BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 53, Number 3, July 2016, Pages 507–512
http://dx.doi.org/10.1090/bull/1525
Article electronically published on January 14, 2016

Lectures on the Riemann zeta function, by H. Iwaniec, University Lecture Series, Vol. 62, American Mathematical Society, Providence, RI, 2014, viii+119 pp., ISBN 978-1-4704-1851, List price US$40, Member price US$32

I hope that Hilbert hasn't jinxed us with his prediction of the future. He purportedly said during a lecture that a proof of the Riemann Hypothesis was forthcoming within a few years, that some in the room might live to see a proof of Fermat's Last Theorem, but that the proof of the irrationality of $2^{\sqrt{2}}$ was centuries away. But, Gelfond and Schneider proved the latter a few years later and Wiles proved Fermat's Last Theorem within that century. So one hopes that Hilbert did not have it exactly backwards!

Analytic number theorists like to claim that the Riemann Hypothesis is the most important unsolved problem in mathematics. With Fermat's Last Theorem and the Poincaré Conjecture out of the way, it is not so difficult to make that argument convincingly! But also the mounting evidence in its favor and the ubiquity of L-functions that also have Riemann Hypotheses convince us of the centrality of this problem. The fact that it has been more than 150 years since Riemann posed his problem adds to the argument. Henryk Iwaniec's new book on the Riemann zeta function gives a fascinating perspective on the subject that will be relished by beginners and experts alike. Like his other beautiful books, Iwaniec gives us a modern treatment of something of great interest to contemporary mathematicians.

I wish I could say that the proof of the Riemann Hypothesis is just around the corner, but I'm afraid that would be extremely misleading. But what I can say is that there is more evidence than ever that it is true. The burning question about the Riemann Hypothesis is "Why is it seemingly so hard?" The best answer I can give is that the Riemann zeta function is a complicated function; more complicated than the other functions we generally encounter. It starts out reasonably elementarily. It is defined by a series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

valid for $s = \sigma + it$ with $\sigma > 1$. It can also be expressed as a product over primes

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1};$$

Euler discovered this analytic formulation of the fundamental theorem of arithmetic. Then Riemann discovered the analytic continuation and functional equation, that

$$\xi(s) = \frac{1}{2} s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

is an entire function of order one which satisfies

$$\xi(s) = \xi(1-s).$$

Riemann showed that $\xi(s)$ has zeros—lots of them—in the critical strip $0 \le \sigma \le 1$ and that these zeros are intimately connected with the prime numbers. He proved that the zeros get denser as one moves up the critical strip but conjectured that

2000 *Mathematics Subject Classification.* Primary 11M06, 11M26.

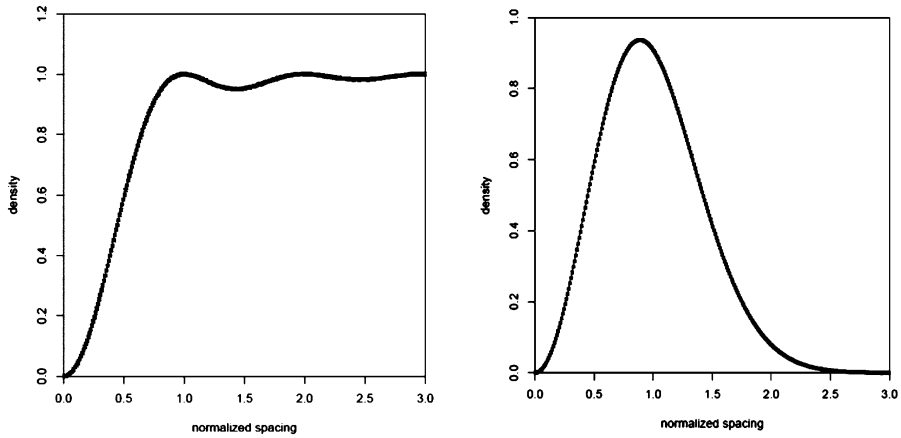
they all are on the critical line $\sigma = 1/2$, the famous Riemann Hypothesis (RH for short). In work since Riemann, we do know that if there are zeros off the line, they are rare. Classical density results, which are nicely covered in Iwaniec's new book, show that zero is an exceptional value for $\zeta(s)$ when the real part of s is not $1/2$.

We know lots of examples of functions that have all of their zeros on a single line: $\sin z$, $1/\Gamma(z)$, the Bessel function $J_0(z)$, and so on. Pólya and Szego's *Problems and Theorem in Analysis* is full of examples of functions and a variety of techniques to prove that various functions have all real zeros, say. But these example functions typically have the feature that their zeros are well spaced on the line in question. The zeros are in approximate arithmetic progression. On the contrary we believe that the zeros of the Riemann zeta function have a very different spacing, namely a spacing like the eigenvalues of a large random Hermitian matrix. The zeros repel each other—they do not like to be too close together—and they have a distribution that is pleasing to the eye. But they sometimes get very close together and sometimes are very far apart, following the laws of distribution discovered by Wigner and Dyson and the researchers studying Random Matrix Theory (RMT) in the 1950s. And this is the tricky part. Every once in a while the zeros are really close together—so close that it is hard to distinguish whether one is observing a double zero or perhaps two zeros that are very close together but *off* the critical line. So whatever method we eventually find to prove the Riemann Hypothesis has to be capable of handling such subtleties. And so far we do not have anything remotely close. It is remarkable in some sense that we can prove that a positive proportion of zeros are on the line. In fact there are two distinct such proofs: the first by Selberg [Sel] and the second by Levinson [Lev]. Iwaniec's treatment of the Levinson method for proving zeros on the critical line is especially useful and has never previously appeared in book form. Moreover, Iwaniec gives a new treatment that is more general than what appears in the literature and which gives hints of links to sieve theory—of which Iwaniec is a grandmaster.

Neither Selberg's nor Levinson's method is likely to lead to a proof of RH. However, the lack of a proof of RH has not held us back from understanding $\zeta(s)$ in different ways.

We now believe that the distributions of RMT are actually everywhere—like the Gaussian distribution which has so dominated our thoughts on averages for the past few centuries. And the fact that they seem to so clearly model the behavior of the Riemann zeta function suggests that there must be a spectral interpretation of the zeros, as apparently suggested by Hilbert and Pólya, independently. We just need to identify the operator.

Indeed the often-told story of Montgomery and Dyson in 1972 realizing that the pair correlation of the zeros of the Riemann zeta function matched that of the pair correlation of eigenvalues of large random Hermitian matrices ushered in an era of uncovering just how important the language and tools of RMT are for the study of the Riemann zeta function and L-functions (see [HLM]). In some sense this connection was not really appreciated until Odlyzko's calculations of zeros of $\zeta(s)$ at heights around 10^{23} and his subsequent spectacular graphs in Figure 1 showing the agreement between the two pair correlations. This was followed a bit later by Hejhal's calculation that the triple correlations also match and Rudnick and Sarnak's discovery that the n-correlationa also match. Then Katz and Sarnak [KaSa] investigated the finer distributions of L-functions in families, in the function field setting where the Riemann Hypothesis is known (Deligne's theorem!), and they



(A) Pair correlation (B) Nearest neighbor

FIGURE 1. Odlyzko's graphics with scatterplots of the distribution of zeros of $\zeta(s)$ on top of the relevant RMT distributions.

revealed that not only the unitary group enters (as the symmetry group for Hermitian matrices) but also the classical orthogonal and symplectic groups with Haar measure and distributions of the eigenvalues of their members enter the picture.

Independently other researchers were looking into a different set of statistics, mainly the distribution of *values* of L-functions in families. This effort was concentrated on understanding moments, for example

$$\int_0^T |\zeta(1/2+it)|^{2k}\ dt.$$

Initially these were useful to obtain some progress toward the Lindelöf Hypothesis that for every $\epsilon > 0$,

$$\zeta(1/2+it) \ll (1+|t|)^{\epsilon}.$$

Pointwise bounds on $\zeta(1/2+it)$ are useful for estimating the error terms in averages of arithmetic functions; and for other families of L-functions such estimates are useful for proving the equidistribution of some arithmetic sequence. What was not known until recently is that these moments have a very beautiful simple structure of their own, one which also utilizes RMT in a fundamental way.

Hardy and Littlewood showed in 1918 that

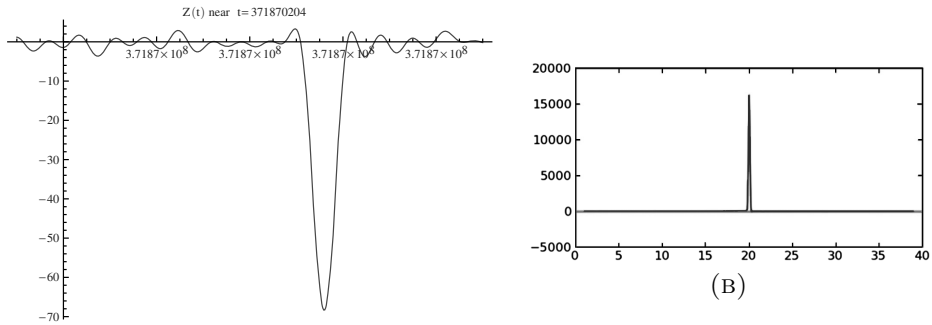
$$\frac{1}{T}\int_0^T |\zeta(1/2+it)|^2\ dt \sim \log T,$$

and Ingham in 1926 that

$$\int_0^T |\zeta(1/2+it)|^4\ dt \sim \frac{1}{2\pi^2}\log^4 T = 2 \sim 2\prod_p \left(1-\frac{1}{p}\right)\left(1+\frac{1}{p}\right)\frac{\log^4 T}{4!}.$$

Conrey and Ghosh conjectured in the early 1990s that

$$\int_0^T |\zeta(1/2+it)|^6\ dt \sim 42\prod_p \left(1-\frac{1}{p}\right)^4\left(1+\frac{4}{p}+\frac{1}{p^2}\right)\frac{\log^9 T}{9!},$$



(A) (B)

FIGURE 2. Hardy's Z-function in selected intervals. Graphics by J. Bober and G. Hiary.

and Conrey and Gonek in 1998 that

$$\int_0^T |\zeta(1/2+it)|^8 \, dt \sim 24024 \prod_p \left(1-\frac{1}{p}\right)^9 \left(1+\frac{9}{p}+\frac{9}{p^2}+\frac{1}{p^3}\right) \frac{\log^{16} T}{16!}.$$

We see that the power on the $\log T$ goes up quadratically, another indication of the wild behavior of $\zeta(s)$. Figure 2 shows Hardy's $Z(t)$ function (for which $|Z(t)| = |\zeta(1/2+it)|$) in two intervals and illustrates the difficulty of determining moments, which are largely governed by the few very large values.

Hiary [Hia] has improved the algorithm of Odlyzko and Schönhage and has computed some spectacularly large values of $\zeta(1/2+it)$ at a height around $t = 10^{30}$ (see Bober's web page [JoBob] for some great graphics on large values of $\zeta(s)$).

In 2002 Keating and Snaith, using RMT arrived at a general conjecture

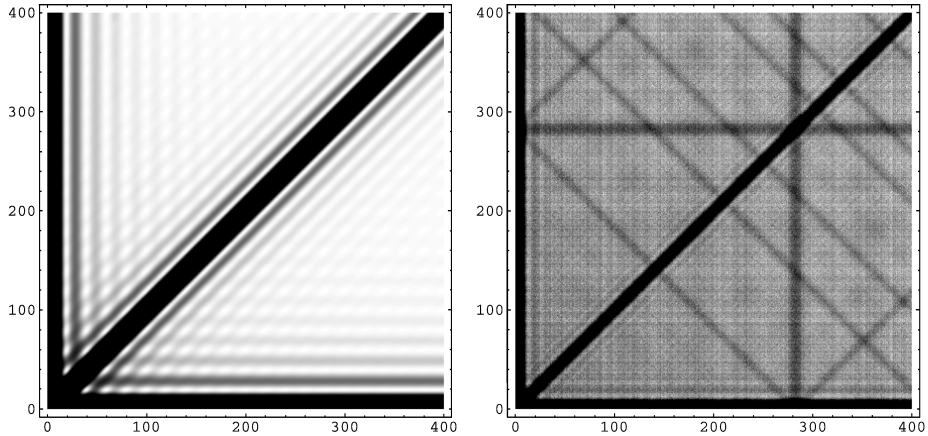
$$\int_0^T |\zeta(1/2+it)|^{2k} \, dt \sim g_k \prod_p \left(1-\frac{1}{p}\right)^{(k-1)^2} \sum_{j=0}^{k-1} \frac{\binom{k-1}{j}^2}{p^j} \frac{\log^{k^2} T}{k^2!},$$

where

$$g_k = \frac{k^2!}{1^1 \cdot 2^2 \cdot \ldots \cdot k^k \cdot (k+1)^{k-1} \cdot \ldots \cdot (2k-1)^1};$$

see [KS]. Note that $g_1 = 1$, $g_2 = 2$, $g_3 = 42$, and $g_4 = 24024$. While this conjecture, accomplished by computing the $2k$th moment averaged over the unitary group $U(N)$ of the characteristic polynomials of the matrices therein, was spectacular and followed on earlier connections with RMT found by Montgomery and Dyson, Odlyzko, Rudnick and Sarnak, Hejhal, and Katz and Sarnak, it suffered from the problem that it was not testable numerically. This issue was solved later when a conjecture for the $2k$th moments with all the main terms and with a power of T savings was produced by Conrey, Farmer, Keating, Rubinstein, and Snaith [CFKRS]. In addition this conjecture, affectionately known as "the recipe" could produce conjectures for any integral moments of any family of L-functions. Subsequently the ratios conjecture, which was an extension of the recipe, was produced by Conrey, Farmer, and Zirnbauer [CFZ] to predict averages such as

$$R_\zeta(\alpha,\beta,\gamma,\delta) = \frac{1}{T}\int_0^T \frac{\zeta(s+\alpha)\zeta(1-s+\beta)}{\zeta(s+\gamma)\zeta(1-s+\delta)} \, dt,$$



(A) Unitary (B) Riemann

FIGURE 3. Triple correlation of unitary matrices and of Riemann zeros. Graphics by N. C. Snaith.

where $s = 1/2 + it$ and $\Re\gamma, \Re\delta > 0$. The ratios conjecture asserts that there is an $\eta > 0$ such that this average is

$$\int_0^T \left(\frac{\zeta(1+\alpha+\beta)\zeta(1+\gamma+\delta)}{\zeta(1+\alpha+\delta)\zeta(1+\beta+\gamma)} A_\zeta(\alpha,\beta,\gamma,\delta) \right.$$
$$\left. + \left(\frac{t}{2\pi}\right)^{-\alpha-\beta} \frac{\zeta(1-\alpha-\beta)\zeta(1+\gamma+\delta)}{\zeta(1-\beta+\delta)\zeta(1-\alpha+\gamma)} A_\zeta(-\beta,-\alpha,\gamma,\delta) \right) dt + O\left(T^{1-\eta}\right),$$

where A is given as a convergent product over primes:

$$A_\zeta(\alpha,\beta,\gamma,\delta) = \prod_p \frac{\left(1-\frac{1}{p^{1+\gamma+\delta}}\right)\left(1-\frac{1}{p^{1+\beta+\gamma}}-\frac{1}{p^{1+\alpha+\delta}}+\frac{1}{p^{1+\gamma+\delta}}\right)}{\left(1-\frac{1}{p^{1+\beta+\gamma}}\right)\left(1-\frac{1}{p^{1+\alpha+\delta}}\right)}.$$

Such a formula provides a different conceptual way to determine the pair correlation of the zeros of ζ (and extensions of this formula to more complicated averages can be used for other correlations). But the formula also highlights the deviations from RMT that appear in such correlations. After all RMT does not know anything about finite primes! In particular, lower order terms in the pair correlation of Montgomery can be determined (i.e., conjectured). Figure 3 shows a graphic that compares the triple correlation of (unitary) eigenvalues with the triple correlation of zeros of ζ. Notice all the subfeatures, such as diagonal bands, that stretch from ordinates of zeros of $\zeta(s)$ on the vertical axis to ordinates of zeros on the horizontal axis. See [CSn] for many more examples of the ratios conjecture in action.

Research on this wonderful world of L-functions is exploding. There is a program going on right now at ICERM entitled "Computational Aspects of the Langlands Program". A significant effort by scores of people have created the L-functions and Modular Forms Data Base (see `lmfdb.org`) to chart the rich web of interconnections between L-functions of every imaginable variety and their friends and to compute their first few zeros. This includes L-functions arising from Maass forms on GL(2), GL(3), and Sp(4) (see [FKL]), whose coefficients are believed to be transcendental.

In every case RH holds for the initial zeros computed. And we begin to see the vast scope of the subject and this undertaking which will certainly occupy researchers throughout the rest of this century. Iwaniec's lovely new book is a splendid window on what many consider the greatest mathematics problem of all time!

References

[JoBob] Jonathan Bober, `http://www.maths.bris.ac.uk/~jb12407/`

[CFKRS] J. B. Conrey, D. W. Farmer, J. P. Keating, M. O. Rubinstein, and N. C. Snaith, *Integral moments of L-functions*, Proc. London Math. Soc. (3) **91** (2005), no. 1, 33–104, DOI 10.1112/S0024611504015175. MR2149530 (2006j:11120)

[CFZ] Brian Conrey, David W. Farmer, and Martin R. Zirnbauer, *Autocorrelation of ratios of L-functions*, Commun. Number Theory Phys. **2** (2008), no. 3, 593–636, DOI 10.4310/CNTP.2008.v2.n3.a4. MR2482944 (2009j:11138)

[CSn] J. B. Conrey and N. C. Snaith, *Applications of the L-functions ratios conjectures*, Proc. Lond. Math. Soc. (3) **94** (2007), no. 3, 594–646, DOI 10.1112/plms/pdl021. MR2325314 (2009a:11179)

[FKL] David W. Farmer, Sally Koutsoliotas, and Stefan Lemurell, *Maass forms on* GL(3) *and* GL(4), Int. Math. Res. Not. IMRN **22** (2014), 6276–6301. MR3283005

[Hia] Ghaith Ayesh Hiary, *Fast methods to compute the Riemann zeta function*, Ann. of Math. (2) **174** (2011), no. 2, 891–946, DOI 10.4007/annals.2011.174.2.4. MR2831110 (2012g:11154)

[KaSa] Nicholas M. Katz and Peter Sarnak, *Zeroes of zeta functions and symmetry*, Bull. Amer. Math. Soc. (N.S.) **36** (1999), no. 1, 1–26, DOI 10.1090/S0273-0979-99-00766-1. MR1640151 (2000f:11114)

[KS] J. P. Keating and N. C. Snaith, *Random matrix theory and* $\zeta(1/2 + it)$, Comm. Math. Phys. **214** (2000), no. 1, 57–89, DOI 10.1007/s002200000261. MR1794265 (2002c:11107)

[Lev] Norman Levinson, *More than one third of zeros of Riemann's zeta-function are on* $\sigma = 1/2$, Advances in Math. **13** (1974), 383–436. MR0564081 (58 #27837)

[LMFDB] *The L-functions and modular forms database*, `http://www.lmfdb.org/`

[HLM] H. L. Montgomery, *The pair correlation of zeros of the zeta function*, Analytic number theory (Proc. Sympos. Pure Math., Vol. XXIV, St. Louis Univ., St. Louis, Mo., 1972), Amer. Math. Soc., Providence, R.I., 1973, pp. 181–193. MR0337821 (49 #2590)

[Sel] Atle Selberg, *On the zeros of Riemann's zeta-function*, Skr. Norske Vid. Akad. Oslo I. **1942** (1942), no. 10, 59. MR0010712 (6,58a)

Brian Conrey

American Institute of Mathematics, San Jose, California;
and University of Bristol, United Kingdom

E-mail address: `conrey@aimath.org`