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Lecture notes on cluster algebras, by Robert J. Marsh, Zürich Lectures in Advanced Mathematics, European Mathematical Society (EMS), Zürich, ii+117 pp., ISBN 978-3-03719-130-9

1. Algebras

Algebras occur in almost all areas of mathematics and every mathematician knows several examples; e.g., the algebra $k[x]$ of polynomials over a field k, the algebra $\mathbb{C}$ of complex numbers over the reals, or the algebra $\mathrm{Mat}_n(R)$ of n-by-n matrices over a ring R.

1.1. **How to define an algebra?** There is no general answer to this question. It depends. Some algebras are defined on a given set by introducing the three operations that make up an algebra: addition, multiplication, and scaling by elements of the ground ring. For example, $\mathbb{C}$ as an algebra over $\mathbb{R}$ is given by vector addition and scaling, and complex multiplication; the matrix algebra $\mathrm{Mat}_n(R)$ is given by defining the matrix operations.

Other algebras are defined as a quotient of a given algebra by an ideal of relations. Here one may think for example of the symmetric algebras and the exterior algebras as quotients of the tensor algebras.

Yet another way of defining algebras is by constructing a subset inside a known algebra which is closed under the operations. In this case, the algebra is often defined by specifying a set of generators and taking its closure under addition, multiplication, and scaling.

Cluster algebras are of the last kind. They are subalgebras of a field of rational functions in several variables generated by a possibly infinite set of generators, which itself has to be constructed recursively.

1.2. **Why are algebras interesting?** Again there is no general answer. It depends. Sometimes one is interested in the elements of an algebra because of their action on some other object. The elements of the matrix algebra $\mathrm{Mat}_n(k)$ are acting on the n-dimensional k-vector space as linear transformations, or the elements of the endomorphims algebra $\mathrm{End}_R M$ of a module M over a ring R act on the module M.

Sometimes algebras are introduced to provide an algebraic framework for a problem that one wants to study. For example, group algebras are motivated by the study of group representations; enveloping algebras of Lie algebras are motivated by the study of representations of the non-associative Lie algebras.

Sometimes the algebra defines a geometric object, for example the algebra of functions on an algebraic variety.

A cluster algebra can fit each of the above criteria, depending on your point of view. They provide an algebraic framework for the study of canonical bases in Lie theory, and some cluster algebras arise as algebras of functions on a variety such as the coordinate ring of a Grassmannian.

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But the most fascinating feature of a cluster algebra, and this is not a typical property for an algebra, is that is has a rich underlying combinatorial structure. This combinatorial structure, although complicated, is very natural in the sense that it appears, sometimes in a hidden way, in many different areas of mathematics and physics. Cluster algebras have deep connections to the representation theory of finite-dimensional algebras, hyperbolic geometry, algebraic geometry and Poisson geometry [24], dynamical systems, string theory, and knot theory.

2. What is a cluster algebra?

Cluster algebras are defined by Fomin and Zelevinsky [19] as subalgebras of a field of rational functions in several variables $x_1, \ldots, x_n$, by constructing a set of generators. These generators are called *cluster variables.* Starting from the n initial variables $x_1, \ldots, x_n$, the set of all cluster variables is constructed recursively by a procedure called mutation. This mutation procedure is determined by the choice of a quiver[1] Q with n vertices which has no oriented cycles of length one or two.

The first mutation μ_i, where $i = 1, \ldots, n$, transforms the initial *cluster* $(x_1, \ldots, x_n)$ by replacing the variable x_i by the variable

$$x_i' = \frac{1}{x_i}\left(\prod_{i \to j} x_j + \prod_{j \to i} x_j\right), \tag{2.1}$$

where the first product runs over all arrows in the quiver Q that start at the vertex i and the second product runs over all arrows ending at i. Thus each mutation μ_i produces a new cluster of n cluster variables $(x_1, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_n)$ by exchanging the old variable x_i for a new variable x_i' that is given as a rational function of the initial cluster $(x_1, \ldots, x_n)$. Its defining equation (2.1) is called the *exchange relation.*

Moreover, the mutation μ_i also changes the quiver Q locally around the vertex i. Thus μ_i produces a new cluster and a new quiver. From this new data, one can again perform n mutations and so on. Mutating once more at the same position i will bring us back to the initial cluster, but mutating in other directions will, in general, produce new cluster variables. The computations involved in the mutation procedure are typically very difficult, and one of the main problems in the theory of cluster algebras is to find good computable formulas for the cluster variables.

2.1. **Laurent phenomenon and positivity.**

By definition, the cluster variables are rational functions, but Fomin and Zelevinsky proved in [19] that they are actually Laurent polynomials (meaning that the denominator consists of a single monomial) with integer coefficients. This is called the *Laurent phenomenon.* The coefficients are now known to be non-negative [25, 30], but this had been an open problem for more than ten years. Several different bases have been constructed for cluster algebras of different types; see [4, 22, 25, 29, 33].

2.2. **Finite type and finite mutation type.**

A cluster algebra is said to be *of finite type* if the number of its cluster variables is finite and *of finite mutation type* it the number of quivers that can be obtained from Q by a finite sequence of mutations is finite. Finite type cluster algebras are of finite mutation type but the converse is not true.

[1] A quiver is an oriented graph.

A remarkable result shown in [20] classifies the finite type cluster algebras as precisely those whose initial quiver can be obtained by mutation from a quiver of Dynkin type $\mathbb{A}_n, \mathbb{D}_n$, or $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$. Notice that the Dynkin diagrams also classify semisimple Lie algebras as well as the hereditary algebras that have only finitely many indecomposable modules up to isomorphism.

For the finite mutation type, it is easy to see that every cluster algebra of rank $n = 1, 2$ is of finite mutation type. For $n \geq 3$, it is shown in [15] that a cluster algebra is of finite mutation type if and only if its quiver is the adjacency quiver of a triangulation of a Riemann surface with marked points or it is one of eleven exceptional types. For example, the Dynkin type $\mathbb{A}_n$ corresponds to a disk with $n + 3$ marked points on the boundary, the type $\mathbb{D}_n$ corresponds to a disk with one puncture and n marked points on the boundary, and the types $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$ are among the eleven exceptional types. The cluster algebras associated to surfaces are said to be of surface type.

3. Relations to other areas

Cluster algebras are related to many areas of mathematics and physics. Here we will only focus on the relation to hyperbolic geometry and representation theory.

3.1. **Surface type.** The relation between cluster algebras and hyperbolic geometry has been established in [16–18, 23]. To every oriented Riemann surface with boundary and marked points one can associate a cluster algebra in such a way that the cluster variables correspond bijectively to arcs in the surface, clusters correspond to ideal triangulations, mutations to flips of diagonals in quadrilaterals, and exchange relations to generalized Ptolemy relations.

Explicit combinatorial formulas for cluster variables in terms of perfect matchings of certain graphs, called snake graphs, have been obtained in [32]. The Laurent polynomial expansion of the cluster variable is given as a sum over all perfect matchings of the weight and the height of the matching. For surfaces without punctures these formulas were used in [33] to define canonical bases for the cluster algebra in terms of curves in the surface.[2]

Moreover, the relations in the cluster algebra are given geometrically by locally replacing a crossing $\times$ of two segments of curves with the sum of segments $\asymp$ and $\supset\subset$, respectively; see [34]. Thus the curves in the surface completely determine the algebraic and combinatorial structure of the cluster algebra. In [11–13] this structure was reinterpreted purely in terms of the snake graphs, bypassing the arcs in the surface and providing a more efficient method for computation.

3.2. **Cluster category.** The relation to representation theory was established by the introduction of cluster categories in [5] (and [8] for type $\mathbb{A}$).[3] Cluster categories provide a categorification of cluster algebras. The cluster variables correspond bijectively to the indecomposable (rigid) objects in the cluster category; clusters correspond to tilting objects and exchange relations to exact triangles in the category.[4]

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[2]In the case were there is only one marked point, this uses [10].

[3]Generalizations were given later in [2, 36] using quivers with potentials developed in [14]. Another connection to representation theory via preprojective algebras was given in [21]. A different type of categorification was introduced in [26, 35].

[4]An object T in the cluster category is called *rigid* if $\mathrm{Ext}^1(T, T) = 0$, and it is *tilting* if it is maximal with respect to this property.

This connection to cluster algebras has triggered a spectacular development in representation theory introducing or advancing the theory of cluster-tilted algebras [6, 8], 2-Calabi–Yau categories [27], τ-tilting theory [1], relation extensions [3], and more. On the other hand, the categorification provided a better understanding of cluster algebras, for example via expansion formulas for cluster variables in terms of Euler–Poincaré characteristics of quiver Grassmannians, [7, 9], and construction of bases for the cluster algebra [22].

4. The book

Marsh's book is a short and very well-written introduction to cluster algebras that will be valuable to graduate students as well as researchers in many disciplines. It can be used as a textbook for a graduate topics course or for self-study. Detailed references are given where proofs are not included.

After a short introductory section, the book gives a concise definition of cluster algebras and even presents an alternative description of the exchange relations using polynomials. This approach was actually the original definition of Fomin and Zelevinsky and was used in the first proof of the Laurent phenomenon. Marsh points out very nicely that some flexibility is needed in the definition of cluster algebras when it comes to the question of invertibility of the coefficients in a cluster algebra.

Marsh then gives a short introduction to reflection groups, which is interesting in its own right, and uses it to present the finite type classification of cluster algebras.

The book then touches upon various aspects of cluster algebras: the introduction of generalized associahedra associated to cluster algebras of finite type; periodicity related to cluster algebras, including periodicity of integer sequences, mutations sequences, as well as a sketch of Keller's proof of Zamolodchikov's periodicity conjecture. The author also discusses cluster algebras of finite mutation type and their classification using triangulated surfaces. The last chapter contains a self-contained account of the cluster structure of the homogeneous coordinate ring of the Grassmannian $Gr(k, n)$ following Scott [37].

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