} \lambda^{p-p_1-1} d\lambda \right) d\nu + \int_Y |f|^{p_2} \left(\int_{|f|}^\infty \lambda^{p-p_2-1} d\lambda \right) d\nu \right)^{1/p} \\ &\leq C \|f\|_{L^p(Y,\nu)}. \end{aligned}$$

It remains to consider the case $p_1 < p_2 = \infty$. We similarly decompose $f = f_{1,\lambda} + f_{2,\lambda}$ with $f_{1,\lambda} = f1_{|f|>c\lambda}$ for suitable small c to be determined momentarily. Then

$$\|Tf_{2,\lambda}\|_{L^\infty(X,\sigma,S)} \leq \|f_{2,\lambda}\|_{L^\infty(Y,\nu)} < c\lambda.$$

It follows from (3.7) that with sufficiently small c

$$\mu(S(Tf) > \lambda) \leq \mu(S(Tf_{1,\lambda}) > \lambda/C) + \mu(S(Tf_{2,\lambda}) > \lambda/C) = \mu(S(Tf_{1,\lambda}) > \lambda/C).$$

Consequently,

$$\begin{aligned} \|Tf\|_{L^p(X,\sigma,S)} &\leq C \left(\int_0^\infty \lambda^{p-1} \mu(S(Tf) > \lambda) d\lambda \right)^{1/p} \\ &\leq C \left(\int_0^\infty \lambda^{p-1} \mu(S(Tf_{1,\lambda}) > \lambda/C) d\lambda \right)^{1/p}. \end{aligned}$$

Then we proceed as before to obtain

$$\|Tf\|_{L^p(X,\sigma,S)} \leq C \left(\int_Y |f|^{p_1} \left(\int_{|f|}^\infty \lambda^{p-p_1-1} d\lambda \right) d\nu \right)^{1/p} \leq C \|f\|_{L^p(Y,\nu)}. \quad \square$$

The following is a simple variant of a classical fact about measures: If a measure ν on a space is absolutely continuous with respect to another measure μ , and if the Radon–Nikodym derivative of ν with respect to μ is bounded, then the total mass of ν can be estimated by the total mass of μ .

Proposition 3.6. *Assume (X, σ, S) is an outer measure space, and assume that about every point in X there is an open ball for which there exists $E \in \mathbf{E}$ which contains the ball. Let ν be a positive Borel measure on X . Assume that for every $f \in \mathcal{B}(X)$ and for every $E \in \mathbf{E}$ we have*

$$\int_E |f| d\nu \leq CS(f)(E)\sigma(E).$$

Then, for every $f \in \mathcal{B}(X)$ with finite $\|f\|_{L^\infty(X,\sigma,S)}$, we have

$$\left| \int_X f d\nu \right| \leq C \|f\|_{L^1(X,\sigma,S)},$$

where the implicit constant C in particular is independent of $\|f\|_{L^\infty(X,\sigma,S)}$.

Proof. We may assume that $\mu(S(f) > \lambda)$ is finite for every $\lambda > 0$, or there is nothing else to prove. For each $k \in \mathbb{Z}$ consider a set F_k such that

$$\text{out sup}_{F_k^c} S(f) \leq 2^k,$$

$$\mu(F_k) \leq 2\mu(S(f) > 2^k).$$

Cover F_k by a countable subcollection \mathbf{E}_k of \mathbf{E} such that

$$\sum_{E \in \mathbf{E}_k} \sigma(E) \leq 2\mu(F_k).$$

Let $F = \bigcup_k F_k$, and note that for every sufficiently small open ball B about a point in X we can find $E \in \mathbf{E}$ such that $B \subset E$, thus

$$\int_{B \cap F^c} |f| 1_{F^c} d\nu \leq CS(f 1_{F^c})(E)\sigma(E) = 0.$$

Hence

$$\int_X |f| d\nu = \int_F |f| d\nu.$$

Since we may assume $F_k = \emptyset$ for sufficiently large k , we have

$$\begin{aligned} \left| \int_X f \, d\nu \right| &\leq \sum_k \int_{F_k \setminus \bigcup_{l>k} F_l} |f| \, d\nu \leq \sum_k \sum_{E \in \mathbf{E}_k} \int_{E \setminus \bigcup_{l>k} F_l} |f| \, d\nu \\ &\leq \sum_k \sum_{E \in \mathbf{E}_k} S(f1_{F_{k+1}^c})\mu(E) \leq C \sum_k \sum_{E \in \mathbf{E}_k} 2^k \sigma(E) \\ &\leq C \sum_k 2^k \mu(S(f) > 2^k) \leq C \|f\|_{L^1(X, \sigma, S)}. \end{aligned}$$

This completes the proof of the proposition. □

4. CARLESON EMBEDDING, PARAPRODUCTS, AND THE $T(1)$ THEOREM

This section contains classical results rephrased in the language of outer measure spaces utilizing Example 3 of Section 2.2. Readers interested in reviewing the classical theory are referred to [17]. A novelty of our approach is the interpretation of Carleson embedding theorems as boundedness of certain maps from a classical L^p to an outer L^p space. As a consequence, outer Hölder’s inequality can be used to prove various multilinear estimates such as paraproduct estimates or a core version of a $T(1)$ theorem [5].

4.1. Carleson embeddings. We consider the upper half-plane $X = \mathbb{R} \times (0, \infty)$, we let \mathbf{E} be the collection of tents (see Figure 1)

$$T(x, s) = \{(y, t) \in \mathbb{R} \times (0, \infty) : t < s, |x - y| < s - t\},$$

and we set $\sigma(T(x, s)) = s$ as in Example 3.

Define for $1 \leq p < \infty$ the sizes

$$S_p(F)(T(x, s)) := \left(s^{-1} \int_{T(x, s)} |F(y, t)|^p \, dy \frac{dt}{t} \right)^{1/p},$$

where we have used standard Lebesgue integration in $\mathbb{R} \times (0, \infty)$, and

$$S_\infty(F)(T(x, s)) := \sup_{(y, t) \in T(x, s)} |F(y, t)|.$$

Let ϕ be a smooth function on the real line supported in $[-1, 1]$, and define for a locally integrable function f on the real line

$$(4.1) \quad F_\phi(f)(y, t) := \int f(x) t^{-1} \overline{\phi(t^{-1}(y - x))} \, dx.$$

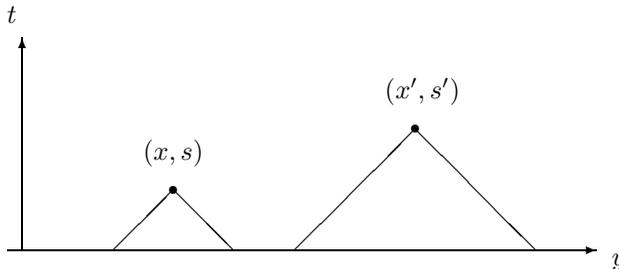


FIGURE 1. The tents $T(x, s)$ and $T(x', s')$.

The mapping $f \rightarrow F_\phi(f)$ is an embedding of a space of functions on the real line into a space of functions in the upper half-plane reminiscent of *Carleson embeddings*. Thus we call the following estimates *Carleson embedding theorems*, even though traditionally this notion is reserved for special instances and applications of such estimates. In particular, if ν is a Borel measure on the upper half-plane satisfying the so-called Carleson measure condition $\nu(T(x, s)) \leq Ms$, then one could deduce from Theorem 4.1 a typical version of the classical Carleson embedding theorem, as follows. Below the first and last L^p norm are classical Lebesgue norms, while the second and third L^p norms are outer L^p norms over an outer measure generated by ν and σ and the tent collection.

$$\begin{aligned} \|F_\phi(f)\|_{L^p(X,\nu)} &\leq \|F_\phi\|_{L^p(X,\nu,S_\infty)} \\ &\leq M\|F_\phi\|_{L^p(X,\sigma,S_\infty)} \\ &\leq C_{p,\phi}M\|f\|_p. \end{aligned}$$

Theorem 4.1. *Let $1 < p \leq \infty$. We have for ϕ as above*

$$(4.2) \quad \|F_\phi(f)\|_{L^p(X,\sigma,S_\infty)} \leq C_{p,\phi}\|f\|_p.$$

If in addition $\int \phi = 0$, then

$$(4.3) \quad \|F_\phi(f)\|_{L^p(X,\sigma,S_2)} \leq C_{p,\phi}\|f\|_p.$$

Proof. We first prove estimate (4.2). The estimate will follow by Marcinkiewicz interpolation (in Proposition 3.5) between weak endpoint bounds at $p = \infty$ and $p = 1$. Clearly, we have for all $(y, t) \in X$,

$$|F_\phi(f)(y, t)| \leq \|f\|_\infty\|\phi\|_1.$$

Hence,

$$S_\infty(F_\phi)(T(x, s)) \leq \|f\|_\infty\|\phi\|_1$$

for every tent $T(x, s)$, and this implies the L^∞ bound. To prove the weak type estimate at L^1 , fix f and $\lambda > 0$. Consider the set $\Omega \subset \mathbb{R}$ where the Hardy–Littlewood maximal function Mf of f is larger than $c_\phi\lambda$ for some constant c_ϕ that depends on ϕ and is specified later. The set Ω is open and thus the disjoint union of at most countably many open intervals $(x_i - s_i, x_i + s_i)$ for $i = 1, 2, \dots$. Let E be the union of the tents $T(x_i, s_i)$. Then the geometry of tents implies that for $(x, s) \notin E$ none of the intervals $(x_i - s_i, x_i + s_i)$ may contain the interval $(x - s, x + s)$ and hence there is a point $y \in (x - s, x + s)$ such that $Mf(y) \leq c_\phi\lambda$. Then we see from a standard estimate of ϕ by a superposition of characteristic functions of intervals of length at least $2s$,

$$F_\phi(f)(x, s) \leq C_\phi Mf(y) \leq \lambda,$$

the latter by appropriate choice of c_ϕ . Hence

$$\text{out sup}_{E^c} S_\infty(F_\phi) \leq \lambda.$$

On the other hand, by the Hardy–Littlewood maximal theorem,

$$\mu(E) \leq \sum_i s_i \leq |\{x : Mf(x) \geq c_\phi\lambda\}| \leq C_\phi\|f\|_1\lambda^{-1}.$$

This proves the weak type estimate at L^1 and completes the proof of estimate (4.2).

We turn to estimate (4.3), which is proven similarly by Marcinkiewicz interpolation between weak endpoint bounds at ∞ and 1. Note first that if ϕ has integral

zero, then the map F_ϕ goes under the name of *continuous wavelet transform* and is well known to be a multiple of an isometry in the following sense:

$$\int_0^\infty \int_{\mathbb{R}} |F_\phi(g)(y, t)|^2 dy \frac{dt}{t} = C_\phi \|g\|_2^2$$

for every $g \in L^2(\mathbb{R})$. This fact goes under the name of Calderón’s reproducing formula or Calderón’s resolution of the identity; see for example [6]. It can be proven by a calculation similar to our reduction of Theorem 6.1 to Lemma 6.2 below.

Consider a tent $T(x, s)$. For (y, t) in the tent, we see from compact support of ϕ that

$$F_\phi(f)(y, t) = F_\phi(f1_{[x-3s, x+3s]})(y, t).$$

Applying Calderón’s reproducing formula with $g = f1_{[x-3s, x+3s]}$ gives

$$\int \int_{T(x, s)} |F_\phi(f)(y, t)|^2 dy \frac{dt}{t} \leq C_\phi \|f1_{[x-3s, x+3s]}\|_2^2 \leq C_\phi s \|f\|_\infty^2.$$

Dividing by s gives

$$S_2(F_\phi(f))(T(x, s)) \leq C_\phi \|f\|_\infty,$$

which proves the desired estimate for $p = \infty$.

To prove the weak type bound at $p = 1$, fix $f \in L^1(\mathbb{R})$ and $\lambda > 0$ and consider again the set $\Omega = \{x : Mf(x) > c_\phi \lambda\}$, which is the disjoint union of open intervals $(x_i - s_i, x_i + s_i)$. Consider the Calderón–Zygmund decomposition of f at level $c_\phi \lambda$,

$$f = g + \sum_i b_i,$$

which is uniquely determined by the demand that for each i the function b_i is supported on $[x_i - s_i, x_i + s_i]$, and has integral zero, while g is constant on this interval. As a consequence, g is bounded by $c_\phi \lambda$, and we have by the previous argument for any tent

$$S_2(F_\phi(g))(T(x, s)) \leq \lambda/2.$$

Let E be the union of tents $T(x_i, 3s_i)$. Let $b = \sum_i b_i$. It remains to show that, with small choice of c_ϕ , for every $(x, s) \in \mathbb{R} \times (0, \infty)$ it holds that

$$S_2(F_\phi(b)1_{E^c})(T(x, s)) \leq \lambda/2.$$

Let B_i denote the compactly supported primitive of b_i . Then we have for $(y, t) \notin E$, using compact support of ϕ ,

$$\begin{aligned} F_\phi(b)(y, t) &= \int b(x)t^{-1}\overline{\phi(t^{-1}(y-x))} dy \\ &= \int \sum_{i:s_i \leq t} b_i(x)t^{-1}\overline{\phi(t^{-1}(y-x))} dx \\ &= \int \sum_{i:s_i \leq t} B_i(x)t^{-2}\overline{\phi'(t^{-1}(y-x))} dx. \end{aligned}$$

Hence

$$|F_\phi(b)(y, t)| \leq \left\| \sum_{i:s_i \leq t} t^{-1}B_i \right\|_\infty \|\phi'\|_1.$$

We claim that the L^∞ norm on the right-hand side is bounded by $4c_\phi \lambda$. Since the B_i are disjointly supported, it suffices to see $\|t^{-1}B_i\|_\infty \leq 4c_\phi \lambda$ for each i with

$s_i \leq t$. However, this follows from $\|b_i\|_1 \leq 4c_\phi \lambda s_i$, which is a standard estimate for the Calderón–Zygmund decomposition. Hence,

$$S_\infty(F_\phi(b)1_{E^c})(T(s, x)) \leq 4c_\phi \lambda.$$

To obtain a bound for S_2 in place of S_∞ , we use log convexity of S_p and a bound on S_1 . Let $T(x, s)$ be a tent, and let b_i be one summand of the bad function. Then we have from considerations of the support of b_i and ϕ ,

$$\begin{aligned} & \int_{(y,t) \in T(x,s) \setminus E} \left| \int_{\mathbb{R}} b_i(z) t^{-1} \overline{\phi(t^{-1}(y-z))} dz \right| dy \frac{dt}{t} \\ & \leq \int_{t \geq s_i} \int_{|y-x_i| \leq 2t} \left| \int_{\mathbb{R}} b_i(z) t^{-1} \overline{\phi(t^{-1}(y-z))} dz \right| dy \frac{dt}{t}. \end{aligned}$$

Using partial integration, we estimate this by

$$\begin{aligned} & \int_{t \geq s_i} \int_{|y-x_i| \leq 2t} \|B_i\|_1 \|\phi'\|_\infty dy \frac{dt}{t^3} \\ & \leq C_\phi \int_{t \geq s_i} \|B_i\|_1 \frac{dt}{t^2} \leq C_\phi \|B_i\|_1 s_i^{-1} \leq C_\phi \|b_i\|_1 \leq \lambda s_i/6. \end{aligned}$$

Adding over the disjointly supported b_i inside $(x - 3s, x + 3s)$, which are all the summands of the bad function possibly contributing to $F_\phi(b)$ on $T(x, s)$, gives

$$S_1(F_\phi(b)1_{E^c})(T(x, s)) \leq \lambda/2.$$

By log convexity we then obtain

$$S_2(F_\phi(b)1_{E^c})(T(x, s)) \leq \lambda/2.$$

Together with the previously established bound for the good function, we obtain by the triangle inequality

$$S_2(F_\phi(f)1_{E^c})(T(x, s)) \leq \lambda,$$

and hence

$$\text{out sup}_{E^c} S_2(F_\phi(f)) \leq \lambda.$$

On the other hand, we have by the Hardy–Littlewood maximal theorem as before

$$\mu(E) \leq C_\phi \|f\|_1 \lambda^{-1}.$$

This completes the proof of the weak type 1 endpoint bound for estimate (4.3) and thus the proof of Theorem 4.1. □

We will need to apply Theorem 4.1 in a slightly modified setting.

For two parameters $-1 \leq \alpha \leq 1$ and $0 < \beta \leq 1$, define

$$F_{\alpha,\beta,\phi}(f)(y, t) := F_\phi(f)(y + \alpha t, \beta t).$$

To estimate the outer L^p norm of $F_{\alpha,\beta,\phi}(f)$, first note that by a simple change of variables

$$s^{-1} \int_{T(x,s)} |F_{\alpha,\beta,\phi}(f)(y, t)|^2 dy \frac{dt}{t} = s^{-1} \int_{T_{\alpha,\beta}(x,s)} |F_\phi(f)(y, t)|^2 dy \frac{dt}{t},$$

where we have defined the modified tent $T_{\alpha,\beta}(x, s)$ to be the set of all points (z, u) such that $(z - \alpha\beta^{-1}u, \beta^{-1}u) \in T(x, s)$. This modified tent is a tilted triangle, it has height βs above the real line and width $2s$ near the real line. The tip of the tilted tent is the point $(x + \alpha s, \beta s)$, which is contained in a rectangle with base

$[x - s, x + s]$ and height s above the x -axis. We construct an outer measure space using the collection of modified tents by setting

$$\sigma_{\alpha,\beta}(T_{\alpha,\beta}(x, s)) := s.$$

We then define for a Borel measurable function G on X

$$S_{\alpha,\beta,2}(G)(T_{\alpha,\beta}(x, s)) := (s^{-1} \int_{T_{\alpha,\beta}(x,s)} |G(y, t)|^2 dy \frac{dt}{t})^{1/2}.$$

We have by transport of structure

$$\|F_{\alpha,\beta,\phi}(f)\|_{L^p(X,\sigma,S_2)} = \|F_\phi(f)\|_{L^p(X,\sigma_{\alpha,\beta},S_{\alpha,\beta,2})}.$$

Given a standard tent $T(x, s)$, we may cover it by a modified tent $T_{\alpha,\beta}(x', s')$ of width $2s' = 4\beta^{-1}s$. Hence

$$\mu_{\alpha,\beta}(T(x, s)) \leq C\beta^{-1}\mu(T(x, s)).$$

Moreover, a modified tent $T_{\alpha,\beta}(x, s)$ is contained in a standard tent $T(x', s')$ of width $2s' = 4s$. Hence

$$S_{\alpha,\beta,2}(G)(T_{\alpha,\beta}(x, s)) \leq CS_2(G)(T(x', s')).$$

Thus Proposition 3.2 applied to the identity map on X gives

$$\|F_\phi(f)\|_{L^p(X,\sigma_{\alpha,\beta},S_{\alpha,\beta,2})} \leq C\beta^{-1/p}\|F_\phi(f)\|_{L^p(X,\sigma,S_2)}.$$

We have thus proven the following

Corollary 4.2. *Assume the setup as above. Let $1 < p \leq \infty$ and $-1 \leq \alpha \leq 1$ and $0 < \beta \leq 1$, and assume $\int \phi = 0$. Then*

$$\|F_{\alpha,\beta,\phi}(f)\|_{L^p(X,\sigma,S_2)} \leq C_{p,\phi}\beta^{-1/p}\|f\|_p.$$

We shall need a slightly better dependence on the parameter β in the last corollary. This is stated in the following lemma, where explicit values for ϵ are not difficult to obtain but unimportant for our purpose.

Lemma 4.3. *Assume the setup as above. Let $1 < p \leq \infty$ and $-1 \leq \alpha \leq 1$ and $0 < \beta \leq 1$, and assume $\int \phi = 0$. Then there exists $\epsilon > 0$ such that we have*

$$\|F_{\alpha,\beta,\phi}(f)\|_{L^p(X,\sigma,S_2)} \leq C_{p,\phi}\beta^{-1/p+\epsilon}\|f\|_p.$$

Proof. This lemma follows by various applications of Marcinkiewicz interpolation using the bounds of Corollary 4.2 and an improved weak type 2 bound,

$$\|F_{\alpha,\beta,\phi}(f)\|_{L^{2,\infty}(X,\sigma,S_2)} = \|F_\phi(f)\|_{L^{2,\infty}(X,\sigma_{\alpha,\beta},S_{\alpha,\beta,2})} \leq C_\phi\|f\|_2,$$

where the right-hand side does not depend on β . To see this bound, fix $f \in L^2(\mathbb{R})$ and $\lambda > 0$. Consider the collection \mathbf{I} of all open intervals $(x - s, x + s)$ on the real line such that

$$s^{-1} \int_{T_{\alpha,\beta}(x,s)} |F_\phi(f)(y, t)|^2 dy \frac{dt}{t} > \lambda^2.$$

The union $\bigcup_{I \in \mathbf{I}} I$ is an open set which can be written as the disjoint union of countably many open intervals $(x_i - s_i, x_i + s_i)$. If we set $E = \bigcup_i T_{\alpha,\beta}(x_i, s_i)$, then it is clear that

$$S_{\alpha,\beta,2}(F_\phi(f)1_{E^c})(T_{\alpha,\beta}(x, s)) \leq \lambda$$

for each (x, s) . Hence, it suffices to show

$$(4.4) \quad \sum_i s_i \leq C\lambda^{-2}\|f\|_2^2.$$

We first show that if $\mathbf{I}_1 \subset \mathbf{I}$ is a collection of disjoint intervals, then

$$\sum_{I_1 \in \mathbf{I}_1} |I_1| \leq C\lambda^{-2}\|f\|_2^2.$$

It is clear that such \mathbf{I}_1 has to be countable. Enumerate the intervals in \mathbf{I}_1 as $(x'_1 - s'_1, x'_1 + s'_1), (x'_2 - s'_2, x'_2 + s'_2), \text{etc.}$ Then we have by choice of the collection \mathbf{I}

$$\sum_i s'_i \leq \sum_i \lambda^{-2} \int_{T_{\alpha,\beta}(x'_i, s'_i)} |F_\phi(f)(y, t)|^2 dy \frac{dt}{t}.$$

However, the tents $T_{\alpha,\beta}(x'_i, s'_i)$ are pairwise disjoint, and hence

$$\sum_i s'_i \leq \lambda^{-2} \int_0^\infty \int_{\mathbb{R}} |F_\phi(f)(y, t)|^2 dy \frac{dt}{t}.$$

By Calderón’s reproducing formula, the latter is bounded by

$$C_\phi \lambda^{-2} \|f\|_2^2.$$

The above estimate shows in particular that $\sup_{I \in \mathbf{I}} |I|$ is finite, and we may select any $I_1 \in \mathbf{I}$ such that $|I_1| > \frac{1}{2} \sup_{I \in \mathbf{I}} |I|$. Let \mathbf{I}' be the collection of intervals in \mathbf{I} that does not intersect (or contain) I_1 . Then select any $I_2 \in \mathbf{I}'$ such that its length is more than half of $\sup_{I \in \mathbf{I}'} |I|$. Iterating this argument, we obtain a sequence I_1, I_2, \dots of disjoint intervals in \mathbf{I} . We claim that

$$\bigcup_{I \in \mathbf{I}} I \subset \bigcup_j 5I_j,$$

where, for any $m > 0$, we define mI_j to be the interval of length $m|I_j|$ with the same center as I_j . Certainly this claim will imply (4.4).

Suppose, towards a contradiction, that there exists $I \in \mathbf{I}$ such that $I \not\subset \bigcup_j 5I_j$. We first claim that I intersects one of the intervals I_1, I_2, \dots . Indeed, since $|I_j| \rightarrow 0$ as $j \rightarrow \infty$, there exists $j \geq 1$ such that $|I| > 2|I_{j+1}|$, which means I was not available for selection after step j , i.e., I has to intersect one of the intervals I_1, \dots, I_j . Now, let $k \geq 1$ be the smallest index such that $I \cap I_k \neq \emptyset$. It follows that I is available for selection after step $k - 1$, and hence $|I_k| \geq \frac{1}{2}|I|$; therefore,

$$I \subset 5I_k,$$

which contradicts the above assumption.

This proves the lemma. □

4.2. Paraproducts and the $T(1)$ theorem. A classical paraproduct is a bilinear operator, which after pairing with a third function becomes a trilinear form that is essentially of the type

$$\Lambda(f_1, f_2, f_3) = \int_{\mathbb{R} \times (0, \infty)} \prod_{j=1}^3 F_{\phi_j}(f_j)(x, t) dx \frac{dt}{t}$$

with three compactly supported smooth functions ϕ_1, ϕ_2, ϕ_3 of which two have vanishing integral while the third does not necessarily have vanishing integral. By symmetry we assume ϕ_1 and ϕ_2 to have vanishing integral. Paraproducts also

appear in different forms in the literature, for example discretized versions of the above integral or versions involving only two embedding maps F_i . In the latter case the third embedding can typically be inserted after using some manipulations on the integral expression.

Assuming f_j are bounded, and thus $F_{\phi_j}(f_j)$ are bounded as well, we obtain by an application of Proposition 3.6 the estimate

$$|\Lambda(f_1, f_2, f_3)| \leq C \prod_{j=1}^3 \|F_{\phi_j}(f_j)\|_{L^1(X, \sigma, S_1)}.$$

By the classical Hölder inequality, we may estimate S_1 by the product of S_2 and S_2 and S_∞ ; therefore, by the outer Hölder inequality, namely Proposition 3.4, we obtain

$$|\Lambda(f_1, f_2, f_3)| \leq C \|F_{\phi_1}(f_1)\|_{L^{p_1}(X, \sigma, S_2)} \|F_{\phi_2}(f_2)\|_{L^{p_2}(X, \sigma, S_2)} \|F_{\phi_3}(f_3)\|_{L^{p_3}(X, \sigma, S_\infty)}$$

for exponents $1 < p_1, p_2, p_3 \leq \infty$. By applying the Carleson embedding theorems, we obtain

$$|\Lambda(f_1, f_2, f_3)| \leq C \|f_1\|_{p_1} \|f_2\|_{p_2} \|f_3\|_{p_3},$$

which reproduces classical paraproduct estimates. Note that the last estimate does not depend on the L^∞ bounds on f_j , and thus easily extends to unbounded functions. With well-known and not too laborous changes in the above arguments, one can also reproduce classical BMO bounds in place of $p_1 = \infty$ or $p_2 = \infty$.

We now state a simplified version of the classical $T(1)$ theorem originating in [5].

Theorem 4.4 ($T(1)$ theorem). *Let ϕ be some nonzero smooth function supported in $[-1, 1]$ with $\int \phi = 0$, and define for $x \in \mathbb{R}$ and $s \in (0, \infty)$*

$$\phi_{x,s}(y) = s^{-1} \phi(s^{-1}(y - x)).$$

Assume T is a bounded linear operator in $L^2(\mathbb{R})$ such that for all x, y, s, t ,

$$(4.5) \quad |\langle T(\phi_{x,s}), \phi_{y,t} \rangle| \leq \frac{\min(t, s)}{\max(t, s, |y - x|)^2}.$$

Then we have for the operator norm of T the bound

$$\|T\|_{L^2 \rightarrow L^2} \leq C$$

for some constant C depending only on ϕ and in particular not on T . Moreover, for $1 < p < \infty$,

$$\|Tf\|_p \leq C_p \|f\|_p$$

for some constant C_p depending only on ϕ and p .

To compare this with more classical formulations of the $T(1)$ theorem, the assumption (4.5) is typically deduced from Calderón–Zygmund kernel estimates if $|x - y| > t + s$, and thus the two test functions $\phi_{x,s}$ and $\phi_{y,t}$ are disjointly supported. It is deduced from one of the assumptions $T(1) = 0$ and $T^*(1) = 0$ and a weak boundedness assumption whether s or t is within a factor of 2 of the maximum of s , t , and $|y - x|$, and thus the two test functions are close. The assumptions $T(1) = 0$ and $T^*(1) = 0$ can be obtained from more general assumptions $T(1) \in \text{BMO}$ and $T^*(1) \in \text{BMO}$ by subtracting paraproducts from T first. A detailed exposition of the $T(1)$ theorem can be found in [17].

Proof. We note from Calderón’s reproducing formula that

$$f = C \int_0^\infty \int_{\mathbb{R}} F(x, s) \phi_{x,s} dx \frac{ds}{s}$$

with a weakly absolutely convergent integral in L^2 and $F = F_\phi(f)$ as defined in (4.1). Thus we may write with the analogous notation $G = F_\phi(g)$

$$\langle T(f), g \rangle = \int_0^\infty \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}} F(x, s) \langle T(\phi_{x,s}), \phi_{y,t} \rangle \overline{G(y, t)} dx dy \frac{ds}{s} \frac{dt}{t}.$$

Here we implicitly used boundedness of T and the Schwarz kernel theorem to move T inside the integral representation of f . Note that we have again expressed the form $\langle T(f), g \rangle$ in terms of the functions F and G on the outer space X , which leads toward the use of embedding theorems. However, we cannot apply Hölder’s inequality directly, but we first have to suitably express the double integral over the space X as superposition of single integrals over X .

Set

$$r := \max(s, t, |y - x|).$$

We split the domain of integration into the two regions $r > |x - y|$ and $r = |x - y|$ and estimate the two integrals separately. Splitting the first region further into two symmetric regions (overlapping in a set of measure zero), we may restrict our attention to the region $s = r$. We estimate the integral over this region by

$$\begin{aligned} & \left| \int_0^\infty \int_{\mathbb{R}} \int_0^s \int_{x-s}^{x+s} F(x, s) \langle T(\phi_{x,s}), \phi_{y,t} \rangle \overline{G(y, t)} dy \frac{dt}{t} dx \frac{ds}{s} \right| \\ & \leq C \int_0^\infty \int_{\mathbb{R}} \int_0^s \int_{x-s}^{x+s} |F(x, s) \overline{G(y, t)}| dy dt dx \frac{ds}{s^3} \\ & = C \int_0^1 \int_{-1}^1 \int_0^\infty \int_{\mathbb{R}} |F(x, s) \overline{G(x + \alpha s, \beta s)}| dx \frac{ds}{s} d\alpha d\beta. \end{aligned}$$

In the last line we have changed variables setting $y - x = \alpha s$ and $t = \beta s$. Setting $G_{\alpha,\beta}(x, s) = G(x + \alpha s, \beta s)$, we estimate the last display, using Propositions 3.6 and outer Hölder’s inequality with dual exponents $1 < p, p' < \infty$, Proposition 3.4,

$$\begin{aligned} & \leq C \int_0^1 \int_{-1}^1 \|FG_{\alpha,\beta}\|_{L^1(X, \sigma, S_1)} d\alpha d\beta \\ & \leq C \int_0^1 \int_{-1}^1 \|F\|_{L^p(X, \sigma, S_2)} \|G_{\alpha,\beta}\|_{L^{p'}(X, \sigma, S_2)} d\alpha d\beta. \end{aligned}$$

The norm of F can be estimated by Theorem 4.1, while the norm of G can be estimated by Lemma 4.3. Hence, we can estimate the last display by

$$\leq C \int_0^1 \int_{-1}^1 \beta^{-1/p'} \|f\|_p \|g\|_{p'} d\alpha d\beta \leq C \|f\|_p \|g\|_{p'}.$$

The region $r = |y - x|$ we also split into symmetric regions, first restricting to $r = y - x$ and $r = x - y$. By symmetry, it suffices to estimate the region $r = y - x$. We now split further into $t \leq s$ and $s \leq t$.

For the subregion $t \leq s$, we obtain the estimate

$$\begin{aligned} & \left| \int_0^\infty \int_{\mathbb{R}} \int_0^r \int_0^s F(x, s) \langle T(\phi_{x,s}), \phi_{x+r,t} \rangle \overline{G(x+r, t)} \frac{dt}{t} \frac{ds}{s} dx dr \right| \\ & \leq \int_0^\infty \int_{\mathbb{R}} \int_0^r \int_0^s |F(x, s) \overline{G(x+r, t)}| dt \frac{ds}{s} dx \frac{dr}{r^2} \\ & = \int_0^1 \int_0^\alpha \int_0^\infty \int_{\mathbb{R}} |F(x, \alpha r) \overline{G(x+r, \beta r)}| dx \frac{dr}{r} d\beta \frac{d\alpha}{\alpha} \\ & = \int_0^1 \int_0^\alpha \int_0^\infty \int_{\mathbb{R}} |F_{0,\alpha}(x, r) \overline{G_{1,\beta}(x, r)}| dx \frac{dr}{r} d\beta \frac{d\alpha}{\alpha}, \end{aligned}$$

where we have used the notation $F_{0,\alpha}$ and $G_{1,\beta}$ as above. We use Lemma 4.3 twice to estimate the last display by

$$\leq C \int_0^1 \int_0^\alpha \alpha^{\epsilon-1/p} \beta^{\epsilon-1/p'} \|f\|_p \|g\|_{p'} d\beta \frac{d\alpha}{\alpha} \leq C \|f\|_p \|g\|_{p'}.$$

The subregion $s \leq t$ could be estimated similarly. This concludes the proof of the L^2 and L^p estimate of Theorem 4.4. □

We conclude this section by pointing at an alternative approach to Calderón–Zygmund operators used in Lerner’s work [13], who essentially controls Calderón–Zygmund operators by a superposition of “sparse” operators. These sparse operators lend themselves to an application of an outer Hölder inequality with spaces $L^\infty(X, \sigma, S_1) \times L^p(X, \sigma, S_\infty) \times L^{p'}(X, \sigma, S_\infty)$ in lieu of the above $L^p(X, \sigma, S_2) \times L^{p'}(X, \sigma, S_2)$ or implicit $L^\infty(X, \sigma, S_\infty) \times L^p(X, \sigma, S_2) \times L^{p'}(X, \sigma, S_2)$.

5. GENERALIZED TENTS AND CARLESON EMBEDDING

In this section we introduce a new outer measure space whose underlying set is the upper three-space. The extra dimension relative to the classical tent spaces is a frequency parameter, which arises due to modulation symmetries in problems of time-frequency analysis. In contrast, the upper half-plane merely represents dilation and translation symmetries. The generalized Carleson embedding theorem below is new, though its proof is an adaption of standard recipes in time-frequency analysis. The novelty lies in the concise formulation of an essential part of time-frequency analysis and in the absence of any discretization in the formulation of Theorem 5.1.

This section is the most technical one of the present paper, as it is devoted to a proof of Theorem 5.1 and its discrete variant, Theorem 5.3. We point out that the application of Theorem 5.1 to the bilinear Hilbert transform discussed in the final section can be understood without detailed reading of the proof in the present section.

Let X be the space $\mathbb{R} \times \mathbb{R} \times (0, \infty)$ with the usual metric as subspace of \mathbb{R}^3 . Let $0 < |\alpha| \leq 1$ and $|\beta| \leq 0.9$ be two real parameters and define for a point (x, ξ, s) in X the generalized tent

$$(5.1) \quad T_{\alpha,\beta}(x, \xi, s) := \{(y, \eta, t) \in X : t \leq s, |y - x| \leq s - t, |\alpha(\eta - \xi) + \beta t^{-1}| \leq t^{-1}\}.$$

For a first understanding the reader may focus on the example $\alpha = 1$ and $\beta = 0$. In this case the condition on the frequency variable η becomes $-t^{-1} \leq \eta - \xi \leq t^{-1}$ which is symmetric around ξ , as can be seen below. The general case with other

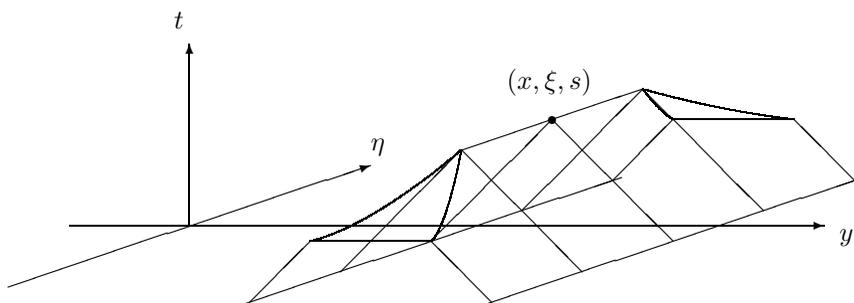


FIGURE 2. The generalized tent $T_{1,0}(x, \xi, s)$

(α, β) leads to a condition $At^{-1} \leq \eta - \xi \leq Bt^{-1}$ for some $A < 0 < B$ depending on α, β , and will correspond to an asymmetric variant of Figure 2.

The projection of the generalized tent onto the first two variables is a classical tent as in Example 3. We are only concerned with generalized tents in this section and will omit the adjective “generalized” when referring to $T_{\alpha,\beta}(x, \xi, s)$. The collection \mathbf{E} of all tents generates an outer measure if we set

$$\sigma(T_{\alpha,\beta}(x, \xi, s)) = s.$$

By a similar argument as in Example 3, σ satisfies (2.2), and hence the outer measure μ is an extension of the function σ on \mathbf{E} .

To define a size on Borel functions on X , we use further auxiliary tents

$$(5.2) \quad T^b(x, \xi, s) := \{(y, \eta, t) \in X : t \leq s, |y - x| \leq s - t, |\eta - \xi| \leq bt^{-1}\}.$$

For $0 < b < 1$ and a Borel measurable function F on X , we define

$$(5.3) \quad \begin{aligned} S^b(F)(T_{\alpha,\beta}(x, \xi, s)) &:= (s^{-1} \int_{T_{\alpha,\beta}(x, \xi, s) \setminus T^b(x, \xi, s)} |F(y, \eta, t)|^2 dy d\eta dt)^{1/2} \\ &\quad + \sup_{(y, \eta, t) \in T_{\alpha,\beta}(x, \xi, s)} |F(y, \eta, t)|. \end{aligned}$$

One easily checks that this size satisfies the properties required in Definition 2.3. The size S^b increases as b decreases.

The following is a version of a Carleson embedding theorem in the setting of generalized tents. We normalize the Fourier transform of a Schwartz function ϕ on the real line as

$$\widehat{\phi}(\xi) = \int_{\mathbb{R}} e^{-i\xi x} \phi(x) dx.$$

Theorem 5.1 (Generalized Carleson embedding). *Let $0 < |\alpha| \leq 1$ and $|\beta| \leq 0.9$. Let $0 < b \leq 2^{-8}$. Let ϕ be a Schwartz function with Fourier transform $\widehat{\phi}$ supported in $(-2^{-8}b, 2^{-8}b)$, and let $2 \leq p \leq \infty$. Define for $f \in L^p(\mathbb{R})$ the function F on X by*

$$F(y, \eta, t) := \int_{\mathbb{R}} f(x) e^{in(y-x)} t^{-1} \phi(t^{-1}(y-x)) dx.$$

There is some constant C depending only on $\alpha, \beta, b, \phi,$ and $p,$ such that if $p > 2,$

$$\|F\|_{\mathbf{L}^p(X, \sigma, S^b)} \leq C\|f\|_p,$$

and if $p = 2,$

$$\|F\|_{\mathbf{L}^{2, \infty}(X, \sigma, S^b)} \leq C\|f\|_2.$$

By symmetry there is no restriction to assume $0 < \alpha,$ and we shall do so.

The dependence of the constant C on the function $\phi,$ conditioned on the fixed support condition on $\widehat{\phi},$ factors as dependence on the constant

$$\sup_x [|\phi(x)|(1 + |x|)^3 + |\phi'(x)|(1 + |x|)^2].$$

We do not claim that this explicit regularity of ϕ is sharp for the above theorem to hold.

From now on we fix the parameters α and $\beta,$ and for simplicity of notation we write T for $T_{\alpha, \beta}.$

It is convenient to work with a discrete variant of Theorem 5.1. Fix the parameter $0 < b \leq 2^{-8}.$ We introduce the discrete subset X_Δ of points $(x, \xi, s) \in X$ such that there exist integers $k, n, l \in \mathbb{Z}$ with

$$x = 2^{k-4}n, \quad \xi = 2^{-k-8}bl, \quad s = 2^k.$$

We denote by \mathbf{E}_Δ the collection of all tents $T(x, \xi, s)$ with $(x, \xi, s) \in X_\Delta.$ This is a discrete subcollection of $\mathbf{E}.$ However, each tent in \mathbf{E}_Δ by itself still forms a continuum in $X.$

We generate an outer measure μ_Δ using \mathbf{E}_Δ as generating collection, setting as before $\sigma_\Delta(T(x, \xi, s)) = s$ for each tent in $\mathbf{E}_\Delta.$

The following lemma will be used to relate this new measure to the previous one.

Lemma 5.2. *If $(x', \xi', s') \in X,$ then there exists a $(x, \xi, s) \in X_\Delta$ such that the tent $T(x, \xi, s)$ contains (x', ξ', s') “centrally” in the sense*

$$\begin{aligned} 2^{-3}s &< s' \leq 2^{-2}s, \\ |x' - x| &\leq 2^{-4}s, \\ |\xi' - \xi| &\leq 2^{-8}bs^{-1}. \end{aligned}$$

Moreover, there exist two points, $(x, \xi_-, s) \in X_\Delta$ and $(x, \xi_+, s) \in X_\Delta,$ so that the corresponding tents contain (x', ξ', s') centrally and satisfy

$$\begin{aligned} T(x', \xi', s') &\subset T(x, \xi_-, s) \cup T(x, \xi_+, s), \\ T(x', \xi', s') \cap T^b(x, \xi_-, s) \cap T^b(x, \xi_+, s) &\subset T^b(x', \xi', s'). \end{aligned}$$

Proof. The interval $[2^2s', 2^3s')$ contains a unique point of the form 2^k with $k \in \mathbb{Z}.$ We set $s = 2^k.$ Then there is a point x of the form $2^{k-4}n$ with some $n \in \mathbb{Z}$ such that $|x - x'| \leq 2^{k-4} = 2^{-4}s.$ Likewise, there is a point ξ of the form $2^{-k-8}bl$ with $l \in \mathbb{Z}$ such that $|\xi - \xi'| \leq 2^{-k-8}b = 2^{-8}bs^{-1}.$

Informally, this point ξ may be chosen on either side of $\xi'.$ Precisely, we may choose $\xi_- \leq \xi'$ and $\xi_+ \geq \xi'$ with $|\xi_- - \xi'| \leq 2^{-8}bs^{-1}$ and $|\xi_+ - \xi'| \leq 2^{-8}bs^{-1}.$ If $(y, \eta, t) \in T(x', \xi', s'),$ then we have

$$\begin{aligned} t &\leq s' \leq 2^{-2}s, \\ |y - x| &\leq |y - x'| + |x' - x| \leq s' - t + 2^{-4}s \leq s - t. \end{aligned}$$

If in addition $\alpha(\eta - \xi') + \beta t^{-1} \geq 0$, then (recall that $\alpha > 0$)

$$-t^{-1} \leq \alpha(\xi' - \xi_+) \leq \alpha(\eta - \xi_+) + \beta t^{-1} \leq \alpha(\eta - \xi') + \beta t^{-1} \leq t^{-1},$$

while if in addition $\alpha(\eta - \xi') + \beta t^{-1} \leq 0$, then

$$-t^{-1} \leq \alpha(\eta - \xi') + \beta t^{-1} \leq \alpha(\eta - \xi_-) + \beta t^{-1} \leq \alpha(\xi' - \xi_-) \leq t^{-1}.$$

Hence $(y, \eta, t) \in T(x, \xi_-, s) \cup T(x, \xi_+, s)$. Now, in addition, let (y, η, t) be an element of $T^b(x, \xi_-, s) \cap T^b(x, \xi_+, s)$. If $\eta \geq \xi'$, then

$$-bt^{-1} < 0 \leq \eta - \xi' \leq \eta - \xi_- \leq bt^{-1},$$

while if $\eta \leq \xi'$, then

$$-bt^{-1} \leq \eta - \xi_+ \leq \eta - \xi' \leq 0 < bt^{-1}.$$

Hence, $(y, \eta, t) \in T^b(x', \xi', s')$. This completes the proof of the lemma. □

As a consequence of this lemma, if T is a tent in \mathbf{E} , then we find two tents T^+ , T^- in \mathbf{E}_Δ such that

$$T \subset T^+ \cup T^-,$$

$$\sigma_\Delta(T^+) + \sigma_\Delta(T^-) \leq C\sigma(T).$$

This implies that for every subset $X' \subset X$

$$\mu(X') \leq \mu_\Delta(X') \leq C\mu(X').$$

Hence, the outer measures μ and μ_Δ are equivalent.

Moreover, we have for the same tents and every Borel function F ,

$$S^b(F)(T) \leq C[S_\Delta^b(F)(T^+) + S_\Delta^b(F)(T^-)],$$

where we have defined

$$S_\Delta^b(F)(T') := S^b(F)(T')$$

for any tent T' in \mathbf{E}_Δ .

This implies that for every $1 \leq p \leq \infty$

$$C^{-1}\mathbf{L}^p(X, \sigma, S^b) \leq \mathbf{L}^p(X, \sigma_\Delta, S_\Delta^b) \leq \mathbf{L}^p(X, \sigma, S^b),$$

$$C^{-1}\mathbf{L}^{p,\infty}(X, \sigma, S^b) \leq \mathbf{L}^{p,\infty}(X, \sigma_\Delta, S_\Delta^b) \leq \mathbf{L}^{p,\infty}(X, \sigma, S^b).$$

Hence Theorem 5.1 is equivalent to the following discrete version.

Theorem 5.3 (Generalized Carleson embedding, discrete version). *Let $0 < \alpha \leq 1$ and $-0.9 \leq \beta \leq 0.9$. Let $0 < b \leq 2^{-8}$. Let ϕ be a Schwartz function with Fourier transform $\widehat{\phi}$ supported in the interval $(-2^{-8}b, 2^{-8}b)$, and let $2 \leq p \leq \infty$. Define for $f \in L^p(\mathbb{R})$ the function F on X by*

$$F(y, \eta, t) := \int_{\mathbb{R}} f(x)e^{i\eta(y-x)}t^{-1}\phi(t^{-1}(y-x)) dx.$$

There is some constant C depending only on α, β, b, ϕ , and p , such that if $p \neq 2$,

$$\|F\|_{\mathbf{L}^p(X, \sigma_\Delta, S_\Delta^b)} \leq C\|f\|_p,$$

and if $p = 2$,

$$\|F\|_{\mathbf{L}^{2,\infty}(X, \sigma_\Delta, S_\Delta^b)} \leq C\|f\|_2.$$

Proof of Theorems 5.1 and 5.3. Since both theorems are equivalent, we will only prove the discrete version, Theorem 5.3. Hence, we will only work with the discrete quantities μ_Δ and S_Δ^b , and for simplicity of notation, we omit the subscript Δ . Since b is fixed, we also denote $S := S^b$.

The theorem follows by Marcinkiewicz interpolation (Proposition 3.5) between the endpoint cases $p = 2$ and $p = \infty$.

5.1. The endpoint $p = \infty$. We need to prove that for every $(x, \xi, s) \in X_\Delta$ and every $f \in L^\infty(\mathbb{R})$, we have

$$S(F)(T(x, \xi, s)) \leq C\|f\|_\infty.$$

The size S is defined as a sum of an L^2 portion and an L^∞ portion. It suffices to estimate both portions separately. Note that for all y, η, t we trivially have $|F(y, \eta, t)| \leq \|f\|_\infty \|\phi\|_1$, and this establishes the desired bound on the L^∞ portion of S .

To estimate the L^2 portion of the size we first establish the estimate

$$(5.4) \quad \int_{T(x, \xi, s) \setminus T^b(x, \xi, s)} |F(y, \eta, t)|^2 dy d\eta dt \leq C\|f\|_2^2$$

for every function $f \in L^2(\mathbb{R})$. Fix such a function f , and we may assume by normalization that $\|f\|_2 = 1$. Replacing the domain of integration by a larger region we can estimate the left-hand side of (5.4) by

$$\int_0^\infty \int_{\mathbb{R}} \int_{bt^{-1} \leq |\eta - \xi| \leq 2\alpha^{-1}t^{-1}} |F(y, \eta, t)|^2 d\eta dy dt.$$

It suffices to estimate the integral over the region where $\eta > \xi$, since by symmetry there is an analogous estimate for the integral over region $\eta < \xi$. We replace the integration variable η by γ such that $\eta - \xi = \gamma t^{-1}$. Using Fubini, we are reduced to estimating

$$\int_b^{2\alpha^{-1}} \int_0^\infty \int_{\mathbb{R}} |F(y, \xi + \gamma t^{-1}, t)|^2 dy \frac{dt}{t} d\gamma.$$

We first estimate the inner double integral for fixed γ .

Define for each y, γ, t the bump function $\phi_{y, \gamma, t}$ by

$$\phi_{y, \gamma, t}(x) = e^{-i(\xi + \gamma t^{-1})(y-x)} t^{-1} \overline{\phi(t^{-1}(y-x))}.$$

We are interested in the region $\gamma \geq b$, where the modulated function $\phi_{y, \gamma, t} e^{-i\xi \cdot}$ has integral zero by support consideration of $\widehat{\phi}$. Hence its primitive is absolutely integrable with good bounds, which we will use later when applying partial integration.

We have

$$\begin{aligned} & \left(\int_0^\infty \int_{\mathbb{R}} |\langle f, \phi_{y, \gamma, t} \rangle|^2 dy \frac{dt}{t} \right)^2 \\ & \leq \left\| \int_0^\infty \int_{\mathbb{R}} \langle f, \phi_{y, \gamma, t} \rangle \phi_{y, \gamma, t} dy \frac{dt}{t} \right\|_2^2 \\ & \leq \int_0^\infty \int_{\mathbb{R}} \int_0^\infty \int_{\mathbb{R}} |\langle f, \phi_{y, \gamma, t} \rangle \langle \phi_{y, \gamma, t}, \phi_{z, \gamma, r} \rangle \langle \phi_{z, \gamma, r}, f \rangle| dz \frac{dr}{r} dy \frac{dt}{t}. \end{aligned}$$

Estimating the smaller of the inner products with f by the larger one and using symmetry, we may estimate this by

$$(5.5) \quad \leq 2 \int_0^\infty \int_{\mathbb{R}} |\langle f, \phi_{y,\gamma,t} \rangle|^2 \left[\int_0^\infty \int_{\mathbb{R}} |\langle \phi_{y,\gamma,t}, \phi_{z,\gamma,r} \rangle| dz \frac{dr}{r} \right] dy \frac{dt}{t}.$$

We consider the inner double integral of (5.5). Considering first the region $t \leq r$ and doing partial integration in the inner product

$$\langle \phi_{y,\gamma,t}, \phi_{z,\gamma,r} \rangle = \langle \phi_{y,\gamma,t} e^{-i\xi \cdot}, \phi_{z,\gamma,r} e^{-i\xi \cdot} \rangle.$$

Integrating the first and differentiating the second bump function, we estimate the integral over this region by

$$\begin{aligned} C \int_{\mathbb{R}} \int_t^\infty \int_{\mathbb{R}} (1 + |t^{-1}(y-x)|)^{-2} r^{-2} (1 + |r^{-1}(z-x)|)^{-2} dx \frac{dr}{r} dz \\ \leq C \int_{\mathbb{R}} \int_t^\infty (1 + |t^{-1}(y-x)|)^{-2} r^{-2} dr dx \\ \leq C \int_{\mathbb{R}} t^{-1} (1 + |t^{-1}(y-x)|)^{-2} dx \leq C. \end{aligned}$$

In the region $t \geq r$, we do partial integration in reverse, differentiating the first and integrating the second bump function, to obtain the estimate for the integral over this region by

$$\begin{aligned} C \int_{\mathbb{R}} \int_0^t \int_{\mathbb{R}} t^{-2} (1 + |t^{-1}(y-x)|)^{-2} (1 + |r^{-1}(z-x)|)^{-2} dx \frac{dr}{r} dz \\ \leq C \int_{\mathbb{R}} \int_0^t t^{-2} (1 + |t^{-1}(y-x)|)^{-2} dr dx \\ \leq C \int_{\mathbb{R}} t^{-1} (1 + |t^{-1}(y-x)|)^{-2} dx \leq C. \end{aligned}$$

Inserting these two estimates into (5.5) gives

$$\left(\int_0^\infty \int_{\mathbb{R}} |\langle f, \phi_{y,\gamma,t} \rangle|^2 dy \frac{dt}{t} \right)^2 \leq C \int_0^\infty \int_{\mathbb{R}} |\langle f, \phi_{y,\gamma,t} \rangle|^2 dy \frac{dt}{t},$$

which proves (5.4).

We note that if we restrict the integral on the left-hand side of (5.4) to the region $\eta > \xi$, we may improve the bound on the right-hand side to

$$(5.6) \quad C \|\widehat{f} 1_{(\xi,\infty)}\|_2^2.$$

This follows simply by support considerations on the Fourier transform side.

Now assume that $f \in L^\infty(\mathbb{R})$, and write $f = f_1 + f_2$, where

$$f_1 = f 1_{[x-2s, x+2s]}.$$

By linearity we may split $F = F_1 + F_2$ correspondingly. We have $\|f_1\|_2^2 \leq Cs \|f\|_\infty^2$, so by the above L^2 bound we have

$$(s^{-1} \int_{T(x,\xi,s) \setminus T^b(x,\xi,s)} |F_1(y,\eta,t)|^2 dy d\eta dt)^{1/2} \leq C \|f\|_\infty.$$

It remains to prove the analogous estimate for F_2 . But for $y \in [x-s, x+s]$ and $t < s$, we have

$$F_2(y,\eta,t) \leq \int_{[-s,s]^c} |f_2(y-z)| t^{-1} |\phi(t^{-1}z)| dz \leq C(t/s) \|f\|_\infty,$$

where we have crudely estimated the integral of the tail of ϕ . But then

$$\begin{aligned} & (s^{-1} \int_{T(x,\xi,s) \setminus T^b(x,\xi,s)} |F_2(y, \eta, t)|^2 dy d\eta dt)^{1/2} \\ & \leq C \|f\|_\infty (s^{-1} \int_0^s \int_{\xi-2\alpha^{-1}t^{-1}}^{\xi+2\alpha^{-1}t^{-1}} \int_{x-s}^{x+s} (t/s)^2 dy d\eta dt)^{1/2} \leq C \|f\|_\infty. \end{aligned}$$

This completes the proof of the endpoint $p = \infty$ of Theorem 5.1.

5.2. The endpoint $p = 2$. We need to find for each $\lambda > 0$ a collection $\mathbf{Q} \subset X_\Delta$ such that

$$\sum_{(x,\xi,s) \in \mathbf{Q}} s \leq C \lambda^{-2} \|f\|_2^2,$$

and for every $T' \in \mathbf{E}_\Delta$, we have

$$(5.7) \quad S(F1_{X \setminus E})(T') \leq \lambda,$$

where $E = \bigcup_{(x,\xi,s) \in \mathbf{Q}} T(x, \xi, s)$.

We first reduce to the special case that the support of \hat{f} is compact. Choose an unbounded monotone increasing sequence $\xi_k, k = 0, 1, 2, \dots$ with $\xi_0 = 0$ such that for f_k defined by $\hat{f}_k = \hat{f}[1_{(-\xi_k, -\xi_{k-1})} + 1_{(\xi_{k-1}, \xi_k)}]$ we have

$$\|f_k\|_2 \leq C 2^{-10k} \|f\|_2.$$

Applying the special case to each of the functions f_k with $\lambda_k = 2^{-k} \lambda$, we obtain corresponding collections \mathbf{Q}_k . Then clearly

$$\sum_{k=1}^\infty \sum_{(x,\xi,s) \in \mathbf{Q}_k} s \leq C \sum_k (2^{-k} \lambda)^{-2} 2^{-20k} \|f\|_2^2 \leq C \lambda^{-2} \|f\|_2^2.$$

If E denotes the union of all $T(x, \xi, s)$ with $(x, \xi, s) \in \bigcup_k \mathbf{Q}_k$, then by countable subadditivity of the size S we have for every $T' \in \mathbf{E}_\Delta$

$$S(F1_{X \setminus E})(T') \leq \sum_{k=1}^\infty S(F_k 1_{X \setminus E})(T') \leq \sum_{k=1}^\infty 2^{-k} \lambda \leq \lambda.$$

This completes the reduction to the case that \hat{f} has compact support, and we shall henceforth assume compact support of \hat{f} .

By scaling of outer Lebesgue spaces, we may assume $\|f\|_2 = 1$. Fix $\lambda > 0$. We first set out to cover all points $(y, \eta, t) \in X$ with $|F(y, \eta, t)| > \lambda$ with tents. Note that there is an a priori upper bound on t for any such point since by Cauchy–Schwarz we have directly from the definition of F that

$$|F(y, \eta, t)| \leq C t^{-1/2} \|f\|_2 \|\phi\|_2 \leq C t^{-1/2}.$$

Assume there is a point (y, η, t) with $|F(y, \eta, t)| > \lambda$, then by Lemma 5.2 we find a tent $T(x, \xi, s)$ centrally containing the point (y, η, t) . Because of the upper bound on t and since $(x, \xi, s) \in X_\Delta$ and therefore $s = 2^k$ for some integer k , we may choose (y, η, t) and (x, ξ, s) such that s is maximal. Denote these points by (y_1, η_1, t_1) and (x_1, ξ_1, s_1) and the tent $T(x_1, \xi_1, s_1)$ by T_1 .

We continue to select tents by iterating this procedure. Assume that we have already chosen points $(y_k, \eta_k, t_k) \in X$ and tents $T_k = T(x_k, \xi_k, s_k)$ for all $1 \leq k < n$. Assume there is a point (y, η, t) with $|F(y, \eta, t)| > \lambda$ not contained in the union of the tents T_k with $1 \leq k < n$. Then we choose such a point (y_n, η_n, t_n) and a tent

$T_n = T(x_n, \xi_n, s_n)$ centrally containing (y_n, η_n, t_n) such that s_n is maximal. We have $|F(y_n, \eta_n, t_n)| > \lambda$ and

$$(y_n, \eta_n, t_n) \notin \bigcup_{k=1}^{n-1} T_k.$$

We claim that

$$(5.8) \quad \sum_{k=1}^n s_k \leq C\lambda^{-2}.$$

To see the claim, let K_m be the set of indices k with $1 \leq k \leq n$ such that

$$2^m \lambda \leq |F(y_k, \eta_k, t_k)| \leq 2^{m+1} \lambda.$$

Then we have

$$\sum_{k=1}^n s_k \leq C \sum_{m=0}^{\infty} 2^{-2m} \lambda^{-2} \sum_{k \in K_m} t_k |F(y_k, \eta_k, t_k)|^2.$$

The claim (5.8) will follow if we show for fixed $m \geq 0$

$$(5.9) \quad \sum_{k \in K_m} t_k |F(y_k, \eta_k, t_k)|^2 \leq C.$$

Define

$$(5.10) \quad \phi_k(x) := \phi_{y_k, \eta_k, t_k}(x) := e^{-i\eta_k(y_k - x)} t_k^{-1/2} \overline{\phi(t_k^{-1}(y_k - x))},$$

so that

$$t_k^{1/2} F(y_k, \eta_k, t_k) = \langle f, \phi_k \rangle.$$

Let A denote the left-hand side of (5.9). Assume we can show for every $k \in K_m$

$$(5.11) \quad \sum_{l \in K_m: s_l \leq s_k} (t_l/t_k)^{1/2} |\langle \phi_k, \phi_l \rangle| \leq C.$$

Then we obtain

$$\begin{aligned} A^2 &\leq \left\| \sum_{k \in K_m} \langle f, \phi_k \rangle \phi_k \right\|_2^2 \\ &\leq \sum_{k, l \in K_m} \langle f, \phi_k \rangle \langle \phi_k, \phi_l \rangle \langle \phi_l, f \rangle \\ &\leq 2 \sum_{k, l \in K_m: s_l \leq s_k} |\langle f, \phi_k \rangle \langle \phi_k, \phi_l \rangle \langle \phi_l, f \rangle| \\ &\leq C \sum_{k, l \in K_m: s_l \leq s_k} (t_l/t_k)^{1/2} |\langle f, \phi_k \rangle|^2 |\langle \phi_k, \phi_l \rangle| \leq CA. \end{aligned}$$

Here in the passage from the penultimate to ultimate line, we have used that $k, l \in K_m$ and hence $t_k^{-1/2} |\langle f, \phi_k \rangle|$ and $t_l^{-1/2} |\langle f, \phi_l \rangle|$ are within a factor of 2 of each other. In the last line we have used (5.11). Dividing by A on both sides of the displayed inequality, we have reduced the proof of the desired estimate (5.9) to the proof of (5.11).

To prove (5.11), fix k . If $l \in K_m$ with $s_l \leq s_k$ such that $\langle \phi_k, \phi_l \rangle \neq 0$, then the supports of $\widehat{\phi}_k$ and $\widehat{\phi}_l$ overlap, and hence there are numbers $-2^{-8} \leq \gamma, \delta \leq 2^{-8}$ such that

$$\eta_l + \delta b t_l^{-1} = \eta_k + \gamma b t_k^{-1}.$$

Now suppose that there is another such l' , and we have analogously

$$\eta_{l'} + \delta'bt_{l'}^{-1} = \eta_k + \gamma'bt_k^{-1}.$$

Assume without loss of generality that T_l is selected prior to $T_{l'}$ and thus $s_k^{-1} \leq s_l^{-1} \leq s_{l'}^{-1}$. Then we have by central containment of (y_l, η_l, t_l) in T_l that

$$\begin{aligned} |\alpha(\eta_{l'} - \xi_l) + \beta t_{l'}^{-1}| &\leq |\alpha(\eta_{l'} - \eta_l)| + |\alpha(\eta_l - \xi_l)| + |\beta t_{l'}^{-1}| \\ &\leq bt_l^{-1} + |\eta_l - \eta_{l'}| + |\beta t_{l'}^{-1}|. \end{aligned}$$

Now using the information from the support of the bump functions, we may estimate the latter by

$$\leq bt_l^{-1} + \delta bt_l^{-1} + \delta'bt_{l'}^{-1} + \gamma bt_k^{-1} + \gamma'bt_k^{-1} + |\beta t_{l'}^{-1}| \leq t_{l'}^{-1}.$$

Here we have used again the central containment to estimate the inverse powers of t_k and t_l by that of $t_{l'}$.

This implies that $[x_l - 2^{-8}s_l, x_l + 2^{-8}s_l]$ and $[x_{l'} - 2^{-8}s_{l'}, x_{l'} + 2^{-8}s_{l'}]$ are disjoint. For if they were not disjoint, then, since $s_{l'} \leq s_l$, we would conclude

$$\begin{aligned} |y_{l'} - x_l| &\leq |y_{l'} - x_{l'}| + |x_{l'} - x_l| \\ &\leq 2^{-4}s_{l'} + 2^{-4}s_l < s_l - t_{l'}, \end{aligned}$$

and together with the previous estimate for $\xi_l - \eta_{l'}$ this implied that the point $(y_{l'}, \eta_{l'}, t_{l'})$ was in the tent T_l , contradicting the choice of this point.

The argument above in particular shows that $|x_k - x_l| \geq 2^{-8}s_k$. Let \bar{x} be the midpoint of x_l and x_k , and let H_l and H_k be the half lines emanating from the midpoint containing x_l and x_k , respectively. We then have

$$|\langle \phi_k, \phi_l \rangle| \leq \|\phi_k\|_{L^1(H_k)} \|\phi_l\|_{L^\infty(H_k)} + \|\phi_k\|_{L^\infty(H_l)} \|\phi_l\|_{L^1(H_l)}.$$

Thanks to the rapid decay of the wave packets, that $s_l \leq s_k$, and the fact that $|x_k - x_l| \geq 2^{-8}s_k$, we can estimate the last display by

$$C(s_l s_k)^{-1/2} \int (1 + (\frac{x - x_k}{s_k})^2)^{-2} 1_{[x_l - 2^{-8}s_l, x_l + 2^{-8}s_l]}(x) dx.$$

By disjointness of the intervals $[x_l - 2^{-8}s_l, x_l + 2^{-8}s_l]$ for different l , we obtain

$$\sum_{l \in K_m, s_l \leq s_k} s_l^{1/2} |\langle \phi_k, \phi_l \rangle| \leq C s_k^{-1/2} \int (1 + (\frac{x - x_k}{s_k})^2)^{-1} dx \leq C s_k^{1/2}.$$

This proves (5.11) since t_l and t_k are comparable to s_l and s_k , and hence completes the proof of (5.8).

If the iterative selection of tents T_n stops because of lack of suitable points (y, η, t) with large enough value $F(y, \eta, t)$, then clearly F is bounded by λ outside the union $\bigcup_{k=1}^{n-1} T_k$. If the iterative selection does not stop, we claim that still F is bounded above by λ outside the union $\bigcup_{k=1}^\infty T_k$. Namely, assume the point (y, η, t) is outside this union. Since by (5.8) we have $s_k \rightarrow 0$ as $k \rightarrow \infty$, we have $s_k < t$ for some k . By maximal choice of s_k , we have $F(y, \eta, t) \leq \lambda$. This proves the desired bound on F . In the case of infinitely many selected tents T_k , it also follows by a limiting argument from (5.8) that $\sum_{k=1}^\infty s_k \leq C\lambda^{-2}$.

Summarizing, we have found a collection \mathbf{Q}_0 of tents such that

$$\sum_{T \in \mathbf{Q}_0} \sigma(T) \leq C\lambda^{-2},$$

and if we set

$$E = \bigcup_{T \in \mathbf{Q}_0} T,$$

then we have

$$F(y, \eta, t) \leq \lambda$$

for all points (y, η, t) in the complement of E . In what follows, we shall no longer need the selected tents explicitly, and hence we shall free the symbols T_k, x_k, ξ_k, s_k to have new meanings in the further selection process.

We need to select tents of large L^2 portion of the size. Given a number ξ , typically arising as second parameter of a tent $T(x, \xi, s)$, we split the space X into upper half

$$X_\xi^+ = \{(y, \eta, s) \in X : \eta \geq \xi\}$$

and lower half $X_\xi^- = X \setminus X_\xi^+$. We first focus on X_ξ^+ .

Call a point $(x, \xi, s) \in X_\Delta$ bad, if

$$(5.12) \quad s^{-1} \int_{(T(x, \xi, s) \cap X_\xi^+) \setminus (T^b(x, \xi, s) \cup E)} |F(y, \eta, t)|^2 dy d\eta dt \geq 2^{-8} \lambda^2.$$

By the estimate (5.4) we obtain an a priori upper bound $2^{k_{\max}}$ for the third component s of any bad point (x, ξ, s) . Given such an upper bound, the parameter ξ becomes a multiple of $2^{-8-k_{\max}}b$ and is thus a discrete parameter. Since \widehat{f} has compact support, we obtain from observation (5.6) an upper bound for ξ depending on the support of \widehat{f} . Hence there is a maximal possible value ξ_{\max} for the second component of a bad point. We choose some bad point (x_1, ξ_1, s_1) with $\xi_1 = \xi_{\max}$ which maximizes s_1 under the constraint $\xi_1 = \xi_{\max}$. Define the tents $T_1 = T(x_1, \xi_1, s_1)$ and $T_1^b = T^b(x_1, \xi_1, s_1)$, and define $X_1^+ = X_{\xi_1}^+$. Note that by maximizing s_1 for fixed ξ_1 and x_1 , we guarantee that (5.12) is sharp up to a factor of 2, and hence the selected tent satisfies an upper bound

$$s^{-1} \int_{(T(x, \xi, s) \cap X_\xi^+) \setminus (T^b(x, \xi, s) \cup E)} |F(y, \eta, t)|^2 dy d\eta dt \leq \lambda^2.$$

Now we iterate this selection: Assume we have already chosen points $(x_k, \xi_k, s_k) \in X_\Delta$ for $1 \leq k < n$, and we have defined tents T_k, T_k^b for $1 \leq k < n$. Define $E_n = E \cup \bigcup_{k=1}^{n-1} T_k$. We update the definition of a bad point (x, ξ, s) to be a point in X_Δ with

$$s^{-1} \int_{(T(x, \xi, s) \cap X_\xi^+) \setminus (T^b(x, \xi, s) \cup E_n)} |F(y, \eta, t)|^2 dy d\eta dt \geq 2^{-8} \lambda^2.$$

Again, there is a (possibly new) maximal value ξ_{\max} for the second component ξ of a bad point. We pick one bad point (x_n, η_n, s_n) with $\xi_n = \xi_{\max}$ which maximizes the value of s_n among all bad points (x, ξ, s) with $\xi = \xi_{\max}$. Then we define the tents $T_n = T(x_n, \xi_n, s_n)$ and $T_n^b = T^b(x_n, \xi_n, s_n)$, and we define $X_n^+ = X_{\xi_n}^+$. This completes the n -th selection step.

We introduce the notation

$$T_n^* = (T_n \cap X_n^+) \setminus (T_n^b \cup E_n).$$

We claim the analogue of (5.8), namely

$$(5.13) \quad \sum_{k=1}^n s_k \leq C\lambda^{-2}.$$

To prove (5.13), it suffices to show

$$\sum_{k=1}^n \int_{T_k^*} |F(y, \eta, t)|^2 dy d\eta dt \leq C.$$

With $\phi_{y,\eta,t}$ defined analogously to (5.10), we may write for the left-hand side of the last display

$$A := \sum_{k=1}^n \int_{T_k^*} |\langle f, \phi_{y,\eta,t} \rangle|^2 dy d\eta \frac{dt}{t}.$$

Then we have by Cauchy–Schwarz

$$(5.14) \quad \begin{aligned} A^2 &\leq \left\| \sum_{k=1}^n \int_{T_k^*} \langle f, \phi_{y,\eta,t} \rangle \phi_{y,\eta,t} dy d\eta \frac{dt}{t} \right\|_2^2 \\ &= \sum_{k,l=1}^n \int_{T_k^* \times T_l^*} \langle f, \phi_{y,\eta,t} \rangle \langle \phi_{y,\eta,t}, \phi_{y',\eta',t'} \rangle \langle \phi_{y',\eta',t'}, f \rangle dy d\eta \frac{dt}{t} dy' d\eta' \frac{dt'}{t'} \\ &= \sum_{k,l=1}^n \int_{T_k^* \times T_l^* : B^{-1}t \leq t' \leq Bt} + \dots + 2 \sum_{k,l=1}^n \int_{T_k^* \times T_l^* : Bt' \leq t} \dots, \end{aligned}$$

where the large number $B = 2^8 \alpha^{-1} b^{-1}$ determines the cutoff in the last line between the diagonal and off-diagonal parts, the latter being estimated by twice the upper triangular part using symmetry. In the diagonal term we use symmetry to estimate the smaller of the inner products with f by the larger one and obtain the upper bound

$$\begin{aligned} &2 \sum_{k,l=1}^n \int_{T_k^* \times T_l^* : B^{-1}t \leq t' \leq Bt} |\langle f, \phi_{y,\eta,t} \rangle|^2 |\langle \phi_{y,\eta,t}, \phi_{y',\eta',t'} \rangle| dy d\eta \frac{dt}{t} dy' d\eta' \frac{dt'}{t'} \\ &\leq 2A \sup_{k,(y,\eta,t) \in T_k^*} \left(\sum_{l=1}^n \int_{T_l^* : B^{-1}t \leq t' \leq Bt} |\langle \phi_{y,\eta,t}, \phi_{y',\eta',t'} \rangle| dy' d\eta' \frac{dt'}{t'} \right) \\ &\leq 2A \sup_{k,(y,\eta,t) \in T_k^*} \left(\int_{B^{-1}t \leq t' \leq Bt} \int_{\mathbb{R}^2} |\langle \phi_{y,\eta,t}, \phi_{y',\eta',t'} \rangle| dy' d\eta' \frac{dt'}{t'} \right). \end{aligned}$$

Here we have used that the regions T_l^* are pairwise disjoint. Integrating over the t' -interval of bounded dt'/t' -measure estimates the previous display by

$$\leq CA \sup_{k,(y,\eta,t) \in T_k^*} \left(\sup_{B^{-1}t \leq t' \leq tB} \int_{\mathbb{R}^2} |\langle \phi_{y,\eta,t}, \phi_{y',\eta',t'} \rangle| dy' d\eta' \right).$$

For the η' integration we use that $\widehat{\phi}_{y,\eta,t}$ is supported on an interval of length t^{-1} and $t \sim t'$:

$$\leq CA \sup_{k,(y,\eta,t) \in T_k^*} \sup_{B^{-1}t \leq t' \leq tB} \sup_{\eta'} t^{-1} \int_{\mathbb{R}} |\langle \phi_{y,\eta,t}, \phi_{y',\eta',t'} \rangle| dy'.$$

For the y' integration we use that $\phi_{y,\eta,t}$ is an L^2 normalized wave packet adapted to an interval of length t . This estimates the last display by CA .

Turning to the off diagonal term in (5.14), we estimate it with Cauchy–Schwarz and the upper bound on the selected tents by

$$2 \sum_{k=1}^n \left(\int_{T_k^*} | \langle f, \phi_{y,\eta,t} \rangle |^2 dy d\eta \frac{dt}{t} \right)^{1/2} H_k^{1/2} \leq C \sum_{k=1}^n \lambda s_k^{1/2} H_k^{1/2},$$

where H_k is equal to

$$\int_{T_k^*} \left(\sum_{l=1}^n \int_{T_l^*: Bt' \leq t} | \langle \phi_{y,\eta,t}, \phi_{y',\eta',t'} \rangle \langle \phi_{y',\eta',t'}, f \rangle | dy' d\eta' \frac{dt'}{t'} \right)^2 dy d\eta \frac{dt}{t}.$$

Using the pointwise bound on $F(y, \eta, t)$ outside E , we can estimate H_k by

$$\int_{T_k^*} \left(\sum_{l=1}^n \int_{T_l^*: Bt' \leq t} c\lambda t'^{1/2} | \langle \phi_{y,\eta,t}, \phi_{y',\eta',t'} \rangle | dy' d\eta' \frac{dt'}{t'} \right)^2 dy d\eta \frac{dt}{t}.$$

Let $(y, \eta, t) \in T_k^*$ and $(y', \eta', t') \in T_l^*$ with $Bt' \leq t$. Assume that the inner product $\langle \phi_{y,\eta,t}, \phi_{y',\eta',t'} \rangle$ is not zero. Then we have

$$\eta' + \gamma'(t')^{-1} = \eta + \gamma t^{-1}$$

for some

$$-2^{-8}b \leq \gamma, \gamma' \leq 2^{-8}b,$$

and hence

$$| \eta - \eta' | \leq 2^{-4}b(t')^{-1}.$$

By definition of the reduced domains we have

$$\alpha(\eta - \xi_k) + \beta t^{-1} \leq t^{-1},$$

$$b(t')^{-1} \leq \eta' - \xi_l.$$

This gives

$$\begin{aligned} \xi_k - \xi_l &= (\eta - \eta') - (\eta - \xi_k) + (\eta' - \xi_l) \\ &\geq -2^{-4}b(t')^{-1} - \alpha^{-1}(1 - \beta)t^{-1} + b(t')^{-1}. \end{aligned}$$

Using $Bt' \leq t$ the last display strictly larger than 0 and hence the tent T_k has been chosen prior to T_l . Since (y', η', t') is in the reduced tent T_l^* , it is not in E_l and hence not in T_k . But $t' < t \leq s_k$ and

$$\begin{aligned} &| \alpha(\eta' - \xi_k) + \beta(t')^{-1} | \\ &\leq | \alpha(\eta' - \eta) | + | \alpha(\eta - \xi_k) + \beta(t)^{-1} | + | \beta(t')^{-1} - \beta(t^{-1}) | \\ &\leq 2^{-4}b(t')^{-1} + t^{-1} + | \beta t^{-1} | + | \beta(t')^{-1} | \leq (t')^{-1}, \end{aligned}$$

and hence we need to have

$$| y' - x_k | \geq s_k - t'.$$

This implies

$$(5.15) \quad | y' - x_k | \geq s_k - t.$$

Now pick a further point $(y'', \eta'', t'') \in T_l^*$ with $Bt'' \leq t$ and nonzero inner product $\langle \phi_{y,\eta,t}, \phi_{y'',\eta'',t''} \rangle$. We have again

$$| \eta - \eta'' | \leq 2^{-4}b(t'')^{-1}$$

and

$$b(t'')^{-1} \leq \eta'' - \xi_{l'}.$$

Now we assume $Bt'' \leq t'$. Then we conclude

$$|\eta' - \eta''| \leq 2^{-2}b(t'')^{-1}$$

and

$$\begin{aligned} \xi_l - \xi_{l'} &= (\xi_l - \eta') + (\eta' - \eta'') + (\eta'' - \xi_{l'}) \\ &\geq -2\alpha^{-1}(t')^{-1} - 2^{-2}b(t'')^{-1} + b(t'')^{-1} > 0. \end{aligned}$$

Hence T_l was chosen prior to $T_{l'}$, and in particular (y'', η'', t'') is not in T_l . But we have $t'' < t' \leq s_l$ and

$$\begin{aligned} &|\alpha(\eta'' - \xi_l) + \beta(t'')^{-1}| \\ &\leq |\alpha(\eta'' - \eta')| + |\alpha(\eta' - \xi_l) + \beta(t')^{-1}| + |\beta(t'')^{-1} - \beta(t')^{-1}| \\ &\leq 2^{-2}b(t'')^{-1} + (t')^{-1} + \beta(t'')^{-1} + \beta(t')^{-1} \leq (t'')^{-1}. \end{aligned}$$

Since (y'', η'', t'') is not in T_l we conclude

$$|y'' - x_l| > s_l - t'' > s_l - t' \geq |y' - x_l|,$$

and in particular $y'' \neq y'$.

To summarize our finding, fix $(y, \eta, t) \in T_k^*$. Then for fixed $y' \in \mathbb{R}$, the minimal and maximal values of parameters t' with $Bt' \leq t$ such that there exists l and η' with $(y', \eta', t') \in T_l^*$ and $\langle \phi_{y, \eta, t}, \phi_{y', \eta', t'} \rangle \neq 0$ are at most a factor B apart. It follows that for every y' there exists an interval $I(y') = [T(y'), BT(y')]$ such that we need $t' \in I(y')$ for such l, η' to exist.

Using also (5.15) and disjointness of the reduced domains T_l^* , we may thus estimate H_k by

$$\begin{aligned} C \int_{T_k^*} \left(\int_{|y' - x_x| > s_k - t} \int_{I(y')} \int_{\mathbb{R}} \lambda t^{1/2} |\langle \phi_{y, \eta, t}, \phi_{y', \eta', t'} \rangle| d\eta' \frac{dt'}{t'} dy' \right)^2 dy d\eta \frac{dt}{t} \\ \leq C \int_{T_k^*} \left(\int_{|y' - x_x| > s_k - t} \sup_{t' \in I(y')} \int_{\mathbb{R}} \lambda t^{1/2} |\langle \phi_{y, \eta, t}, \phi_{y', \eta', t'} \rangle| d\eta' dy' \right)^2 dy d\eta \frac{dt}{t}. \end{aligned}$$

Further, by trivial reasoning with the Fourier support of the bump functions, if we fix y' and t' as in this integral, then there is an interval of length $2t'^{-1}$ which must contain η' for the inner product $\langle \phi_{y, \eta, t}, \phi_{y', \eta', t'} \rangle$ to be nonzero. Using the estimate

$$\langle \phi_{y, \eta, t}, \phi_{y', \eta', t'} \rangle \leq C \left(\frac{t'}{t}\right)^{1/2} \left(1 + \frac{|y' - y|}{t}\right)^{-2},$$

we obtain for the previous display the upper bound

$$\begin{aligned} C \int_{T_k^*} \left(\int_{|y' - x_k| > s_k - t} \lambda \left(\frac{1}{t}\right)^{1/2} \left(1 + \frac{|y' - y|}{t}\right)^{-2} dy' \right)^2 dy d\eta \frac{dt}{t} \\ \leq C \int_{T_k^*} \left(\lambda t^{1/2} \left(1 + \frac{s_k - |y - x_k|}{t}\right)^{-1} \right)^2 dy d\eta \frac{dt}{t} \\ \leq C \int_0^{s_k} \int_{x_k - s_k}^{x_k + s_k} \int_{\xi_k - 2\alpha^{-1}t^{-1}}^{\xi_k + 2\alpha^{-1}t^{-1}} \lambda^2 t \left(1 + \frac{s_k - |y - x_k|}{t}\right)^{-2} d\eta dy \frac{dt}{t} \\ \leq C \lambda^2 s_k. \end{aligned}$$

This completes our estimation of (5.14), and we have shown

$$\frac{1}{4}A^2 \leq CA + C \sum_{k=1}^n \lambda^2 s_k \leq CA,$$

where in the last inequality we have used the lower bound on the selected tents. Dividing by A proves the desired estimate for A and completes the proof of (5.13) for the newly selected tents.

If the selection of tents stops lacking any further (x, ξ, s) with

$$s^{-1} \int_{(T(x,\xi,s) \cap X_\xi^+) \setminus (T_{x,\xi,s}^b \cup E_n)} |F(y, \eta, t)|^2 dy d\eta dt \geq 2^{-8} \lambda^2,$$

then clearly the converse inequality holds for all $(x, \xi, s) \in X_\Delta$. If the selection of tents does not stop, we collect T_k for all $k \in \mathbb{N}$ and write $E_{(1)} = E \bigcup_{k=1}^\infty E_k$. Note that ξ_k is a decreasing sequence, and as noted before the possible values of ξ_k are in the discrete lattice $\mathbb{Z}b2^{-8-k_{\max}}$. If $\xi_k \rightarrow -\infty$, then for every $(x, \xi, s) \in X_\Delta$

$$s^{-1} \int_{(T(x,\xi,s) \cap X_\xi^+) \setminus (T_{x,\xi,s}^b \cup E_{(1)})} |F(y, \eta, t)|^2 dy d\eta dt \leq 2^{-8} \lambda^2.$$

Namely, assume not. Then $\xi > \xi_k$ for some k , and this would contradict the choice of T_k .

Now assume ξ_k does not tend to $-\infty$. Then the sequence stabilizes, meaning it eventually becomes constant, at some value $\xi_{(1)}$. We shall then choose further tents, and for emphasis we rename the previously selected tents into $T_k =: T_{(1),k} = T(x_{(1),k}, \xi_{(1),k}, s_{(1),k})$.

Call a point $(x, \xi, s) \in X_\Delta$ bad if

$$s^{-1} \int_{(T(x,\xi,s) \cap X_\xi^+) \setminus (T^b(x,\xi,s) \cup E_{(1)})} |F(y, \eta, t)|^2 dy d\eta dt \geq 2^{-8} \lambda^2,$$

and let ξ_{\max} be the maximal possible value of the second component of a bad point. Note that ξ_{\max} is strictly less than $\xi_{(1)}$. For if not, then $\xi_{\max} = \xi_{(1)}$ by choice of the previously selected tents. Since $s_k \rightarrow 0$ by (5.13), we have $s > s_k$ for some k and some bad point (x, ξ_{\max}, s) . This however contradicts the choice of the tent $T_{(1),k}$. We then choose a bad point $(x_{(2),1}, \xi_{(2),1}, s_{(2),1})$ such that $\xi_{(2),1} = \xi_{\max}$ and $s_{(2),1}$ is maximal among all such choices. We then iterate this selection process as before, obtaining tents $T_{(2),k} = T(x_{(2),k}, \xi_{(2),k}, s_{(2),k})$. Our proof of (5.13) applies verbatim to yield

$$\left(\sum_{k=1}^\infty s_{(1),k}\right) + \left(\sum_{k=1}^n s_{(2),k}\right) \leq C\lambda^{-2}.$$

We now continue this double recursion in the obvious manner. If at some point the recursion stops or yields for some fixed m a sequence $\xi_{(m),k}$ tending to $-\infty$, then by the previous discussions we are left with no bad points. If the double iteration does not stop, we obtain a double sequence of tents $T_{(m),k}$ with

$$\sum_{m=1}^\infty \sum_{k=1}^\infty s_{(m),k} \leq C\lambda^{-2}.$$

Moreover, the sequence $\xi_{(m)}$ of stabilizing points decreases to $-\infty$, since they are strict monotone decreasing and in a discrete lattice. We can then observe that there are no bad points outside $\bigcup_{m=1}^\infty E_{(m)}$.

Summarizing, we have found a collection \mathbf{Q}_+ of tents such that

$$\sum_{T \in \mathbf{Q}_+} \sigma(T) \leq C\lambda^{-2},$$

and if we set

$$E_+ = E \cup \bigcup_{T \in \mathbf{Q}_+} T,$$

then we have

$$s^{-1} \int_{(T(x, \xi, s) \cap X_\xi^+) \setminus T^b(x, \xi, s)} |F(y, \eta, t) 1_{E_+^c}(y, \eta, t)|^2 dy d\eta dt \leq 2^{-8}\lambda^2$$

for all $(x, \xi, \delta) \in X_\Delta$.

We may repeat the above argument symmetrically to obtain a collection \mathbf{Q}_- of tents such that

$$\sum_{T \in \mathbf{Q}_-} \sigma(T) \leq C\lambda^{-2},$$

and if we set

$$E_- = E \cup \bigcup_{T \in \mathbf{Q}_-} T,$$

then we have

$$s^{-1} \int_{(T(x, \xi, s) \cap X_\xi^-) \setminus T^b(x, \xi, s)} |F(y, \eta, t) 1_{E_-^c}(y, \eta, t)|^2 dy d\eta dt \leq 2^{-8}\lambda^2$$

for all $(x, \xi, \delta) \in X_\Delta$.

Setting finally $\mathbf{Q} = \mathbf{Q}_0 \cup \mathbf{Q}_+ \cup \mathbf{Q}_-$ we have clearly found the desired collection of tents. This completes the proof of the endpoint $p = 2$ of Theorem 5.1. \square

6. THE BILINEAR HILBERT TRANSFORM

The most immediate application of Theorem 5.1 is to prove basic estimates for the bilinear Hilbert transform. Another possible application is toward Carleson’s theorem [3] on almost everywhere convergence of Fourier series. However, the latter application requires more work, as Carleson’s operator lacks the symmetry that is exhibited by the bilinear Hilbert transform and, therefore, needs an additional embedding theorem. Hence we decided to restrict our attention to the bilinear Hilbert transform, which suffices to illustrate some key points of time-frequency analysis originating in Carleson’s work on convergence of Fourier series.

Let $\beta = (\beta_1, \beta_2, \beta_3)$ be a vector in \mathbb{R}^3 with pairwise distinct entries. For three Schwartz functions f_1, f_2, f_3 on the real line, we define

$$\Lambda_\beta(f_1, f_2, f_3) := p.v. \int_{\mathbb{R}} \left[\int_{\mathbb{R}} \left[\prod_{j=1}^3 f_j(x - \beta_j t) \right] dx \right] \frac{dt}{t}.$$

Note that the inner integral produces a Schwartz function in the variable t , to which we apply the tempered distribution $p.v.1/t$. By a change of variables, scaling t and translating x , we may and do restrict attention to vectors β which have unit length and are perpendicular to $(1, 1, 1)$. The resulting one-parameter family of trilinear forms is dual to a family of bilinear operators called *bilinear Hilbert transforms*. To obtain explicit expressions for these bilinear operators, one applies another translation in the x variable to make one of the components, say β_i , vanish. After

interchanging the order of integrals, one obtains an explicit pairing of a bilinear operator in $f_j, j \neq i$, with the function f_i .

Let α be a unit vector perpendicular to $(1, 1, 1)$ and β . The vector α is unique up to reflection at the origin, and it has only nonzero components by the assumption that β has pairwise distinct components. Note also that $|\beta_j| \leq 0.9$ for each j . For if one component of β_j in absolute value exceeds 0.9, then since β is perpendicular to $(1, 1, 1)$, at least one further component has to exceed 0.45 in absolute value. But then the vector cannot be a unit vector.

The following a priori estimate for Λ_β originates in [10].

Theorem 6.1. *For a unit vector β perpendicular to $(1, 1, 1)$ with pairwise distinct entries, and for $2 < p_1, p_2, p_3 < \infty$ with $\sum_j \frac{1}{p_j} = 1$, there is a constant C such that for all Schwartz functions f_1, f_2, f_3 we have*

$$|\Lambda_\beta(f_1, f_2, f_3)| \leq C \prod_{j=1}^3 \|f_j\|_{p_j}.$$

We give a new proof of this theorem based on Theorem 5.1 and an outer Hölder inequality. This proof is analogous to the previously presented proof of boundedness of paraproducts. In our approach, much of the difficulty in proving bounds for the bilinear Hilbert transform has been moved into the proof of the generalized Carleson embedding theorem. What remains to be done is relatively easier and, in particular, conceptually quite simple. It is the strength of our approach that the main difficulty is packaged into a cleanly separated module; previous approaches do not suggest the formulation of as clean a statement as Theorem 5.1. In particular, our proof is the first one to succeed without the passage to a discrete model operator. This avoids a cumbersome setup of choices of the discretization.

One can prove a version of Theorem 6.1 with a constant independent of β (see [9]), but only at the expense of considerable additional work. One may also extend the range of exponents (see [11]). It would be interesting to discuss these results in the context of outer measure theory, but this is beyond the scope of the present paper.

Proof of Theorem 6.1. Define for $j = 1, 2, 3$

$$(6.1) \quad F_j(y, \eta, t) := \int_{\mathbb{R}} f_j(x) e^{i\eta(y-x)} t^{-1} \phi(t^{-1}(y-x)) dx,$$

where ϕ is a real valued Schwartz function such that $\widehat{\phi}$ is nonnegative, nonvanishing at the origin, and supported in $[-\epsilon, \epsilon]$ for suitably small ϵ . It will suffice to choose $\epsilon = 2^{-16}$.

The estimate of Theorem 6.1 can be reformulated by means of the functions F_j .

Lemma 6.2. *Under the assumptions of Theorem 6.1, there is a constant C depending only on β, p_1, p_2, p_3 , and ϕ as above such that*

$$(6.2) \quad \left| \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}} \prod_{j=1}^3 F_j(y, \alpha_j \eta + \beta_j t^{-1}, t) d\eta dy dt \right| \leq C \prod_{j=1}^3 \|f_j\|_{p_j}.$$

We postpone the proof of Lemma 6.2 and proceed to deduce Theorem 6.1 from Lemma 6.2.

Inserting the definition of F_j and using that α and β are perpendicular to $(1, 1, 1)$, we obtain for the integral on the left-hand side of (6.2),

$$\int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}} t^{-3} \prod_{j=1}^3 \left[\int_{\mathbb{R}} f_j(x_j) e^{-i\beta_j t^{-1} x_j} e^{-i\alpha_j \eta x_j} \phi(t^{-1}(y - x_j)) dx_j \right] d\eta dy dt.$$

Recall that the integral of the Fourier transform of a Schwartz function φ in \mathbb{R}^3 over the line through the origin spanned by α is proportional to the integral of the Schwartz function itself over the perpendicular hyperplane through the origin spanned by $(1, 1, 1)$ and β :

$$(6.3) \quad \int_{\mathbb{R}} \widehat{\varphi}(\eta\alpha) d\eta = c \int_{\mathbb{R}} \int_{\mathbb{R}} \varphi(u(1, 1, 1) + v\beta) du dv.$$

To apply this fact, we observe that the inner triple integral of the previous display over x_1, x_2, x_3 is the value of the Fourier transform of a certain Schwartz function in \mathbb{R}^3 at the point $\eta\alpha \in \mathbb{R}^3$, and the integral in η is then the integral of this Fourier transformation over the line spanned by α . Hence, we obtain up to a nonzero constant factor for that display:

$$\int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} [t^{-3} e^{-it^{-1}v} \prod_{j=1}^3 f_j(u + \beta_j v) \phi(t^{-1}(y - u - \beta_j v))] du dv dy dt.$$

Here we have used again in the argument of the exponential function that α, β and $(1, 1, 1)$ are pairwise orthogonal and that β has unit length. Changing the order of integration so that the y integration becomes innermost, we obtain for the last display

$$\int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}} \left[\prod_{j=1}^3 f_j(u + \beta_j v) \right] t^{-2} e^{-it^{-1}v} \psi(t^{-1}v) du dv dt,$$

where

$$\psi(w) := \int_{\mathbb{R}} \prod_{j=1}^3 \phi(z - \beta_j w) dz.$$

We claim that there are nonzero constants a and b such that for any Schwartz function g on the real line we have

$$(6.4) \quad \int_0^\infty \int_{\mathbb{R}} g(v) t^{-2} e^{-it^{-1}v} \psi(t^{-1}v) dv dt = ag(0) + bp.v. \int g(t) \frac{dt}{t}.$$

This claim turns the left-hand side of (6.2) into a nontrivial linear combination of

$$\int_{\mathbb{R}} \left[\prod_{j=1}^3 f_j(u) \right] du$$

and

$$p.v. \int \left[\int f_1(u - \beta_1 t) f_2(u - \beta_2 t) f_3(u - \beta_3 t) du \right] \frac{dt}{t}.$$

Since L^p bounds for the former follow by Hölder’s inequality, we can deduce L^p bounds for the latter from L^p bounds as in (6.2). This will complete the reduction of Theorem 6.1 to Lemma 6.2, once we have verified the above claim.

To see the claim, it suffices to verify that the left-hand side of (6.4) can be written as a nonzero multiple of

$$\int_{-\infty}^0 \widehat{g}(\zeta) d\zeta,$$

since the characteristic function of the left half-line is known to be a nontrivial linear combination of the Fourier transform of the Dirac delta distribution and the principal value integral against dt/t . Using Plancherel, we identify the left-hand side of (6.4) as nonzero multiple of

$$\int_0^\infty \int_{\mathbb{R}} \widehat{g}(\zeta) \widehat{\psi}(1-t\zeta) d\zeta \frac{dt}{t}.$$

The claim will thus follow by Fubini if we can establish that $\widehat{\psi}$ is proportional to a function that is nonnegative, nonzero at 0, and supported in $[-1/2, 1/2]$. We have

$$\widehat{\psi}(\eta) = \int_{\mathbb{R}} \int_{\mathbb{R}} \prod_{j=1}^3 \phi(z - \beta_j w) e^{i\beta_j \eta(z - \beta_j w)} dz dw.$$

This is an integral of a Schwartz function in \mathbb{R}^3 over the plane spanned by $(1, 1, 1)$ and β , which by the observation (6.3) again may be written as multiple of the integral of the Fourier transform of the Schwartz function over the line spanned by α :

$$\int_{\mathbb{R}} \prod_{j=1}^3 \widehat{\phi}(\alpha_j \xi - \beta_j \eta) d\xi.$$

Since α and β are perpendicular unit vectors and the support of $\widehat{\phi} \otimes \widehat{\phi} \otimes \widehat{\phi}$ is in a neighborhood of 0 with diameter less than $1/10$, this integral is nonzero only if $|\eta|$ is smaller than $1/2$. Moreover, $\widehat{\psi}$ is evidently nonnegative real and nonzero at 0. This completes the proof of the claim and the reduction of Theorem 6.1 to Lemma 6.2. □

Proof of Lemma 6.2. We consider the space $X = \mathbb{R} \times \mathbb{R} \times (0, \infty)$ and the outer measure generated by the collection \mathbf{E} of all tents

$$T(x, \xi, s) := \{(y, \eta, t) \in X : t < s, |y - x| < s - t, |\eta - \xi| \leq t^{-1}\}$$

parameterized by $(x, \xi, s) \in X$ and the premeasure $\sigma(T(x, \xi, s)) = s$.

Define a size S by setting

$$S(G)(T(x, \xi, s)) = s^{-1} \int_{T(x, \xi, s)} |G(y, \eta, t)| dy d\eta dt$$

for each $G \in \mathcal{B}(X)$.

By a straightforward application of Proposition 3.6, we may estimate the left-hand side of (6.2) by

$$C \|G_1 G_2 G_3\|_{L^1(X, \sigma, S)},$$

where we have defined G_j for $j = 1, 2, 3$ by

$$G_j(y, \eta, t) := F_j(y, \alpha_j \eta + \beta_j t^{-1}, t).$$

We intend to apply a threefold Hölder's inequality, which requires us to define three appropriate sizes S_j . Set

$$b = 2^{-8} \min_{i \neq j} |\beta_i - \beta_j|.$$

Since no two components of β are equal, we have $b > 0$. Define for each $1 \leq j \leq 3$ and $(x, \xi, s) \in X$ the region

$$T^{(j)}(x, \xi, s) := \{(y, \eta, t) \in X : t \leq s, |y - x| \leq s - t, |\alpha_j^{-1}(\eta - \xi) - \alpha_j^{-1}\beta_j t^{-1}| \leq bt^{-1}\}.$$

For fixed (x, ξ, s) the three regions $T^{(j)}(x, \xi, s)$ are pairwise disjoint, by symmetry it suffices to establish this for $j = 1, 2$. To get a contradiction, assume that we have η, t with

$$|\alpha_1^{-1}(\eta - \xi) - \alpha_1^{-1}\beta_1 t^{-1}|, |\alpha_2^{-1}(\eta - \xi) - \alpha_2^{-1}\beta_2 t^{-1}| \leq bt^{-1}.$$

Multiplying by $|\alpha_1|, |\alpha_2| \leq 1$, respectively, and comparing yields $|\beta_1 - \beta_2| \leq 2b$. This however is a contradiction to the choice of b and thus proves that the regions $T^{(j)}(x, \xi, s)$ are pairwise disjoint.

We now observe for each $T = T(x, \xi, s)$ with similar notation $T^{(j)} = T^{(j)}(x, \xi, s)$:

$$\begin{aligned} sS(G)(T) &= \int_T |G(y, \eta, t)| \, dy \, d\eta \, dt \\ &= \int_{T \setminus (T^{(1)} \cup T^{(2)} \cup T^{(3)})} |G(y, \eta, t)| \, dy \, d\eta \, dt + \sum_{j=1}^3 \int_{T \cap T^{(j)}} |G(y, \eta, t)| \, dy \, d\eta \, dt \\ &\leq \prod_{j=1}^3 \left(\int_{T \setminus T^{(j)}} |G_j(y, \eta, t)|^3 \, dy \, d\eta \, dt \right)^{1/3} \\ &\quad + \sum_{j=1}^3 \sup_{(y, \eta, t) \in T^{(j)}} |G_j(y, \eta, t)| \prod_{k \neq j} \left(\int_{T \setminus T^{(k)}} |G_k(y, \eta, t)|^2 \, dy \, d\eta \, dt \right)^{1/2}. \end{aligned}$$

Define the size

$$S_j(G)(T) := (s^{-1} \int_{T \setminus T^{(j)}} |G(y, \eta, t)|^2 \, dy \, d\eta \, dt)^{1/2} + \sup_{(y, \eta, t) \in T} |G(y, \eta, t)|.$$

Then we conclude from the previous considerations that

$$S(G)(T) \leq 4 \prod_{k=1}^3 S_k(G_k)(T),$$

where we have with log convexity estimated L^3 norms by L^2 and L^∞ norms.

By the outer Hölder inequality (Proposition 3.4) we obtain for the left-hand side of (6.2) the bound

$$C \prod_{j=1}^3 \|G_j\|_{L^{p_j}(X, \sigma, S_j)}$$

with exponents p_j as in Lemma 6.2. It remains to show for each j that

$$\|G_j\|_{L^{p_j}(X, \sigma, S_j)} \leq C \|f_j\|_{p_j}.$$

This follows from the generalized Carleson embedding (Theorem 5.1) after a reparameterization of the space X under the homeomorphism

$$\Phi_j : X \rightarrow X, (y, \eta, t) \mapsto (y, \alpha_j \eta + \beta_j t^{-1}, t).$$

Note that Φ_j maps $T_{\alpha_j, \beta_j}(x, \alpha_j^{-1}\xi, s)$ as defined in (5.1) to $T(x, \xi, s)$ as above, and it maps $T^b(x, \alpha_j^{-1}\xi, s)$ to $T^{(j)}(x, \xi, s)$ as above, and we have $F_j \circ \Phi_j = G_j$. This completes the proof of Lemma 6.2. \square

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