BOOK REVIEWS

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Elements of mathematics: from Euclid to Gödel, by John Stillwell, Princeton University Press, Princeton, 2016, xiv+422 pp., ISBN 978-0-691-17168-5, US $39.95.

Stillwell's *Elements of mathematics* is a wonderful book that aims to "give a birds-eye view of elementary mathematics", to explain why it is elementary, and then to briefly indicate where and when nonelementary (advanced) aspects of mathematics were developed. Stillwell organizes his book into eight different fields of mathematics—arithmetic, computation, algebra, geometry, calculus, combinatorics, probability, and logic—noting, of course, that these areas overlap to some extent. After an introductory chapter dealing with the most elementary parts of each of these topics, each further chapter discusses aspects of the mathematical topic that Stillwell considers to be elementary, considers some interesting problems, and provides proofs of a few results. Occasionally, he shows where elementary mathematics ends and advanced mathematics begins. He gradually develops the point that advanced mathematics consists of ideas requiring the concept of real numbers and, more generally, the concept of infinity. In fact, his chapter on logic shows how these concepts relate to one another and why we must consider them as advanced. In the book's final chapter, Stillwell provides more details on advanced aspects of each of the eight mathematical fields.

The title of this book is meant to recall the title of the most famous mathematical text of all time, the *Elements* of Euclid, dating from around 300 BCE. In fact, Euclid's book had the same general goal as the current one: to present what Euclid considered to be elementary mathematics, mathematics that could be based on a very limited set of axioms. Euclid's work covered only two of Stillwell's eight areas, namely, geometry and arithmetic, because only those two fields could be set out in an organized fashion at the time. Stillwell's "computation" existed, of course, in Greece, but it was not thought to be an area worthy of real study. On the other hand, Stillwell's "logic" was studied in Greece, particularly by Aristotle before Euclid's time and Chrysippus afterwards. Euclid himself used the logic of these thinkers in his work but did not analyze it at all. Whether "calculus" was an area studied in Greece can be debated. Certainly, Euclid dealt occasionally with the idea of a tangent and showed that the ratio of areas of circles to one another is the same as that of the squares on their diameters. (Since Euclid did not do any computations, he made no attempt to give a formula such as $A = \frac{\pi}{4}d^2$.) He also gave indications of how to find volumes of certain solids. A bit later Archimedes found additional

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areas and volumes but did not claim to have a systematic method of calculating these in general. And Apollonius was able to find tangents to various conic sections but did not attempt to generalize this method to other curves. Stillwell's other areas, algebra, combinatorics, and probability, were not areas to which the Greeks made any contribution.

But did the Greeks study advanced mathematics at all, according to Stillwell's definition, or was everything elementary? And a similar question can be asked about those civilizations that developed algebra, combinatorics, and probability. Stillwell himself gives brief historical remarks for each of his eight mathematical areas but does not directly answer the question. Each of Stillwell's chapters covers a wide historical swath, often containing material from ancient times up until the nineteenth or twentieth centuries. So according to Stillwell, elementary mathematics is not a historical construct at all. Elementary mathematical topics were being considered in all epochs of the history of mathematics. But how about advanced mathematics? Does advanced mathematics only consist of mathematical ideas in the past century or two, or were some of these advanced ideas worked out in ancient or medieval or Renaissance mathematics? The focus of this review, then, will be on analyzing Stillwell's meaning of elementary mathematics and challenging it at various points. Elementary mathematics in the present day ought to mean ideas that can be explained to secondary students or students in the first two years of university. We will see that this generally means that elementary mathematics is mathematics whose origin is in the nineteenth century or earlier and that advanced mathematics consists of ideas that historically have proved difficult. We will explain this in more detail as we proceed through the chapters.

We begin with arithmetic, that is the so-called "higher arithmetic", which we generally call number theory. Stillwell's chapter on arithmetic covers a long historical time frame, beginning with the Euclidean algorithm and ending with the Pell equation. But there seems to be a good deal of arithmetic missing. One of the first topics that was considered by the Greeks was that of a figured number. In fact, the Pythagoreans understood how to construct square numbers by adding successive odd numbers together, and they also understood the relationship between triangular numbers and oblong numbers. This topic of figured numbers came up again in the work of Nicomachus (first century CE), who extended the idea to pentagonal and hexagonal numbers. And then later on, much detail was worked out in medieval Islamic civilization, especially by ibn Mun'im (thirteenth century), who showed the relationships among the different figured numbers and how to calculate sums of sequences of these numbers. Prime numbers, however, are the focus of Stillwell's work, and he shows how Euclid's basic definition was expanded in the nineteenth century to encompass quadratic integers, particularly Gaussian integers. These numbers have the property of unique factorization into primes and thus can be treated very similarly to ordinary integers. Of course, the Gaussian integers are not the only quadratic domain with unique factorization. It would have been useful to the reader to see some examples of a nonunique-factorization quadratic domain. But to then go further with this topic, however, requires the notion of an ideal, an idea that is reasonable to call advanced. These were invented by Dedekind in the 1870s to try to restore unique factorization to algebraic integers.

Computation is the next topic. Stillwell's basic aim in this chapter is to determine the feasibility of certain computations, such as finding $79^{37} \pmod{107}$. This leads to a discussion of P and NP problems and then to a description of a

Turing machine. It seems to me that Turing machines themselves are not an elementary concept. True, they are designed to encompass the meaning of what a computation is, but the fact that this idea was only discovered in 1936 leads me to believe that one should not consider it elementary at all. The concept of P and NP problems is quite a deep result, because proving that a given problem belongs to one or the other of these classes is often a long and difficult process. But then Stillwell leaves elementary mathematics to discuss some unsolvable problems. Of course, historically, computation has always been central to mathematical problem solving. In fact, it was the introduction of the Hindu-Arabic place-value system into Europe that helped European mathematicians produce important advances in the fifteenth century. Basic arithmetical calculations are generally easier in this system, but it took Europe several centuries to fully understand this. Similarly, even though the place-value system had earlier spread to the Islamic domains from its origins in India, it also took some time for the methods to be accepted there as well. Fibonacci's 1202 treatise *Liber abaci* was but one of the roots by which the place-value system reached Europe. Other routes include the Hebrew work by Abraham ibn Ezra, in which the author used Hebrew letters for the first nine digits, but then an "O" for zero. Most of the texts written by the *maestri di abbaci* in Italy in the thirteenth and fourteenth centuries began with a study of computation and demonstrated the methods for arithmetic computations, some of which differ from the standard algorithms used today.

What is elementary algebra? In normal discourse, this means the algebra that is studied in high school dealing mainly with the solving of equations as well as the manipulations necessary to achieve that goal. In general, these equations are limited to linear and quadratic ones, or systems of these, since solving third degree equations often requires a somewhat complicated formula discovered in Italy early in the sixteenth century. The techniques necessary for solving single quadratic equations have their roots in ancient Mesopotamia, while the process of solving systems of linear equations goes back to China around the beginning of our era. Islamic mathematicians, beginning in the ninth century, developed this algebra in detail, although using words and not symbols, and several of their elementary textbooks were translated into Latin and helped teach algebra to western Europeans. Once several Italian mathematicians found procedures for solving third and fourth degree equations, mathematicians began to search for similar methods for higher degree equations. When there was no success for this quest, it began to dawn on people that perhaps there was no general solution to a fifth degree equation by radicals. Eventually, this result was proved by Niels Abel in 1826. But the search for solutions involved looking at permutations of roots, and so the idea of a permutation became one of the notions leading to abstract group theory, a notion that was finally axiomatized in the 1870s. Meanwhile, Galois had discovered the relationship between the solvability of a polynomial equation with radicals and the structure of the group of permutations on these roots. Since this relationship involved what are now known as fields, that notion was also axiomatized about the same time as the notion of a group, and, of course, the field axioms essentially include the group axioms for both addition and multiplication. Initially, all fields were subfields of the field of complex numbers, but it was eventually realized that the set of residues modulo any prime number formed a finite field. Then other finite fields were considered that were extensions of these prime fields. Stillwell wants to relegate groups to advanced mathematics, while considering fields to be elementary. But it seems to me that, because these concepts are so closely related and are usually taught together, both of these should be considered elementary, as well as the corresponding study of rings. Now there are certainly topics in group theory, as well as in ring theory and field theory, that must be considered advanced, but in my opinion the basics are elementary. In fact, one of the motivations for defining an abstract group in the first place was Kronecker's noticing that Gauss's arguments in his theory of forms and Kummer's arguments in working out details of his theory of ideal complex numbers were quite analogous. As Kronecker wrote,

> The very simple principles on which Gauss's method rests are applied not only in given context but also frequently elsewhere, in particular in the elementary parts of number theory. This circumstance shows, and it is easy to convince oneself, that these principles belong to a more general, abstract realm of ideas. It is therefore appropriate to free their development from all unimportant restrictions, so that one can spare oneself from the necessity of repeating the same argument in different cases.

Now, it is true that Kronecker's "principles" were the basis of the theory of commutative groups. But Hamilton had already shown a context for noncommutative multiplication in his quaternions (which certainly seem to be an elementary concept), and Arthur Cayley, even before the work of Kronecker, had already abstracted the general axioms of a (possibly noncommutative) group from a consideration of certain transformations. In fact, Cayley then used rather simple arguments to determine all the possible groups of order up to 8. Other mathematicians soon took up the same idea, and the notion of a group became part of mainstream mathematics by the early 1880s, with a special name, Abelian, given to groups with commutative multiplication. Although Stillwell claims (p. 147) that noncommutative multiplication is difficult because it is quite close to the operations of a Turing machine and because the "word problem" for noncommutative multiplication is unsolvable, calculations in finite noncommutative groups are quite straightforward, and students in a first abstract algebra course generally have no problems with them.

Now, the elementary algebra that Stillwill describes is what we might call "algorithmic", since solving equations is generally done through some algorithm. But it was the joining of algebra with geometry, which occurred shortly after Viète had developed a flexible algebraic symbolism, that gave algebra new dynamic power and made it essential to the developments in physics and astronomy in the late sixteenth and early seventeenth centuries. Stillwell does deal with analytic geometry, certainly an elementary subject in his chapter on geometry, but does not discuss some other aspects of geometry that I would think must be considered as elementary. Certainly, large portions of Euclid's *Elements* are elementary, almost by definition, and Stillwell emphasizes some of Euclid's ideas about angles, areas, volumes, and constructions. But he then turns to the use of algebra in dealing with geometry, instead of discussing many other geometric ideas that could be accomplished without algebra. In particular, Stillwell leaves out entirely any discussion of conic sections, certainly the most important species of curves dealt with in antiquity and ones with numerous applications. It is an interesting topic, and much can be developed without the algebraic machinery of the seventeenth century. Furthermore, although in the chapter on combinatorics Stillwell gives a neat graph-theoretic proof of Euclid's result that there are only five regular polyhedra, there is much else that can be done with these solids. For example, there are elementary ways of showing the relationships among the surface areas and volumes of these solids, some of which were detailed in the spurious Book XIV of the *Elements*, with many more being worked out by several Arabic- and Hebrew-speaking authors of the Middle Ages. The proofs of these relationships are a bit complicated, but they are elementary in that they do not involve any difficult or very deep concepts. One might argue that Euclid's proposition XII-7 determining the volume of a tetrahedron is nonelementary, because its proof relies on the method of exhaustion, but since the result itself was known in Mesopotamia and probably in Egypt and since the method of exhaustion avoids any actual use of infinity, I think it reasonable to call the result elementary.

One major elementary topic that Stillwell seems to ignore is trigonometry, which could be considered part of either geometry or calculus, depending on what aspects of the theory one was interested in. In any case, I think that the ideas of trigonometry are elementary, given that they begin with basic principles of similarity and then use some results of Euclidean geometry for further development. In fact, calculating a sine table is a useful exercise for students in that it requires the solution of a quadratic equation to get beyond the simple angles of 30, 45, and 60 degrees. And then, once one has proven Ptolemy's theorem, again just using the notion of similar triangles, one can develop the addition and subtraction theorems for the sine as well as the half angle theorem. While using no calculation more advanced than the extraction of square roots, one can determine the sine of any angle that is a multiple of three degrees to any accuracy one desires, and then one can easily be convinced that the sine function is nearly linear for small angles. It is then easy to conjecture the impossibility of trisecting an angle using Euclidean means. But one can also see the usefulness of linear interpolation to calculate a reasonably accurate value of the sine of one degree. Islamic and Jewish mathematicians of the medieval period improved on Ptolemy in finding new ways to imiprove the accuracy of this computation—and they were certainly using elementary mathematics. To get to analytic trigonometry, we must turn to Stillwell's next subject, calculus.

So what is elementary calculus? Here, Stillwell basically argues that elementary calculus is the calculus of the elementary functions. He begins with the notion of the derivative and proceeds to calculate the derivative of powers of x, noting that any limits involved in this determination are so obvious that it is unnecessary to give a formal definition of that concept. By then considering the sum, difference, product, and quotient rules for derivatives, as well as the inverse function rule and the chain rule, all of which have quite intuitive proofs, the differentiation of rational and explicit algebraic functions is now "elementary". But in developing the integral calculus, Stillwell chooses to use the derivative of the area function. This gives a neat demonstration of the fact that the area under $y = t^k$ from 0 to x is $\frac{x^{k+1}}{k+1}$. But given that these areas were known before the notion of derivative was well understood, it seems more elementary to calculate these areas directly using a formula for the sum of integral powers, and the essential part of this formula can be developed without a lot of trouble. The advantage in teaching how to calculate areas this way is that it makes the fundamental theorem of calculus virtually obvious for powers of x and for polynomials in general. Be that as it may, Stillwell presents the standard development of the natural logarithm via areas and then develops the exponential function by using the inverse function property. He then develops the power series for the logarithm from the definition as an integral. Although he classified this development as elementary, he notes that a development of the power series for the exponential function is advanced, claiming that any justification of the series requires advanced calculus. Yet it seems that if you can develop a power series for the logarithm, one really ought to be able to develop one for the exponential, as well as for the other basic elementary functions, the sine, the cosine, the tangent, and their inverses.

As Stillwell notes, Leonhard Euler wrote one of the first books designed explicitly to prepare students for calculus, the *Introduction to the analysis of the infinite* in 1748. Thus, for Euler, all the material in this book was elementary in the sense that it would be taught to students before they began the study of calculus. The central theme of the book is the development of power series (thus the title). In particular, Euler derives the power series of all the elementary functions mentioned above, as well as other functions. Why does Stillwell then consider these to be advanced topics, while Euler considered them elementary? The basic reason is that Euler's derivations were not rigorous in the modern sense because he made use of both infinitely small and infinitely large numbers. Now, such numbers do not exist, but Euler had a wonderful intuition as to how these "impossible" numbers would behave. To make Euler's arguments correct for today, one certainly needs concepts from advanced calculus, as Stillwell suggests. But it seems to me that these power series are not only beautiful, but they lead to so many interesting ideas that it would be unjust to wait for their development until some later time. For example, once the power series for sine, cosine, and e^x are known, it seems rather obvious that these functions must be related. With a bit of a detour into the complex numbers (which should be another elementary topic), one discovers the result $e^{ix} = \cos x + i \sin x$ and its amazing corollary $e^{i\pi} = -1$. Then, one might want to look at what happens when the alternating signs in the power series for sine and cosine are all replaced by plus signs. With the knowledge of the power series for the sine and cosine, one can try to divide and come up with a power series for the tangent. Yes, one cannot prove at an elementary level that these power series converge and that term-by-term differentiation or integration is a valid method of differentiating or integrating, but the relationships the student sees will be a great motivator for studying the rigorous ways of developing these ideas at a later date.

In the section on combinatorics in his opening chapter, Stillwell develops the basic combinatorial principles, the methods of counting permutations and combinations. Of course, counting combinations leads to the development of the Pascal triangle, and Stillwell notes that the triangle was developed in China and India long before Pascal. But it would be useful if he showed that in both medieval Islamic mathematics and medieval Hebrew mathematics these ideas were considered and that various mathematicians used methods close to our own proofs by induction to justify the combinatorial formulas. Stillwell spends most of the combinatorics chapter, however, on the combinatorial study of graphs, culminating with a proof of the Euler polyhedron formula $V - E + F = 2$.

One of the earliest probability problems to be discussed was the problem of "division of stakes": If a game between two players cannot be finished, how should one distribute the stakes, knowing how many more points each player needs to win? This problem was discussed in a famous series of letters between Pascal and Fermat, with both ultimately agreeing on the solution. Stillwell details the solution procedure for one example of this problem in his introductory chapter. But in his chapter on probability, he rapidly moves on to deal with probability distributions and the law of large numbers. But there are some important topics in probability that are missing. In particular, there is no discussion of the notion of expectation, a central idea in determining fairness in any game of chance. The earliest text on probability, *On calculation in games of chance*, published by Christian Huygens in 1657, discusses this notion in detail and gives numerous examples and exercises, some of which appeared frequently in later texts. For example, Huygens discussed another classic problem, how to determine the number of tosses of two dice necessary to have at least an even chance of getting a double six. Stillwell does discuss the Bell curve, although he considers this advanced mathematics. Yet the normal distribution modeled by this curve occurs so frequently and is so often discussed in an elementary statistics class that one should really consider this to be part of elementary mathematics, even though proving that this curve is the graph of ke^{-x^2} is rather difficult. The proof was actually accomplished by De Moivre in the mid-eighteenth century. Stillwell includes a brief section on mean, variance, and standard deviation, but does not mention any other statistical ideas. The increasing importance of statistics in our society, however, would seem to warrant that any survey of elementary mathematics includes discussions of the basic principles of statistics and, in particular, methods of statistical inference.

Stillwell's final chapter in his survey of elementary mathematics is on logic. He had earlier discussed mathematical induction, but here he presents an outline of propositional logic, along with some examples of the use of mathematical induction. Although the symbolism of propositional logic is a relatively modern topic, it is interesting to note that this logic is the logic of Euclid's *Elements* and that the basic ideas of propositional logic were discovered by Chrysippus in the third century BCE. The remainder of the chapter deals with advanced topics, including Peano arithmetic, the real number system, and infinite sets, all of which were studied in detail in the late nineteenth century. I agree that this material is certainly "advanced", although there are many aspects of these subjects that can profitably discussed at an elementary level. After all, mathematicians dealt with the real number system for centuries before it became necessary to figure out a definition for it. And most of the results of that definition are relatively obvious whenever one discusses the real number system in a calculus course, for example.

Although I disagree to some extent with Stillwell's definition of "elementary", I still highly recommend this book to anyone who wants a solid survey of a wide range of basic mathematical topics, topics that will show up in numerous mathematics courses at secondary or lower university level. And many of these topics are those that anyone taking advanced mathematics courses (however one defines that) will need to know, whether or not he or she has seen them earlier. Finally, as Stillwell himself suggests, some of the topics are those that even someone with an advanced degree may not be familiar with. Stillwell's clear exposition will help even a college professor gain a solid understanding of such topics. Stillwell also provides a long bibliography so that readers can pursue the topics further. However, it is a bit bothersome that he lists in this bibliography a good number of quite old mathematical documents. For example, he lists Brahmagupta's *Brâhma-sphuta-siddhânta* of 628 CE, even though it is difficult to find a copy of this in the original Sanskrit. He does mention that there is a partial English translation by Colebrooke in 1817, but it is also difficult for someone to find a particular topic in that translation. It would have been far better for Stillwell to have either given a page reference in Colebrooke or, better, a page reference in a modern analysis of Brahamagupta's work, for the solution of the Pell equation $x^2 - 92y^2 = 1$. Similarly, he references Levi ben Gershon's 1321 work, *Maasei Hoshev*, in its 1909 German translation. But again. to find the pages in either the original Hebrew or the German translation on which mathematical induction is used would be extremely difficult. It would have been far better to have referenced a modern discussion of Levi's work. However, this is a minor quibble. I think that Stillwell's book will itself become a modern classic and a reference work for anyone trying to learn basic topics in any of the major fields of mathematics.

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