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THE LEGACY OF VAUGHAN JONES IN II_1 FACTORS

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Abstract. We describe Vaughan Jones's ground-breaking discovery that symmetries of II_1 factors, as encoded by their subfactors, are quantized and have a natural index that can be non-integral. We then comment on the impact his revolutionary work had in the study of II_1 factors.

It is with some emotion that I open this series of contributions dedicated to the mathematical legacy of Vaughan Jones, a dear colleague and friend for more than forty years. His passing in September 2020, just short of his 68th birthday, was untimely and totally unexpected. A huge loss for the mathematical community, for his many friends, and for his family.

Vaughan was a mathematician of exceptional originality and breadth. His work had a huge impact on developments in several areas of mathematics and mathematical physics, bringing together disparate areas such as analysis of operator algebras on Hilbert spaces, low dimensional topology, statistical mechanics, and quantum field theory. The articles in this issue will give an idea of the extraordinary influence of this work.

Yet Vaughan was so much more than his mathematics. His wonderful personality comes across quite well from the article "Memories of Vaughan Jones" in the *Notices of the AMS* [4], where a large number of colleagues and friends shared personal memories on him. Although harder to convey, his rather unique style of research in mathematics, based on openness and generosity in sharing ideas, which by itself played an important role in the explosion of mathematics related to his work, has been much related in [4] and will also transpire from articles in this issue.

The crucial work that led to all these striking connections was Vaughan's ground-breaking discovery in [24] that symmetries of II_1 factors (a special class of von Neumann algebras), as encoded by their *subfactors*, are *quantized* and generate quantized groups, a completely new type of structure, endowed with a dimension function and an index, that can be nonintegral.

In what follows I will try to explain in more detail this extremely important discovery. I will also comment on impact, but since the influence of his work in low dimensional topology, statistical mechanics, and mathematical physics is so amply presented in the other articles in this issue of the *Bulletin of the AMS*, I will focus on the influence it had in von Neumann algebras and II_1 factors, describing some of the results in what has become *Jones subfactor theory.*

A *von Neumann algebra* is an algebra M generated by a system of self-adjoint operators on a Hilbert space $\mathcal{H}$ (e.g., observables in quantum mechanics) that contains $1 = id_{\mathcal{H}}$ and is closed in the *weak operator* (*wo*) topology given by the seminorms

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$|\langle x(\xi), \eta\rangle|$, $\xi, \eta \in \mathcal{H}$. These properties imply that once an operator x lies in M, then its adjoint x^* and polar decomposition belong to M. If in addition $x = x^*$, then the functional calculus of x with Borel functions belongs to M as well, in particular all its spectral projections. So M has a large set of projections $\mathcal{P}(M)$, which in addition form a complete lattice, as $\bigvee_i p_i, \bigwedge_i p_i \in M$ whenever $\{p_i\}_i \subset M$.

Such algebras were introduced by von Neumann in 1929 ([39]) as part of his effort to create a rigorous framework for quantum mechanics. Any von Neumann algebra can be realized as a "measurable field" of *factors*, i.e., von Neumann algebras that have trivial center. An example of a factor is the algebra $\mathcal{B}(\mathcal{H})$ of all linear bounded operators on the Hilbert space $\mathcal{H}$. Any factor M that contains minimal projections ("atoms") is of this form. But in 1936 Murray and von Neumann discovered that *diffuse* (or *continuous*) factors, i.e., factors without atoms, do exist and they can be of three types, which they labeled II_1, II_∞, and III, according to the way $\mathcal{P}(M)$ behaves under the equivalence $\sim$ given by existence of $x \in M$ having cokernel and range given by the respective projections; see [37]. They showed that all these types appear naturally from free ergodic actions of discrete groups by nonsingular transformations on a measure space, $G \curvearrowright (X, \mu)$, via a crossed product type construction, $M = L^\infty(X, \mu) \rtimes G$, acting on the Hilbert space $L^2(X, \mu) \otimes \ell^2 G$, with the II_1 case corresponding to μ finite with $G \curvearrowright X$ measure preserving. This is known as the *group measure space construction*. Later in [38], they provided a simpler construction, known as *group* II_1 *factors*, obtained as the wo-closure of the algebra generated by the range of the left regular representation of a group G with infinite conjugacy classes (ICC) acting on $\ell^2 G$, and denoted LG.

Abstractly, a II_1 *factor* M is a diffuse factor with the property that $p \sim 1$ for a projection p in $\mathcal{P}(M)$ implies $p = 1$. This is shown in [37] to be equivalent to M being a diffuse factor which admits a trace state τ, i.e., a functional $\tau : M \to \mathbb{C}$ that is positive ($\tau(x^*x) \geq 0$, $\forall x \in M$), with $\tau(1) = 1$ (it is a *state*) and satisfies $\tau(xy) = \tau(yx)$, $\forall x, y \in M$ (it is a *trace*). The factoriality of M entails uniqueness and complete additivity of the trace τ; see [13]. So a II_1 factor can be viewed as a quantized version of a probability space (X, μ) (which hosts observables in classical mechanics), hence the modern term *noncommutative probability space* for the pair (M, τ).[1] But for a II_1 factor M, the "measure" τ is intrinsic to M! The trace τ also allows the definition of an intrinsic Hilbert space $L^2 M$, defined as the completion of M in the norm $\|x\|_2 = \tau(x^*x)^{1/2}$, and which is canonically a bimodule over M using the left and right actions.

The existence of the trace combined with the factoriality of M and the topological property $M = \overline{M}^{wo}$ implies that if $p, q \in \mathcal{P}(M)$, then $p \sim q$ iff $\tau(p) = \tau(q)$. So τ implements a *dimension function* on $\mathcal{P}(M)$. Together with the fact that M is diffuse, it shows that the range of τ on $\mathcal{P}(M)$ covers the whole interval $[0, 1]$. This allows one to define the t-amplification M^t of M (the "$t \times t$ matrix algebra over M"), for any $t > 0$. It also shows that any representation of the II_1 factor as a wo-closed algebra $M \subset \mathcal{B}(\mathcal{H})$, viewed as a *left Hilbert M-module* ${}_M\mathcal{H}$, is of the form $\mathcal{H} \simeq \bigoplus_k L^2 M p_k$, for some projections $\{p_k\}_{k \in K} \subset \mathcal{P}(M)$, with the action of M by left multiplication. Moreover, $\dim({}_M\mathcal{H}) \overset{\text{def}}{=} \sum_k \tau(p_k) \in [0, \infty]$ characterizes the isomorphism class of ${}_M\mathcal{H}$. This summarizes the famous continuous dimension phenomenon brought to light by the discovery of II_1 factors in [37] and [38].

[1] von Neumann refers in [40] to the complete lattice $(\mathcal{P}(M), \vee, \wedge)$ endowed with the dimension function, which encodes (M, τ), as *quantum logic*.

It was realized over the years that II_1 factors arise naturally from a large variety of data, such as measurable groupoids and orbit equivalence relations, and that they provide a category of objects that is closed under inductive limits, tensor products, amalgamated free products, crossed products by groups of automorphisms, amplifications, quantum deformations, etc. They provide a unique environment to do *noncommutative analysis*, where randomness and rigidity phenomena may simultaneously occur, a co-existence that produces a large number of striking phenomena (see e.g., [56]).

Vaughan became acquainted with von Neumann algebras as a student at Auckland University in 1972–1974, where he was interested in both mathematics and physics. He went to the University of Geneva in 1974 with the initial intention to get a PhD in quantum physics, but then switched to mathematics with Andre Haefliger as his adviser. Then in 1975–1976 he met Alain Connes at a conference in Strasbourg and was very impressed. Connes had just completed his seminal work on the structure and classification of factors, notably amenable ones (see below), where he discovered the crucial importance of studying automorphisms of factors.

During 1963–1973, some two decades after Murray and von Neumann had shown that any approximately finite dimensional (AFD) II_1 factor is isomorphic to the so-called *hyperfinite* factor $R = \overline{\bigotimes}_{n=1}^{\infty}(\mathbb{M}_2(\mathbb{C}), tr)_n$ in [38], the suitable notion of amenability for factors was developed in several equivalent ways. This work showed in particular that R and its II_1 subfactors are amenable and that a group factor LG (resp., group measure space factor $L^\infty(X) \rtimes G$) is amenable iff G is amenable. Then in 1975 Connes proved his famous theorem that all amenable II_1 factors (so in particular all II_1 subfactors of R) are isomorphic to R, a fundamental result that became a landmark in the subject (see [9]). In a parallel work, he also classified single automorphisms of R, in particular periodic automorphisms; see[8].

By his own account, Vaughan was captivated right away by "the world of II_1 factors" and these recent developments. He avidly studied all the papers in this area, from Murray and von Neumann's pioneering work to Connes's recent preprints, then gathered a list of ten possible thesis topics and traveled to Paris to show them to Connes, who went rapidly down the list, "No, no, no, maybe, no, . . . , good, . . . ," and the "good" one became Vaughan's PhD thesis. That topic was to generalize Connes classification of periodic automorphisms of R to arbitrary finite groups, a project Vaughan completed in 1979 (see [23]), with Haefliger as his formal thesis adviser and Connes as his informal one.

In this work Vaughan developed a novel algebraic approach to the classification of actions of a finite group Γ on a II_1 factor N, in which he encoded the action $\Gamma \curvearrowright N$ by the isomorphism class of the inclusion of II_1 factors $N \subset M = N \rtimes \Gamma$. Shortly after his thesis, he realized that one can assign a natural notion of *index* to an abstract inclusion of II_1 factors $N \subset M$ (what he called a *subfactor*), defined as $[M : N] := \dim({}_N L^2 M)$, noticing that if $M = N \rtimes \Gamma$, then $[M : N] = |\Gamma|$, and that for an inclusion of group factors $LH \subset LG$ arising from an inclusion of ICC groups $H \subset G$, one has $[LG : LH] = [G : H]$.

He became intrigued by the question of what values this index may take and developed tools to investigate this problem, notably his *basic construction* (explained below) and *local index formula*, showing that if $p_1, \dots, p_n \in M$ is a partition of 1 with projections that commute with N, then $[M : N] = \sum_i \frac{[p_i M p_i : N p_i]}{\tau(p_i)}$. This formula readily implies that the relative commutant (or centralizer) of N in M,

$N' \cap M := \{x \in M \mid xy = yx, \forall y \in N\}$, satisfies $\dim(N' \cap M) \leq [M : N]$, and that if $[M : N] < 4$, then $N' \cap M = \mathbb{C}$. On the other hand, by using the basic construction, Vaughan proved that below $1 + \sqrt{2}$ the index $[M : N]$ could only take the values 1 and 2. Moreover, he realized that one can get subfactors of any index ≥ 4 in the hyperfinite II_1 factor R, by exploiting the Murray–von Neumann result that pRp is isomorphic to R for any nonzero projection $p \in R$. Thus, if one takes an isomorphism $\theta : pRp \simeq (1-p)R(1-p)$ for $p \in \mathcal{P}(R)$ of trace $t \neq 0, 1$, then the subfactor $R(t) = \{x + \theta(x) \mid x \in pRp\} \subset R$ has index $[R : R(t)] = t^{-1} + (1-t)^{-1}$ by the local index formula, thus taking all values $[4, \infty)$ as t runs over the interval $(0, 1)$. Note however that these subfactors are not irreducible, i.e., $R(t)' \cap R \neq \mathbb{C}$.

Vaughan obtained these initial results by early 1980, but then he was stuck, leaving open the problem of what may happen in the interval $[1 + \sqrt{2}, 4]$. He discussed his work with many people (including myself) and even gave talks at conferences on it in the summer of 1981. The general opinion was that under the trivial relative commutant condition $N' \cap M = \mathbb{C}$ (so in particular for values less than 4) the index should be an integer. But towards the end of 1981 I received a letter from Vaughan, dated December 2nd, where he wrote,

> My big news is that I think I solved the index problem. The answer is rather curious: there is a sequence of values beginning $1, 2, 1 + \Phi, 3, \ldots$ (with Φ the golden ratio) and converging to 4, which are the only possible values of the index between 1 and 4, and for which there are subfactors (automatically with trivial relative commutant) of R with these values!

A typical example of Vaughan's understatement and modesty…

In other words, together with the result showing that all values in $[4, \infty)$ can be obtained, which he already had, this states that the index can only take values in $\{4\cos^2(\pi/n) \mid n \geq 3\} \cup [4, \infty)$ and that all these values can be realized as indices of hyperfinite II_1 subfactors, a result I will refer to as the *Jones theorem*. After a few months I received from him a first draft of the preprint with complete proofs of this amazing result (published in [24]). Let me say a few words about this proof.

Vaughan's basic construction, which I mentioned before, is a crucial tool in the proof (it continued to play a fundamental role throughout subfactor theory). Given an inclusion of II_1 factors $N \subset M$ with finite index, this construction associates to it in a canonical way a II_1 factor M_1 that contains M and a projection $e_N \in M_1$ such that:

(a) $e_N x e_N = E_N(x) e_N$, $\forall x \in M$, where $E_N : M \to N$ denotes the unique τ-preserving conditional expectation of M onto N;
(b) $e_N y = y e_N, \forall y \in N$;
(c) $M_1 = \mathrm{span}\{x e_N y \mid x, y \in M\}$;
(d) the (unique) trace state τ_{M_1} on M_1 satisfies $\tau_{M_1}(x e_N y) = \lambda \tau_M(xy)$, $\forall x, y \in M$, where $\lambda = [M : N]^{-1}$.

Moreover, these conditions automatically imply that $[M_1 : M] = [M : N]$.

This means that one can apply again the basic construction for the new subfactor $M \subset M_1$, which has same index as $[M : N]$. So letting $M_{-1} = N$, $M_0 = M$, $e_0 = e_N$, this allows constructing iteratively a whole *tower* of inclusions of II_1 factors, $M_{-1} \subset M_0 \subset_{e_0} M_1 \subset_{e_1} M_2 \subset \cdots$, with each M_{i+1}, $i \geq 0$, being generated by M_i

and a projection e_i of trace $\lambda = [M : N]^{-1}$, having index $[M_{i+1} : M_i] = [M : N]$ and satisfying the properties:

(1) $e_i x e_i = E_{M_{i-1}}(x) e_i$, $\forall x \in M_i$;
(2) $\{e_i\}' \cap M_i = M_{i-1}$;
(3) $\tau(x e_i) = \lambda \tau(x)$, $\forall x \in M_i$.

In particular, the sequence of projections $\{e_i\}_{i \geq 0}$ with the trace τ satisfy the conditions:

(1′) $e_i e_{i\pm1} e_i = \lambda e_i$;
(2′) $[e_i, e_j] = 0$, if $|j - i| > 1$;
(3′) $\tau(x e_{i+1}) = \lambda \tau(x)$, $\forall x \in \mathrm{Alg}(\{e_0, e_1, \ldots, e_i\})$.

By using the fact that $[e_i, N] = 0$ and that the index is multiplicative (if $N \subset P \subset M$ are inclusions of II_1 factors, then $[M : N] = [M : P][P : N]$), one gets that the algebra A_n generated by $\{1, e_0, \ldots, e_{n-1}\} \subset N' \cap M_n$ has dimension at most $[M_n : N] = [M : N]^{n+1}$, so it is in fact a finite dimensional von Neumann subalgebra of M. Due to the relations (1′)–(3′), one deduces that $e_0 \vee \cdots \vee e_n$ is a central projection in A_n. The key point in the argument is that the trace of its complement, $\tau(1 - \bigvee_{i=0}^n e_i)$, which is always a nonnegative number ≤ 1, can be calculated recursively (again due to (1′)–(3′)) and that it is equal to $P_{n+1}(\lambda)$ whenever $P_n(\lambda), P_{n-1}(\lambda) > 0$, where $P_k(t)$, $k \geq 0$, $k \geq -1$, are the polynomials defined by the formulas $P_{-1} = 1, P_0 = 1$, $P_{n+1}(t) = P_n(t) - t P_{n-1}(t)$, $n \geq 0$. A rather elementary analysis of the roots of the polynomials $P_n(t)$ then shows that these conditions force $\lambda = [M : N]^{-1}$ to be either $\leq 1/4$ or of the form $\sec^2(\pi/n)/4$, for some $n \geq 3$.

Thus, the index of any inclusion of II_1 factors $N \subset M$ lies in the set

$$\{4\cos^2(\pi/n) \mid n \geq 3\} \cup [4, \infty),$$

the first four values of which are $1, 2, \frac{3+\sqrt{5}}{2} = 1 + \Phi$, and 3. This takes care of the restriction part in the Jones theorem. But note that this doesn't exclude the possibility that $[M : N]$ can only take the values $1, 2, 3$ if less than 4!

The proof of the existence part in the theorem is equally brilliant. On the one hand, Vaughan proved that if $(\{e_n\}_{n \geq 0}, \tau)$ is any set of projections with a trace satisfying (1′)–(3′), then the von Neumann algebra it generates is approximately finite dimensional and factorial, thus isomorphic to R (by [38]).

Then he noticed that in order to obtain a tower of inclusions of tracial von Neumann algebras (not necessarily factorial!) $N \subset M \subset_{e_0} M_1 \subset \cdots$ with a trace τ on $\bigcup_i M_i$ satisfying (1)–(3) above, and thus also $(\{e_i\}_i, \tau)$ satisfying (1′)–(3′), all one needs is the existence of the first step (the "initial" basic construction) $N \subset M \subset_{e_0 = e_N} M_1$ satisfying (a)–(d) for some scalar $\lambda > 0$, as the rest of the tower and projections will automatically exist and satisfy (1)–(3) for that same λ.

He then proved that given any connected bipartite graph T, and an inclusion of finite dimensional von Neumann algebras $\bigoplus_{k \in K} \mathbb{M}_{n_k}(\mathbb{C}) = N \subset M = \bigoplus_{l \in L} \mathbb{M}_{m_l}(\mathbb{C})$ (direct sums of matrix algebras, called *multi-matrix* algebras in [16]) with the set K equal to the set of "left/odd" vertices of T, the set L equal to the set of "right/even" vertices of T and the multiplicity diagram describing the inclusion given by T (thus $T^t \vec{n} = \vec{m}$ when T is viewed as a $K \times L$ matrix), then the trace τ on M that has weights proportional to the Perron–Frobenius eigenvector of $T^t T$ has exactly the desired property, with the corresponding λ given by $\|T^t T\|^{-1}$. Taking T to be the Coxeter graph A_{n-1}, $n \geq 3$, which is known to have norm $2\cos(\pi/n)$, one obtains for $\lambda^{-1} = 4\cos^2(\pi/n)$ a sequence of projections and a trace satisfying (1′)–(3′) for that λ.

Finally, defining R to be the II_1 factor generated by $(\{e_n\}_{n\geq 0}, \tau)$ and R_λ its subfactor generated by $\{e_n\}_{n\geq 1}$, one obtains an inclusion of hyperfinite II_1 factors $R_\lambda \subset R$ and the axioms (1′)–(3′) used once again easily imply that $[R : R_\lambda] = \lambda^{-1} = 4\cos^2(\pi/n)$.[2]

The elegance and beauty of these arguments is striking (Jurg Frohlich used the word "magical" for it, which I think is perfectly suited!)

As for the result itself, I would argue that it defies all intuition developed prior to Vaughan's work. There was nothing that could predict such phenomena.

There is of course nothing like it in the "classical, commutative world" of function algebras. For an inclusion $L^\infty(X) = N \subset M = L^\infty(Y)$ to have a "good notion" of index, one has to make some assumption of homogeneity of the inclusion (this is somewhat analogous to the trivial centralizer requirement for inclusions of II_1 factors), which basically amounts to requiring that the measurable surjective map $f : Y \to X$ that implements such inclusion satisfies $|f^{-1}(x)|$ constant, for all $x \in X$ (a.e.), making $[M : N]$ equal to this common cardinal.

As explained before, II_1 factors come from some "data" $\mathbb{G}$, which are often geometric/dynamic in nature, like a group (or a groupoid) acting on a space. Denoting the corresponding II_1 factor $L\mathbb{G}$, by analogy with the notation for group factors, any subfactor of finite index of $L\mathbb{G}$ that a "classical intuition" would conceive would come from a subobject $\mathbb{H} \subset \mathbb{G}$ (e.g., an embedding $(H \curvearrowright X) \hookrightarrow (G \curvearrowright Y)$), with a "classical" index $[\mathbb{G} : \mathbb{H}]$ an integer, resulting in $[L\mathbb{G} : L\mathbb{H}] = [\mathbb{G} : \mathbb{H}]$ being an integer.

In turn, a "II_1 factor intuition" may find the continuous part of the Jones spectrum $[4, \infty)$ as "normal", because of the Murray–von Neumann theorem in [38] that all amplifications R^t of the hyperfinite II_1 factor are approximately finite dimensional and thus isomorphic to R. But then the "discrete part" $\{4\cos^2(\pi/n) \mid n \geq 3\}$ looks like an anomaly!

While at first the sequence of numbers $4\cos^2(\pi/n)$ may seem mysterious, the proof of the existence of the subfactors $R_\lambda \subset R$ of index $\lambda^{-1} = 4\cos^2(\pi/n)$ in the Jones theorem provides a hint of the hidden structure behind this discrete part of the spectrum: these numbers coincide with the square norms of the Coxeter graphs A_{n-1}. A second proof to this theorem, that Vaughan obtained in early 1984 (see [27] or [16]; the existence part was obtained independently by Pimsner and myself, see [47], and was much emphasized in [42]), shows that in fact there is a graph-like structure behind both the restrictions and existence parts for index < 4.

Thus, given any subfactor $N \subset M$, the increasing sequence of finite dimensional algebras $\mathbb{C} = M' \cap M \subset M' \cap M_1 \subset M' \cap M_2 \subset M' \cap M_3 \subset \cdots$ is described by a connected pointed bipartite graph, $\Gamma = \Gamma_{N \subset M}$, called the *principal* (or *standard*) *graph* of $N \subset M$, with the inclusions described by $\Gamma, \Gamma^t, \Gamma, \Gamma^t, \ldots,$ starting from the "initial" even vertex $*$ of the pointed graph, corresponding to $\mathbb{C} = M' \cap M$ (e.g., the edges from $*$ describing the inclusion $M' \cap M \subset M' \cap M_1$). Also, when viewed as a matrix, $\Gamma = (a_{kl})_{k,l}$, with a_{kl} the number of edges between the even vertex k and the odd vertex l, the graph of $N \subset M$ satisfies $\Gamma\Gamma^t(\vec{v}) = [M : N]\vec{v}$, where $\vec{v} = (v_k)_k$

[2]One can take the inclusion $R_\lambda = \mathrm{vN}(\{e_i\}_{i\geq 1}, \tau) \subset \mathrm{vN}(\{e_i\}_{i\geq 0}, \tau) = R$ for $\lambda^{-1} \geq 4$ as well, but these subfactors have nontrivial relative commutant by [24], in fact by [44] they are of the above "locally trivial" form $R(t) \subset R$, where $t(1-t) = \lambda$.

is given by the square roots of the indices of irreducible subfactors appearing in the inclusion $M \subset M \subset M_{2n}, n \geq 1$. This implies $\|\Gamma_{N\subset M}\|^2 \leq [M : N]$, with equality whenever the graph is finite (by the Perron–Frobenius theorem). Since the only bipartite graphs of norm < 2 are the Coxeter graphs A_n, D_n, E_6, E_7, E_8, and all have norms in $\{2\cos(\pi/m) \mid m \geq 3\}$, this proves the restrictions in the Jones theorem.

On the other hand, if one takes T to be any graph of norm $2\cos(\pi/n)$ (e.g., $T = A_{n-1}$) and let $B_{-1} \subset B_0$ be an inclusion of multi-matrix algebras with inclusion diagram given by T and trace given by the Perron–Frobenius eigenvector of T^tT, with $B_0 \subset_{e_0} B_1$ the corresponding basic construction, then $u := \alpha e_0 - (1 - e_0)$, with $\alpha = \exp(\frac{2\pi i}{n})$, is a unitary element and the embedding of $(C_0 \subset C_1) := (B_{-1} \subset uB_0u^*)$ into $(B_0 \subset B_1)$ is a *commuting square* (i.e., the corresponding trace preserving expectations commute) that gives rise to commuting square embeddings of the Jones towers $(C_0 \subset C_1 \subset_{e_1} \subset \cdots)$ into $(B_0 \subset B_1 \subset_{e_1} \subset \cdots)$, with the inclusion of the inductive limits $C_\infty \subset B_\infty$ giving a hyperfinite subfactor of index $\|T\|^2 = 4\cos^2(\pi/n)$. This provides the alternative proof of "existence" in the Jones theorem.

The symmetries of a II_1 factor N were initially considered to be its automorphisms, whose importance is as crucial in the study of the structure and classification von Neumann algebras (cf. [8]) as it is for classical spaces. If $\theta \in \mathrm{Aut}(N)$, then the Hilbert N-bimodule L^2N with left-right multiplication given by $x \cdot \xi \cdot y = x\xi\theta(y)$ completely encodes θ. If $N \subset M$ is a subfactor of finite index, then the Hilbert $N - M$ bimodule ${}_NL^2M_M$ and the *-tensor category it generates can thus be viewed as a *quantized symmetry* and the *quantized group* it generates, with an index $\dim({}_Q\mathcal{H}_P) := \dim({}_Q\mathcal{H})\dim(\mathcal{H}_P)$, $P, Q \in \{N, M\}$, that is multiplicative and takes (quantized) values in the semigroup $\{4\cos^2(\pi/n) \mid n \geq 3\} \cup [4, \infty)$.[3]

Such quantized symmetries are specific to noncommutative algebras of observables on the Hilbert space, a fact that anticipated at the outset the importance of this discovery to mathematical physics, plainly confirmed in subsequent years as the article by Dai Evans and Yasu Kawahigashi in this issue shows [15]. That it turned out to be equally important and relevant to low dimensional topology, due to Vaughan's spectacular discovery in 1984 (see [26], [28]) of his polynomial invariant for knots by using the representations of braid groups entailed by the λ-sequence of projections $\{e_i\}_i$ and the trace τ, as the articles in this issue by Lou Kauffman [33] and by Robert Lipschitz and Mikhail Khovanov [34] show, makes it "one of the great jewels of the unity of mathematics", as Alain Connes commented in [7].

The legacy of Vaughan's revolutionary work within the theory of II_1 factors has been particularly deep and enduring. In fact, the whole area of operator algebras has been influenced by him in multiple ways, directly and indirectly. First and foremost, his work showed that the "symmetry picture" of a II_1 factor is much more subtle and complicated than was previously thought. Quantized symmetries were there but were overlooked, and henceforth need to be taken into account.

Vaughan's early 1980s work led right away to a huge number of exciting problems. It turns out that almost any question one would ask about these fascinating structures proves to be hard and challenging, requiring new tools and new ideas.

[3]The idea of Hilbert bimodules as *correlators* (*correspondences*) between von Neumann algebras is due to Connes [10], and it was key to defining a "good representation theory" for II_1 factors (see [12] and [46]). It was linked this way to subfactor theory in [46] and [42].

Just 10–15 years after his initial work, there have already been countless contributions in this area; see the comprehensive book [14] which accounts for part of these developments (see also [16], [44], the preliminary sections in [49] or [50] for basics on subfactor theory, and [2] for general theory of II_1 factors).

A common trait of problems on subfactors is that answers are hard to predict, and the final answer often comes as a surprise. In what follows I will comment on just a few of the results and problems in this area.

One of the very first questions in this subject, posed by Vaughan in [24] and [27], was to find all possible values > 4 of indices of irreducible subfactors (i.e., subfactors with trivial relative commutant), on which he commented in [27], "the current feeling is that there should be a gap between 4 and the next irreducible index value". It has been shown in [47] that for any $\lambda^{-1} > 4$ there is a canonical family of II_1 factors $N \subset M$ of index $[M : N] = \lambda^{-1}$ and trivial relative commutant $N' \cap M = \mathbb{C}$, in fact with all relative commutants $N' \cap M_i$ in the tower $N \subset M \subset_{e_0} \subset \cdots$ being "minimal", in that they are generated by the Jones projections $\{e_i\}_i$ alone. Equivalently, the graph of any of these subfactors is of the form $\Gamma_{N\subset M} = A_\infty$.

Another question that Vaughan stated in [27] was to find all bipartite graphs that can occur as graphs of subfactors. By now one knows that in this generality this problem is out of reach. But many striking obstruction results have been obtained over the years about nonoccurrence of certain graphs as graphs of subfactors. The very first such result in [42] shows that, while the list of bipartite graphs Γ of norm < 2 consists of A_n, D_n, E_6, E_7, E_8, the graphs A_n, D_{even}, E_6, E_8 do occur as graphs of subfactors but D_{odd} and E_7 do not (cf. also [22]). More obstruction results were obtained in [19], where it was shown for instance that any irreducible subfactor $N \subset M$ of index $4 < [M : N] < \frac{5+\sqrt{13}}{2}$ has an A_∞ graph. Since the set $\mathbb{E}^2$ of square norms of (possibly infinite) bipartite graphs is a closed set consisting of an increasing sequence of accumulation points (the first of which being $4 = \lim_n 4\cos^2(\pi/n)$) converging to $2+\sqrt{5}$, followed by the half-line $[2+\sqrt{5}, \infty]$ (see e.g., [16, Appendix]), this implied that if a bipartite graph Γ satisfies $\|\Gamma\|^2 \in (4, \frac{5+\sqrt{13}}{2}) \cap \mathbb{E}^2$, then it cannot be a subfactor graph! More recently, results in [32] give a complete list of subfactor graphs of square norm ≤ 5 (cf. also [1]).

Related to Vaughan's early "subfactor graph question", there has been a parallel interest in understanding not just the graph $\Gamma_{N\subset M}$, but the whole *standard invariant* of a subfactor $N \subset M$, given by the ensemble of higher relative commutants in its tower, $\mathcal{G}_{N\subset M} := (\{M_i' \cap M_j\}_{j\geq i}, \tau)$. By the Jones local index formula and properties of the tower, this is a system of inclusions of multi-matrix algebras endowed with a trace, satisfying the commuting square condition, having the λ-sequence of Jones projections $(\{e_i\}_i, \tau)$ as a "spine" and the principal graph $\Gamma_{N\subset M}$ (and a "sister" dual graph $\Gamma'_{N\subset M}$) describing the inclusions. The higher relative commutants were described in [42] as intertwining spaces between the Hilbert bimodules in consecutive tensor products $\mathcal{H} \otimes_M \mathcal{H}^* \otimes_N \ldots$, $\mathcal{H}^* \otimes_N \mathcal{H} \otimes_M \ldots$, where $\mathcal{H} = {}_N L^2 M_M$, thus characterizing $\mathcal{G}_{N\subset M}$ as a * 2-tensor category (so in a way a "group-like" object), with a complete axiomatization of these objects in the *finite depth* case, corresponding to the graph $\Gamma_{N\subset M}$ being finite.

The result in [48] provides in fact a canonical construction of (nonhyperfinite!) subfactors having "minimal standard invariant", where the higher relative commutants are generated by the Jones projections alone: if $(\{e_i\}_{i\geq 0}, \tau)$ is the λ-sequence of projections for some arbitrary $\lambda^{-1} > 4$, then the lattice $\mathcal{G}_\lambda$ consisting of the multi-matrix algebras $A_{ij} = \{e_k \mid i \leq k < j\}$ is a standard invariant of a subfactor of index λ^{-1}. It is called the *TLJ λ-lattice* (an abbreviation for Temperley, Lieb, and Jones; see [15] in this issue for an explanation of this terminology).

The reconstruction method involved in the proof of this result in [48], which uses tracial amalgamated free products, is pushed further in [51] to obtain an axiomatization as abstract objects of the lattices of commuting squares of tracial multi-matrix algebras $\mathcal{G} = (\{A_{ij}\}_{j \geq i \geq 0}, \tau)$ that can occur as standard invariants of arbitrary subfactors, irrespective of depth. These objects are called *standard λ-lattices* in [51]. The usage of tracial amalgamated free product algebras in this context led to fruitful connections with Voiculescu's free probability theory, due to the effort to adapt the random matrix model in [69] to identify such algebras as free group factors (see [66] and [67]). An alternative version of the construction in [51] allowed us to prove in [59] that any standard λ-lattice $\mathcal{G}$ can be realized as the invariant $\mathcal{G}_{N \subset M}$ of an inclusion of factors $N \subset M$ with $N \simeq M \simeq L\mathbb{F}_\infty$. In other words, any abstract "quantized group" $\mathcal{G}$ can "act" on the free group factor $L\mathbb{F}_\infty$!

Vaughan obtained in [29] a new axiomatization of the standard invariant of a subfactor, described as a two-dimensional diagramatic structure of tangles and concatenations called *planar algebra*. This proved to be a very powerful calculus tool. It led to an avalanche of results (see e.g., [6]), going a long way into understanding these complicated objects. It eventually allowed a complete classification of the planar algebras of index ≤ 5 in [32] (even up to index 5.25 in [1]). Also, a new manner of obtaining the reconstruction in [59], directly from planar algebras, was discovered in [16] and [17], leading to exciting new connections with free probability theory and mathematical physics.

Another type of problem that Vaughan's work triggered, of a more specific nature, is to calculate the *subfactor picture* (i.e., "quantum symmetry picture") of a given II_1 factor $M = L\mathbb{G}$, associated to some given data $\mathbb{G}$. In particular, the question is to calculate the set $\mathscr{C}(M)$ of indices of irreducible subfactors of $M = L\mathbb{G}$. Quite interestingly, there are only two types of existing results where $\mathscr{C}(M)$ could be fully calculated, and they are at opposite ends. Thus, if $M = L\mathbb{F}_\infty$, then $\mathscr{C}(M)$ is the whole Jones spectrum $\{4\cos^2(\pi/n) \mid n \geq 3\} \cup [4, \infty)$ by [59]. In striking contrast, it has been shown in [61] and [62] (cf. also [55] and [43]) that for a large class of groups G, which includes any nonamenable group that is a product of hyperbolic groups, and for any free ergodic probability measure preserving action $G \curvearrowright X$, the group measure space factor $M = L^\infty(X) \rtimes G$ has only subfactors of integer index; in fact all subfactors of M come from "subobjects" $\mathbb{H} \subset \mathbb{G} = (G \curvearrowright X)$! In particular, if $M = L^\infty(X) \rtimes \mathbb{F}_n$, $2 \leq n \leq \infty$, then $\mathscr{C}(M) = \{1, 2, \dots\}$.

Thus, if a II_1 factor M comes from "random-like" data, involving (amalgamated) free products, such as those in [48], [51], [59], and [18], then $\mathscr{C}(M)$ is the full Jones semigroup, while if M arises from a more "geometric background", such as actions of certain groups on the standard probability space, then $\mathscr{C}(M)$ can be extremely rigid, reduced to some subset of the integers $\{1, 2, 3, \dots\}$ (there are examples in [68] with no irreducible subfactors at all; see also [21] and [64]).

The most exciting problem along these lines, formulated already in [24] (cf. also [27], [30]), is to calculate the set $\mathscr{C}(R)$ of indices of irreducible subfactors of the hyperfinite II_1 factor. This case is particularly puzzling, as R can be constructed from very geometric data (finitary, more generally amenable, due to [9]), while at the same time it is the playing field for matrix randomness! It is conjectured in

[58] that any index of an irreducible subfactor of R is the square of the norm of a bipartite graph, i.e., $\mathscr{C}(R) \subset \mathbb{E}^2$. A strategy and some tools are proposed in [58] on how to approach this problem, whose resolution will certainly require some very fine analysis. In fact, it is speculated in [58] that $\mathbb{E}^2 \subset \mathscr{C}(R)$ as well, a problem that is however of a different, more calculatory nature. It amounts to solving the so-called *commuting square (c.sq.) problem* for bipartite graphs (already hinted at in [16], [45], [19]). The c.sq. problem was solved for the graph E_{10}, whose square norm $\|E_{10}\|^2 \approx 4.0265\ldots$ gives the smallest value in $\mathbb{E}^2 \cap (4, \infty)$, with a computer assisted proof (see [19]). Before that, for many years the smallest known value in $\mathscr{C}(R) \cap (4, \infty)$ was $3 + \sqrt{3}$, with an ad hoc c.sq. construction Vaughan had in 1983 (see [16]). There are no known examples of nonalgebraic values $\beta \in \mathscr{C}(R)$ (so not the square norm of a finite graph). Proving a general statement like $\mathbb{E}^2 \subset \mathscr{C}(R)$ will probably require a very ingenious, holistic approach to the c.sq. problem that may include "approximate solutions" to c.sq. and computer-assisted methods. See [58] for a detailed discussion, including motivations behind these conjectures, as well as for many related problems.

An even more challenging problem is to classify all subfactors $N \subset R$ of finite index of the hyperfinite II_1 factor. Since any action $G \curvearrowright^{\sigma} R$ of a finitely generated group G can be encoded (up to cocyle conjugacy) into an inclusion $N_\sigma \subset R$ with $[R : N_\sigma] < \infty$ and any nonamenable group can be shown to have "unclassifiably many" actions on R, this problem cannot be solved in this generality. In fact, most interesting is to classify subfactors $N \subset R$ by their standard invariant, $\mathcal{G}_{N \subset R}$, a problem that has been looked at since mid-1980s. It was proved in [47] that finite depth subfactors of R are indeed completely classified by their standard invariant. So in particular, subfactors $N \subset R$ of index $[R : N] < 4$, which all have finite depth, are classified by $\mathcal{G}_{N \subset R}$. The notion of amenability for standard λ-lattices $\mathcal{G}$ (equivalently planar algebras, or what I informally call "quantized groups") was developed in a series of papers [49], [52], and [53], being described in several equivalent ways. One of them is that the graph $\Gamma_{\mathcal{G}}$ of $\mathcal{G}$ satisfies the Kesten-type condition $\|\Gamma_{\mathcal{G}}\|^2 = \lambda^{-1}$. It was shown in [49], [50], [52], and [53] that subfactors $N \subset R$ with $\mathcal{G}_{N \subset R}$ amenable (i.e., $\|\Gamma_{N \subset R}\|^2 = [R : N]$) are completely classified by their standard invariant (see also [57] and [58]). Since for index $[R : N] = 4$ the amenability condition is automatic, this allowed a listing of all hyperfinite subfactors of index ≤ 4 in [49]. Combined with the list of planar algebras of index ≤ 5 in [32] (cf. also [3] and [1]), it further allowed the classification of all subfactors $N \subset R$ with graph not equal to A_∞ and index at most 5. The classification of hyperfinite subfactors with amenable graph in [49]–[53] generalized the classification up to so-called cocycle conjugacy of free actions of amenable group G on R in [41], in the case G is finitely generated, and it allowed the classification of prime actions of compact Lie groups on R in [65]. It is an open problem whether subfactors of R that have amenable graph provide the largest class of hyperfinite subfactors that can be classified by their standard invariant. To prove this, one needs to show that if a subfactor $N \subset R$ has nonamenable invariant $\mathcal{G}_{N \subset R}$, then there exists a subfactor $N_0 \subset R$ such that $\mathcal{G}_{N_0 \subset R} = \mathcal{G}_{N \subset R}$ but that cannot be conjugated to N via an automorphism of R (the analogous such statement for actions of nonamenable groups $G \curvearrowright R$ was shown in [25]). A problem related to both the classification problem for hyperfinite subfactors and to the calculation of $\mathscr{C}(R)$ is to find the set $\mathscr{G}(R)$ of all "quantized groups" (planar algebras, or standard λ-lattice) $\mathcal{G}$ that can "act" on

R; i.e., that can occur as standard invariant of some subfactor $N \subset R$. It is for instance wide open for what values $\lambda^{-1} > 4$ the TLJ λ-lattice $\mathcal{G}_\lambda$ can act on R. It is however shown in [20] and [68] that if $\mathcal{G}_1, \mathcal{G}_2 \in \mathscr{G}(R)$, then their free product $\mathcal{G}_1 * \mathcal{G}_2$ (as defined in [5] and [6]) belongs to $\mathscr{G}(R)$. Also, by [54], if a standard λ-lattice $\mathcal{G}$ is amenable, then $\mathcal{G} \in \mathscr{G}(R)$.

A useful tool for investigating the structure of a subfactor $N \subset M$ is the *symmetric enveloping* (*SE*) inclusion of II_1 factors $M \otimes M^{op} \subset M \underset{N}{\boxtimes} M^{op}$ defined in [52] and [53], with $M \underset{N}{\to} \boxtimes M^{op}$ constructed in a canonical way from commuting copies of M, M^{op} and a projection e that implements both the expectation of M onto N and of M^{op} onto N^{op} (cf. also [42] for finite depth hyperfinite $N \subset M$, and [36] for general finite depth $N \subset M$). This "quantum double" type construction is used in [51] and [53] to define property (T) for λ-lattices (quantized groups) $\mathcal{G}_{N\subset M}$ and to obtain further characterizations of amenability. The SE-algebra framework is also important for defining in [63] a representation theory for subfactors and their quantized groups and for studying property (T), the Haagerup property, and weak amenability (in Cowling–Haagerup style) for these objects. The SE-inclusion is crucial in defining a notion of cohomology and L^2-Betti numbers for subfactors and their quantized groups in [60], generalizing Atiyah's classical such notions for groups.

I have mentioned only a few results and problems concerning "quantized symmetries" of II_1 factors (subfactor theory). This subject has developed tremendously since the initial discoveries of Vaughan where he constantly played a key role. His extraordinary insight and stimulating personality will be greatly missed.

About the author

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