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GEOMETRIES OF TOPOLOGICAL GROUPS

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ABSTRACT. The paper provides an overarching framework for the study of some of the intrinsic geometries that a topological group may carry. An initial analysis is based on geometric nonlinear functional analysis, that is, the study of Banach spaces as metric spaces up to various notions of isomorphism, such as bi-Lipschitz equivalence, uniform homeomorphism, and coarse equivalence. This motivates the introduction of the various geometric categories applicable to all topological groups, namely, their uniform and coarse structure, along with those applicable to a more select class, that is, (local) Lipschitz and quasimetric structure. Our study touches on Lie theory, geometric group theory, and geometric nonlinear functional analysis and makes evident that these can all be seen as instances of a single coherent theory.

The aim of the present paper is to organise and put into a coherent form a number of old and new results, ideas and research programmes regarding topological groups and their linear counterparts, namely Banach spaces. As the title indicates, our focus will be on geometries by which we understand the various types of geometric structures that a Banach space or a topological group may be equipped with, e.g., Lipschitz structure or the quasimetric structure underlying geometric group theory. We shall attempt to provide a common framework and language for several different currently very active disciplines, including geometric nonlinear functional analysis and geometric group theory, and varied objects, e.g., Banach spaces, finitely generated, Lie, totally disconnected locally compact, and Polish groups. For this reason, it will be useful initially not to restrict the objects we consider.

1. Banach spaces as geometric objects

1.1. **Categories of geometric structures.** Our model example of topological groups, namely, the additive topological group $(X,+)$ underlying a Banach space $(X,\|\cdot\|)$ is perhaps somewhat unconventional. Certainly, the Banach space $(X,\|\cdot\|)$ is far more structured than $(X,+)$ and thus one misses much important information by leaving out the normed linear structure. Moreover, algebraically $(X,+)$ is just too simple to be of much interest. However, Banach spaces are good examples since they are objects that have classically been studied under a variety of different perspectives, e.g., as topological vector spaces, as metric or as uniform spaces. So, apart from their intrinsic interest, Banach spaces will illustrate some of the

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appropriate categories in which to study topological groups and also will provide a valuable lesson in how rigidity results allow us to reconstruct forgotten structure.

The language of category theory will be convenient to formulate the various geometric structures we shall be studying. Recall that to define a category, we need to specify the objects and the morphisms between them. In that way, we derive the concept of isomorphism. Namely, an *isomorphism* between objects X and Y is a morphism $X \xrightarrow{\phi} Y$ so that, for some morphism $Y \xrightarrow{\psi} X$, both $\psi\phi$ and $\phi\psi$ equal the unique identities on X and Y, respectively.

On the other hand, *embedding*, i.e., isomorphism with a substructure, is not readily a categorical notion as it relies on the model theoretical concept of substructure. However, in all our examples, what constitutes a substructure is evident, e.g., a substructure of a topological vector space is a linear subspace with the induced topology, while a substructure of a metric space is just a subset with the restricted metric. So, for example, an embedding of topological vector spaces is linear map $X \xrightarrow{T} Y$, which is a homeomorphism with its image $T[X] \subseteq Y$.

1.2. **Metric spaces.** Recall that a *Banach space* is a complete normed vector space $(X, \|\cdot\|)$. Thus, the norm is part of the given data. For simplicity, **all Banach spaces are assumed to be real**, i.e., over the field $\mathbb{R}$. In the strictest sense, an isomorphism should be a surjective linear isometry between Banach spaces, and the proper notion of morphism is thus *linear isometry*, i.e., a linear operator $X \xrightarrow{T} Y$ so that $\|Tx\| = \|x\|$.

However, instead of *normed vector spaces*, quite often Banach spaces are considered in the weaker category of *topological vector spaces* with morphisms simply being continuous linear operators. The procedure of dropping the norm from a normed linear space while retaining the topology thus amounts to a forgetful functor

$$\mathsf{NVS} \xrightarrow{\mathbf{F}} \mathsf{T}VS$$

from the category of normed vector spaces to the category of topological vector spaces. Similarly, rather than entirely eliminating the norm, we may instead erase the linear structure while recording the induced norm metric and thus obtain a forgetful functor

$$\mathsf{NVS} \xrightarrow{\mathbf{G}} \mathsf{Metric\ Spaces}$$

to the category of metric spaces whose morphisms are (not necessarily surjective) isometries. Observe also that these functors preserve embeddings.

This latter erasure however points to our first rigidity phenomenon, namely, the Mazur–Ulam theorem. Indeed, S. Mazur and S. Ulam [39] showed that, if $X \xrightarrow{\phi} Y$ is a surjective isometry between Banach spaces, then ϕ is necessarily affine, i.e., the map $Tx = \phi(x) - \phi(0)$ is a surjective linear isometry between X and Y. In particular, any two isometric Banach spaces are automatically linearly isometric.

In a more recent breakthrough [22], G. Godefroy and N. J. Kalton established a similar rigidity result for *separable* Banach spaces.

Theorem 1.1 ([22, Corollary 3.3])**.** *If $X \xrightarrow{\phi} Y$ is an isometric embedding from a separable Banach space X into a Banach space Y, then there is an isometric linear embedding of X into Y.*

Observe that the conclusion here is somewhat weaker than in the Mazur–Ulam theorem, since ϕ itself may not be affine. This is for good reasons as, for example, the map $\phi(x) = (x, \sin x)$ is an isometric, but clearly nonaffine embedding of $\mathbb{R}$ into $\ell^\infty(2) = (\mathbb{R}^2, \|\cdot\|_\infty)$. Also, the assumption that X is separable is known to be necessary as there are counterexamples in the nonseparable setting (see [22, Corollary 4.4]).

Although these two rigidity results do not provide us with a functor from the category of metric space reducts of separable Banach spaces to the category of normed vector spaces, they do show that an isomorphism or embedding in the weaker category of metric spaces implies the existence of an isomorphism, respectively, embedding in the category of normed vector spaces.

1.3. **Lipschitz structures.** To venture beyond these simple examples, we consider some common types of maps between metric spaces.

Definition 1.2. A map $X \xrightarrow{\phi} M$ between metric spaces (X, d) and (M, ∂) is

- *Lipschitz* if there is a constant K so that, for all $x, y \in X$,
$$\partial(\phi x, \phi y) \leqslant K \cdot d(x, y);$$
- *Lipschitz for large distances* if there is a constant K so that, for all $x, y \in X$,
$$\partial(\phi x, \phi y) \leqslant K \cdot d(x, y) + K;$$
- *Lipschitz for short distances* if there are constants $K, \delta > 0$ so that
$$\partial(\phi x, \phi y) \leqslant K \cdot d(x, y)$$
whenever $x, y \in X$ satisfy $d(x, y) \leqslant \delta$.

A fact that will become important later on is that our definitions above provide a splitting of being Lipschitz as the conjunction of two weaker conditions. Namely, we have the following simple fact:

$$\phi \text{ is Lipschitz } \Leftrightarrow \phi \text{ is Lipschitz for both large and short distances.}$$

As the composition of two Lipschitz maps is again Lipschitz, the class of metric spaces also forms a category where the morphisms are now Lipschitz maps. Similarly with Lipschitz for both large and short distances. However, for later purposes where there are no canonical metrics, it is better not to treat spaces with specific choices of metrics, but rather equivalence classes of these. We therefore define three equivalence relations, namely, *bi-Lipschitz*, *quasi-isometric*, and *local bi-Lipschitz equivalence* on the collection of all metrics on a set X by letting

$$\begin{aligned}
d \sim_{\mathsf{Lip}} \partial &\Leftrightarrow \big(X, d\big) \underset{\mathsf{id}}{\overset{\mathsf{id}}{\rightleftarrows}} \big(X, \partial\big) \text{ are both Lipschitz} \\
&\Leftrightarrow \exists K \;\; \frac{1}{K} d \leqslant \partial \leqslant K \cdot d, \\
d \sim_{\mathsf{QI}} \partial &\Leftrightarrow \big(X, d\big) \underset{\mathsf{id}}{\overset{\mathsf{id}}{\rightleftarrows}} \big(X, \partial\big) \text{ are Lipschitz for large distances} \\
&\Leftrightarrow \exists K \;\; \frac{1}{K} d - K \leqslant \partial \leqslant K \cdot d + K, \\
d \sim_{\mathsf{locLip}} \partial &\Leftrightarrow \big(X, d\big) \underset{\mathsf{id}}{\overset{\mathsf{id}}{\rightleftarrows}} \big(X, \partial\big) \text{ are Lipschitz for short distances.}
\end{aligned}$$

Example 1.3. The standard euclidean metric $d_1(x,y) = |x-y|$ on $\mathbb{R}$ is locally bi-Lipschitz equivalent with the truncated metric $d_2(x,y) = \min\{1, |x-y|\}$. On the other hand, since the map $x \mapsto \sqrt{x}$ is not Lipschitz for short distances, these are not locally bi-Lipschitz equivalent with the metric

$$d_3(x,y) = \sqrt{|x-y|}.$$

Eventually, when we turn to topological groups, we may occasionally pick out equivalence classes of metrics without being able to choose any particular metric. These thus become objects of the following types.

Definition 1.4. A *Lipschitz*, *quasimetric*, respectively *local Lipschitz space* is a set X equipped with a bi-Lipschitz, quasi-isometric, respectively local bi-Lipschitz equivalence class $\mathcal{D}$ of metrics on X.

In none of these three cases do we have an easy grasp of what the space actually is. By definition, it is *that which is invariant* under a certain class of transformations. On the other hand, morphisms are simpler. Indeed, a *morphism*

$$\big(X, \mathcal{D}_X\big) \xrightarrow{\phi} \big(M, \mathcal{D}_M\big)$$

between two Lipschitz or local Lipschitz spaces is a map $X \xrightarrow{\phi} M$ that is Lipschitz, respectively Lipschitz for short distances, with respect to some or equivalently any choice of metrics from the respective equivalence classes $\mathcal{D}_X$ and $\mathcal{D}_M$. In this way, Lipschitz and local Lipschitz spaces form categories in which the isomorphisms are bijective functions that are Lipschitz (for short distances) with an inverse that is also Lipschitz (for short distances).

Just as maps that are Lipschitz for large distances need not be continuous and hence fail to capture topological notions, isomorphisms between quasimetric spaces should neither preserve topology nor record spaces' cardinality either. In analogy with homotopy equivalence of topological spaces, we therefore adjust the notion of morphism.

Definition 1.5. Two maps $X \xrightarrow{\phi,\psi} M$ from a set X to a metric space (M,d) are *close* if

$$\sup_{x\in X} d\big(\phi x, \psi x\big) < \infty.$$

Observe that whether ϕ and ψ are close depends only on the quasi-isometry class of the metric d on M. We may therefore define morphisms in the category of quasimetric spaces to be closeness classes of Lipschitz for large distances maps between these spaces and where composition is computed by composing representatives of these classes.

As a consequence, a Lipschitz for large distances map $X \xrightarrow{\phi} M$ between two quasimetric spaces is a closeness representative of an *isomorphism* between X and M exactly when there is $M \xrightarrow{\psi} X$, Lipschitz for large distances, so that both $\psi\phi$ and $\phi\psi$ are close to the identities on X and M, respectively, i.e., so that

$$\sup_{x\in X} d\big(\psi\phi(x), x\big) < \infty \quad \text{and} \quad \sup_{z\in M} \partial\big(\phi\psi(z), z\big) < \infty$$

for some/any choice of compatible metrics d, ∂ on X and M.

Whereas motivating the discussion of isomorphisms here, in practice we shall often avoid equivalence classes of metrics and maps and simply work with representatives from these classes. In this way, a map between metric spaces is called a *quasi-isometry* if it is a representative for an isomorphism between the associated quasimetric spaces.

Example 1.6. The map $\mathbb{R}^n \xrightarrow{\phi} \mathbb{Z}^n$ given by $\phi(x_1, \ldots, x_n) = \big(\lfloor x_1 \rfloor, \ldots, \lfloor x_n \rfloor\big)$ is a quasi-isometry whose inverse is the inclusion map $\mathbb{Z}^n \to \mathbb{R}^n$ when both are given the euclidean metric.

It is obvious that every metric d on a set X induces not only a metric space (X, d) but also a Lipschitz, locally Lipschitz, and quasimetric space by taking the respective equivalence classes of the metric. Moreover, because the morphisms in the category of a metric space are (not necessarily surjective) *isometries*, these are also automatically morphisms in the other categories.

On the other hand, whereas not every topological vector space X has a Lipschitz structure compatible with its topology, if X happens to be the reduct of a normed vector space, then all norms compatible with the topology on X are bi-Lipschitz equivalent and thus X is naturally equipped with the Lipschitz structure induced by these norms. This is just a consequence of the simple fact that a continuous linear operator between normed spaces is bounded and therefore Lipschitz.

Although there are counter-examples in the nonseparable case (see [9, Example 7.12]), the outstanding problem regarding Lipschitz structure on Banach spaces is whether this completely determines the linear structure.

Problem 1.7. Suppose X and Y are bi-Lipschitz equivalent separable Banach spaces. Must X and Y also be isomorphic as topological vector spaces?

Even though it is generally felt that the answer should be negative, there are several partial positive results, e.g., [26] and [23]. Foremost among these is the following.

Theorem 1.8 (S. Heinrich and P. Mankiewicz [26, Theorem 2.6])**.** *Suppose X and Y are bi-Lipschitz equivalent separable dual Banach spaces and assume that $X \cong X \oplus X$ and $Y \cong Y \oplus Y$ as topological vector spaces. Then X and Y are isomorphic as topological vector spaces.*

This applies, for example, to reflexive spaces such as ℓ^p and $L^p([0,1])$ for $1 < p < \infty$.

1.4. **Banach spaces as uniform spaces.** Evidently, every map between metric spaces that is Lipschitz for short distances is automatically uniformly continuous. In particular, this means that the uniform structures $\mathcal{U}_d$ and $\mathcal{U}_\partial$ given by two locally Lipschitz equivalent metrics d and ∂ must coincide, i.e., $\mathcal{U}_d = \mathcal{U}_\partial$. However, to give a proper presentation of this and also to motivate the category of coarse spaces, recall the definition of uniform structures.

Definition 1.9 (A. Weil [55])**.** A *uniform space* is a set X equipped with a *filter* $\mathcal{U}$ of subsets $E \subseteq X \times X$, called *entourages*, satisfying

(1) $\Delta \subseteq E$ for all $E \in \mathcal{U}$;
(2) if $E \in \mathcal{U}$, then $E^{-1} = \{(y, x) \mid (x, y) \in E\} \in \mathcal{U}$;
(3) if $E \in \mathcal{U}$, then $F \circ F = \{(x, z) \mid \exists y\, (x, y), (y, z) \in F\} \subseteq E$ for some $F \in \mathcal{U}$.

Here $\Delta = \{(x, x) \mid x \in X\}$ denotes the diagonal in $X \times X$. Recall that if d is an écart (a.k.a. pseudo-, pre-, or semimetric) on a set X (i.e., d is a metric except that possibly $d(x, y) = 0$ for distinct $x, y \in X$), then the induced uniform structure $\mathcal{U}_d$ is the filter generated by the family of entourages

$$E_\alpha = \{(x, y) \mid d(x, y) < \alpha\}$$

for $\alpha > 0$.

Also, a morphism between two uniform spaces $(X, \mathcal{U})$ and $(M, \mathcal{V})$ is simply a uniformly continuous map $X \xrightarrow{\phi} M$, that is, satisfying

$$\forall F \in \mathcal{V} \ \exists E \in \mathcal{U} \colon (x, y) \in E \Rightarrow (\phi x, \phi y) \in F.$$

Again, as the notion of substructure is apparent, we obtain a notion of uniform embeddings, namely, isomorphism with a substructure.

Important early work on the uniform classification of Banach spaces was done by P. Enflo, J. Lindenstrauss, and M. Ribe, who established a number of rigidity results for these. For example, the combined results of Lindenstrass [35] and Enflo [14] establish that if $1 \leqslant p < q < \infty$, then the spaces $L^p([0,1])$ and $L^q([0,1])$ are not uniformly homeomorphic. However, whereas this distinguishes between the L^p spaces, it does not tell an L^p space apart from an arbitrary space. Regarding this, W. B. Johnson, J. Lindenstrauss, and G. Schechtman [28] show that if a Banach space X is uniformly homeomorphic to ℓ^p for some $1 < p < \infty$, then X is actually isomorphic to ℓ^p as topological vector spaces. Considering instead uniform embeddings, let us just mention the result of Enflo [15] stating that not every separable Banach space embeds uniformly into ℓ^2.

For the record, let us mention that, as opposed to the Lipschitz category, it is known that the uniform structure does not determine the linear structure even in the separable case. Namely, by work of Ribe [45], there are examples of separable uniformly homeomorphic Banach spaces that are not isomorphic as topological vector spaces. Similarly, quasimetric structure does not determine uniform structure. Indeed by a result due to Kalton [32] there are separable quasi-isometric Banach spaces that are not uniformly homeomorphic.

1.5. **Banach spaces as coarse spaces.** Although we have not discussed Banach spaces viewed as quasimetric spaces, we shall now consider an even weaker category that abstracts large scale content from metric spaces in a manner similar to how uniform spaces abstract small scale content. In fact, the following definition is an almost perfect large scale counterpart to that of uniform spaces.

Definition 1.10 (J. Roe [46])**.** A *coarse space* is a set X equipped with an *ideal* $\mathcal{E}$ of entourages $E \subseteq X \times X$ satisfying

(1) $\Delta \in \mathcal{E}$;
(2) if $E \in \mathcal{E}$, then $E^{-1} \in \mathcal{E}$;
(3) if $E \in \mathcal{E}$, then $E \circ E \in \mathcal{E}$.

Again, if (X, d) is a pseudometric space, the associated coarse structure $\mathcal{E}_d$ is then the ideal generated by the entourages $E_\alpha = \{(x, y) \in X \times X \mid d(x, y) < \alpha\}$, where now we require $\alpha < \infty$ rather than $\alpha > 0$.

In particular, this means that we can define two maps $Y \xrightarrow{\phi, \psi} X$ from a set Y into a coarse space $(X, \mathcal{E})$ to be *close* if there is an entourage $E \in \mathcal{E}$ so that $(\phi y, \psi y) \in E$ for all $y \in Y$. This conservatively extends the definition of closeness from the case of metric spaces.

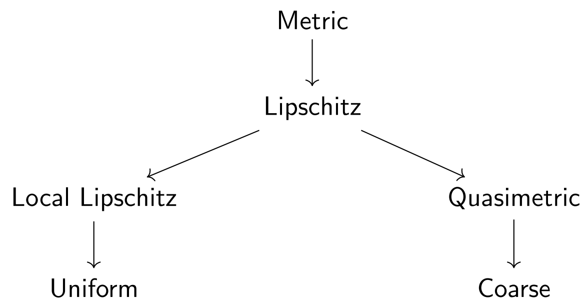

FIGURE 1. Forgetful functors between geometric categories

Definition 1.11. A map $X \xrightarrow{\phi} M$ between two coarse spaces $(X, \mathcal{E})$ and $(M, \mathcal{F})$ is *bornologous* if

$$\forall E \in \mathcal{E} \;\; \exists F \in \mathcal{F}\colon (x, y) \in E \Rightarrow (\phi x, \phi y) \in F.$$

It follows that a map $(X, d) \xrightarrow{\phi} (M, \partial)$ between pseudometric spaces is bornologous if and only if there is a monotone increasing function $\omega\colon \mathbb{R}_+ \to \mathbb{R}_+$ so that

$$\partial(\phi x, \phi y) \leqslant \omega\big(d(x, y)\big)$$

for all $x, y \in X$.

Analogously to the category of quasimetric spaces, morphisms between coarse spaces are closeness classes of bornologous maps, and so two coarse spaces $(X, \mathcal{E})$ and $(Y, \mathcal{F})$ are *coarsely equivalent* (that is, isomorphic as coarse spaces) if there are bornologous maps $X \underset{\phi}{\overset{\psi}{\rightleftarrows}} Y$ so that $\psi\phi$ and $\phi\psi$ are close to the identities on X and Y, respectively.

More concretely, note that a map $X \xrightarrow{\phi} M$ from a metric space (X, d) into a metric space (M, ∂) is a uniform embedding if

$$d(x_n, y_n) \to 0 \Leftrightarrow \partial(\phi x_n, \phi y_n) \to 0$$

for all sequences $x_n, y_n \in X$. In the same manner, $X \xrightarrow{\phi} M$ is a *coarse embedding* if, for all x_n, y_n,

$$d(x_n, y_n) \to \infty \Leftrightarrow \partial(\phi x_n, \phi y_n) \to \infty.$$

A coarse embedding is then a coarse equivalence[1] if furthermore $\phi[X]$ is *cobounded* in M, i.e.,

$$\sup_{a \in M} \mathsf{dist}(a, \phi[X]) < \infty.$$

Because Lipschitz for short distances entails uniformly continuous and Lipschitz for large distances entails bornologous, we obtain a diagram of forgetful functors between the categories of metric, Lipschitz, local Lipschitz, uniform, quasimetric, and coarse spaces as in Figure 1.

Example 1.12 (Near isometries)**.** Consider the category of metric spaces in which morphisms are closeness classes of *near isometries*, i.e., of maps $(X, d) \xrightarrow{\phi} (Y, \partial)$ so

[1]Strictly speaking, ϕ is a closeness representative of a coarse embedding.

that

$$\kappa_\phi = \sup_{x,z\in X} |d(x,z) - \partial(\phi x, \phi z)| < \infty.$$

Then two spaces are isomorphic provided there are near isometries $X \underset{\phi}{\overset{\psi}{\rightleftarrows}} Y$ so that $\psi\phi$ and $\phi\psi$ are close to the identities on X and Y, respectively. Observe that, in this category, it is easy to produce isomorphic spaces that are not isometric and also automorphisms that are not close to any auto-isometries.

We remark that, if X and Y are Banach spaces that are isomorphic in this category, then there is a surjective near isometry $X \xrightarrow{\phi} Y$ so that furthermore $\phi(0) = 0$. Furthermore, by results due to J. Gevirtz [19] and P. M. Gruber [25], for any such ϕ, there is a linear isometry $X \xrightarrow{T} Y$ with

$$\sup_x \|Tx - \phi x\| \leqslant 4\kappa_\phi.$$

In particular, this shows that any isomorphism is close to a surjective linear isometry and hence that the new notion of isomorphism coincides with linear isometry of spaces.

1.6. **Rigidity of morphisms and embeddability.** So far we have encountered rigidity results for isomorphisms and individual objects in the various categories. The following simple fact, on the other hand, will establish rigidity of morphisms.

Lemma 1.13 (General Corson–Klee lemma). *Suppose $X \xrightarrow{\phi} E$ is a map between normed vector spaces so that, for some $\delta, \Delta > 0$ and all $x, y \in X$,*

$$\|x - y\| < \delta \ \Rightarrow\ \|\phi x - \phi y\| < \Delta.$$

Then ϕ is Lipschitz for large distances.

Proof. Given $x, y \in X$, let n be minimal so that $\|x - y\| < n \cdot \delta$. Then there are $v_0 = x, v_1, \ldots, v_n = y$ so that $\|v_i - v_{i+1}\| < \delta$ for all i. It thus follows that

$$\|\phi x - \phi y\| \leqslant \sum_{i=0}^{n-1} \|\phi v_i - \phi v_{i+1}\| < n \cdot \Delta.$$

Therefore, $\|\phi x - \phi y\| < \frac{\Delta}{\delta} \cdot \|x - y\| + \Delta$. □

In particular, both a uniformly continuous and a bornologous map between two Banach spaces is automatically Lipschitz for large distances. Similarly, a uniform homeomorphism or a coarse equivalence between Banach spaces is also a quasi-isometry. On the other hand, since a uniform or coarse subspace of a Banach space need not be the reduct of linear subspace itself, a uniform or coarse embedding between Banach spaces is not in general a quasi-isometric embedding.

Remark 1.14 (Reconstruction functors). The above comments show that, when we restrict our attention to reducts of Banach or just normed vector spaces, there are reconstruction functors going from the categories of uniform, respectively coarse spaces, to quasimetric spaces. Namely, suppose $\mathcal{U}$ is the uniform structure induced from some normed vector space structure on the set X. Then we let $\mathbf{F}(X, \mathcal{U}) = (X, \mathcal{D})$ be the quasimetric space induced by some or, equivalently, any normed vector space structure on the set X that is compatible with the uniformity $\mathcal{U}$.

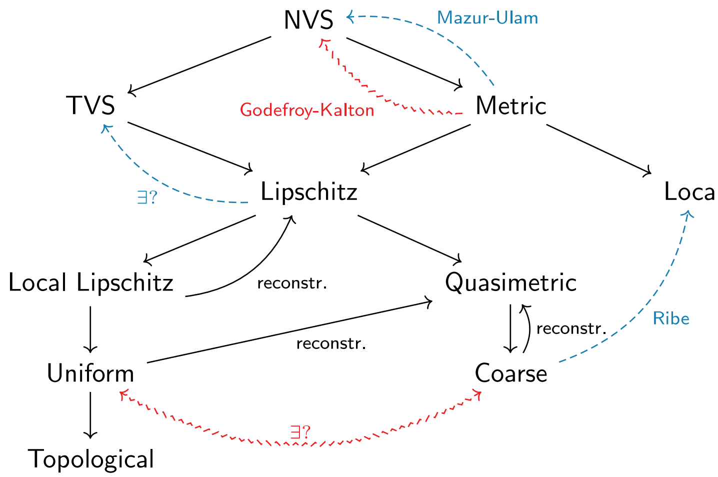

FIGURE 2. Diagram of functors between diverse categories of reducts of separable real Banach spaces. Dashed blue arrows and curly red arrows refer to rigidity results for isomorphisms, respectively, embeddings.

Indeed, if $(X, +, \|\cdot\|)$ and $(X, \oplus, |||\cdot|||)$ are two such normed vector space structures, then

$$(X, \|\cdot\|) \xrightarrow{\text{id}} (X, |||\cdot|||)$$

is a uniform homeomorphism and thus a quasi-isometric equivalence. It thus follows that the quasi-isometric equivalence classes of the norm metrics actually coincide.

Similarly, every map between Banach spaces that is Lipschitz for short distances is automatically Lipschitz for large distances and hence actually Lipschitz (for all distances). So this provides a functor from the category of Banach spaces viewed as local Lipschitz spaces to the category of Banach spaces viewed as Lipschitz spaces.

At this point, we can refer to Figure 2 for a diagram of categories and the functors relating them. All categories refer exclusively to reducts of separable real Banach spaces and the black arrows to functors. Also, dashed blue arrows refer to a rigidity result for isomorphism. For example, an isomorphism in the category of metric spaces induces another isomorphism in the category of normed vector spaces by the Mazur–Ulam theorem.

Again, whereas a functor maps isomorphisms to isomorphisms, it need not preserve embeddings, since the latter notion is not intrinsic to the category. Thus, although a uniform embedding between Banach spaces is bornologous, it need not be a coarse embedding. Nevertheless, we do have rigidity results for embeddings not stemming from functors. Indeed, for separable Banach spaces, by the Godefroy–Kalton theorem, isometric embeddings give rise to other linear isometric embeddings. This rigidity is indicated by a curly red arrow in Figure 2.

Now, even though by [32] there are separable quasi-isometric Banach spaces that are not uniformly homeomorphic, it is an open problem whether the notions of uniform and coarse embeddability between Banach spaces coincide.[2]

[2]The origins of this problem are not entirely clear, but the need for a better understanding of the connection between these notions was noted by Kalton in [31].

Problem 1.15. Are the following two conditions equivalent for all (separable) Banach spaces X and E?

(1) X uniformly embeds into E.
(2) X coarsely embeds into E.

Observe that this is far from being trivial, since it is easy to produce uniform embeddings that are not coarse embeddings, and vice versa. Also, one cannot hope to replace coarse embeddings by quasi-isometric embeddings, since, for example, ℓ^1 embeds into ℓ^2 uniformly, but not quasi-isometrically.

Theorem 1.16. *Assume X and E are Banach spaces and that $E \oplus E$ embeds as a topological vector space into E. Suppose also $X \xrightarrow{\phi} E$ is uniformly continuous and that, for some $\delta, \Delta > 0$,*

$$\|x - y\| > \Delta \ \Rightarrow\ \|\phi x - \phi y\| > \delta.$$

Then there is a simultaneously uniform and coarse embedding $X \xrightarrow{\psi} E$.

Proof. As $E \oplus E$ embeds into E, we may inductively construct three sequences E_n, Z_n, V_n of closed linear subspaces of E so that $E_n \cong Z_n \cong E$ as topological vector spaces and

$$E_{n+1} \oplus Z_{n+1} \subseteq Z_n$$

and

$$V_n = E_1 \oplus E_2 \oplus \cdots \oplus E_n \oplus Z_n.$$

Indeed, we simply begin with an isomorphic copy V_1 of $E \oplus E$ inside of E, and let E_1 and Z_1 be the first and second summand, respectively. Again, pick a copy of $E \oplus E$ inside of Z_1 with first and second summand denoted respectively E_2 and Z_2 and let $V_2 = E_1 \oplus E_2 \oplus Z_2 \subseteq V_1$, etc.

Let also P_n denote the projection of V_n onto the summand E_n along the decomposition above. While each P_n is bounded, there need not be any uniform bound on their norms. Note now that $V_1 \supseteq V_2 \supseteq \cdots$, so we can let $V = \bigcap_{n=1}^{\infty} V_n$, which is a closed linear subspace of E containing all of the E_n. Moreover, the P_n all restrict to bounded projections $P_n \colon V \to E_n$ so that $E_m \subseteq \ker P_n$ whenever $n \neq m$.

Composing ϕ with linear isomorphisms between E and E_n, we get a sequence of uniformly continuous maps $X \xrightarrow{\phi_n} E_n$ satisfying $\|x - y\| > \Delta_n \ \Rightarrow\ \|\phi x - \phi y\| > \delta_n$ for some $\Delta_n, \delta_n > 0$ and bounded projections $P_n \colon V \to E_n$ so that $E_m \subseteq \ker P_n$ for $n \neq m$. By Lemma 1 of [48], this implies that X admits a simultaneously coarse and uniform embedding into V and thus into E. □

Observe that if $X \xrightarrow{\phi} E$ is either a uniform or coarse embedding between Banach spaces, then there are $\Delta, \delta > 0$ as in Theorem 1.16. Therefore, apart from the mild assumption that $E \oplus E$ embeds as a topological vector space into E, we have the implication (1)⇒(2) in Problem 1.15.

Corollary 1.17. *Suppose X and E are Banach spaces so that $E \oplus E$ embeds as a topological vector space into E. Then, if X uniformly embeds into E, X also coarsely embeds into E.*

On the other hand, if a coarse embedding could always be strengthened to be uniformly continuous, then we would essentially have proved the converse direction (2)⇒(1). However, one must contend with the following serious obstruction.

Theorem 1.18 (A. Naor [42, Theorem 1])**.** *There is a bornologous map* $X \xrightarrow{\phi} E$ *between separable Banach spaces that is not close to any uniformly continuous map.*

The above results indicate that the uniform structure of a Banach space is more rigid than the coarse structure. However, once we pass to the underlying topology, almost no information is left. Indeed, it is a result of M. I. Kadets [29] and H. Toruńczyk [53] that any two infinite-dimensional Banach spaces of the same density character are homeomorphic. Furthermore, in combination with a result of R. D. Anderson [2], it follows that all separable infinite-dimensional Banach spaces are all homeomorphic to the countable product of lines, $\mathbb{R}^{\mathbb{N}}$.

Remark 1.19 (Universal spaces). In the various categories above, it is interesting to search for *universal spaces*, that is, separable spaces into which every other separable spaces embeds. For example, a classical result states that, for K an uncountable compact metric space, $C(K)$ is universal in the category NVS; every separable Banach space admits an isometric linear embedding into $C(K)$. Similarly, by a result of I. Aharoni [1], c_0 is universal in the category $\mathsf{Lipschitz}$.

In contradistinction to this, F. Baudier, G. Lancien, and T. Schlumprecht [7] recently showed that there is no infinite-dimensional space that coarsely embeds into all infinite-dimensional spaces. And when combined with a result of Y. Raynaud [43], one sees that the same holds for uniform embeddings.

1.7. **Banach spaces as local objects.** The results of Enflo, Johnson, Lindenstrauss, and Schechtman [14, 28, 35] mentioned earlier show rigidity for the uniform structure of the individual spaces $L^p([0,1])$ and ℓ^p. However, there is also a beautiful rigidity result due to Ribe encompassing all Banach spaces. To explain this, we need a technical concept.

Definition 1.20. A Banach space X is said to be *crudely finitely representable* in a Banach space Y if there is a constant K so that, for every finite-dimensional subspace $E \subseteq X$, there is a finite-dimensional subspace $F \subseteq Y$ and a linear isomorphism $E \xrightarrow{T} F$ with $\|T\| \cdot \|T^{-1}\| \leqslant K$.

We then say that X and Y are *locally isomorphic* in case they are crudely finitely representable in each other. In [44], Ribe then establishes the surprising fact that any two uniformly homeomorphic spaces must be locally isomorphic. In particular, this implies that all local properties of Banach spaces, i.e., that only depend on the finite-dimensional subspaces (up to some uniform constant of isomorphism), are in principle expressible in terms of the uniform structure of the entire space. This, in turn, has motivated to so called *Ribe programme* (see, e.g., Naor [41]) of identifying exclusively metric expressions for these various local invariants of Banach spaces, such as convexity, smoothness, type, and cotype, which furthermore then become applicable not only in the linear setting but to metric spaces in general.

Subsequent proofs of Ribe's theorem go by showing that if X and Y are quasi-isometric separable spaces, then X and Y have bi-Lipschitz equivalent ultrapowers $X^{\mathcal{U}}$ and $Y^{\mathcal{U}}$. Moreover, if $V \xrightarrow{\phi} W$ is a bi-Lipschitz embedding of a separable Banach space V into a Banach space W, then, using differentiation techniques, V embeds as a topological vector space into W^{**}. In particular, the diagonal copy of X in $X^{\mathcal{U}}$ embeds as a topological vector space into $(Y^{\mathcal{U}})^{**}$. Now, by the principle of local reflexivity, $(Y^{\mathcal{U}})^{**}$ is crudely finitely representable in $Y^{\mathcal{U}}$ and, by the nature of ultrapowers, $Y^{\mathcal{U}}$ is crudely finitely representable in Y. Combined, this shows that if X and Y are quasi-isometric separable spaces, then X is crudely finitely representable in Y and vice versa, i.e., X and Y are locally isomorphic. As uniformly homeomorphic or coarsely equivalent spaces are also quasi-isometric, Ribe's theorem follows.

One may think of Banach spaces as objects in the category Local of *local spaces* in the following sense. The objects of the category are simply separable Banach spaces, and we put an arrow $X \to Y$ from X to Y in case X is crudely finitely representable in Y. Observe that, in this way, an arrow $X \to Y$ does not necessarily correspond to the existence of a special type of function from X to Y. However, if X isometrically embeds into Y, then X also linearly isometrically embeds and thus is crudely finitely representable in Y. This means that we obtain a last functor from the category of metric reducts of separable Banach spaces to Local.

In Figure 2, Ribe's theorem is indicated as an arrow from the category Coarse to Local. His original rigidity theorem (that is, the arrow from $\mathsf{Uniform}$ to Local) is then obtained by composition with the functors from $\mathsf{Uniform}$ to $\mathsf{Quasimetric}$ and further onto Coarse.

When we restrict the category Local to infinite-dimensional spaces, we have initial and terminal objects X and Y, that is, so that for every Z there are (trivially unique) arrows

$$X \to Z \to Y.$$

Indeed, by a result of A. Dvoretzky, Hilbert space ℓ^2 is crudely finitely representable in every infinite-dimensional Banach space (see [18]), whereas, by a result of S. Kwapień [34] any space crudely finitely representable in ℓ^2 has type and cotype 2 and must be isomorphic to ℓ^2 as a topological vector space. Thus, up to isomorphism, ℓ^2 is the unique initial object.

On the other hand, Y is a terminal object exactly when ℓ^∞ is crudely finitely representable in Y, which by a result of B. Maurey and G. Pisier [38] is equivalent to Y only having trivial cotype. This shows that, for example, c_0 and the reflexive space

$$(\ell^\infty(2) \oplus \ell^\infty(3) \oplus \cdots)_{\ell^2}$$

are terminal.

For Banach spaces, there are also interesting concepts of minimality of objects, which can be phrased as being an initial object in an appropriate category. Namely, a separable infinite-dimensional Banach space X is said to be *minimal* if X embeds as a topological vector space into all of its infinite-dimensional closed subspaces. Similarly, X is *locally minimal* if X is crudely finitely representable in all its closed infinite-dimensional subspaces. Both of these concepts allow for Ramsey style dichotomies that establish canonical obstructions for containing (locally) minimal subspaces (see [17, Theorems 1.1 and 1.2]). Specifically, every infinite-dimensional Banach space contains an infinite-dimensional closed linear subspace X satisfying one of the following:

(1) X is crudely finitely representable in all its infinite-dimensional subspaces;
(2) X has a Schauder basis $(x_n)_{n=1}^\infty$ so that no infinite-dimensional subspace $Y \subseteq X$ is crudely finitely representable in all tail subspaces $X_m = [x_n]_{n=M}^\infty$ with a uniform constant.

Let us end this section by noting that, particularly through the impetus of J. Bourgain and A. Naor, the nonlinear and metric theory of Banach spaces has blossomed into a very rich theory with deep connections to computer science. An overview of some of these topics can be found in G. Godefroy's survey [21].

2. Geometric structures on topological groups

2.1. **Uniform and local Lipschitz structure.** In the preceding section, we have introduced various geometric structures through the instructive example of Banach spaces. In this case, once the categories are understood, there is no discussion of what the appropriate structure of a Banach space is, since it is just obtained by stripping away information. Also, we saw how one may reconstruct, e.g., affine structure from the metric structure and quasimetric structure from the uniform structure. However, for topological groups that do not a priori have this additional structure, the problem is the reverse. Namely, how and when can we endow the abstract topological group with a canonical structure of a given type.

Recall that a topological group is simply a group G equipped with a topology in which the group operations are continuous. Even a Lie group may just be seen as a locally compact, locally euclidean group (in the light of the solution to Hilbert's fifth problem) and thus simply a special type of topological group without any further differentiable structure. For simplicity, **all topological groups will henceforth be assumed to be Hausdorff**.

Now, apart from being a topological space, a topological group G also has a couple of canonical uniform structures associated with it. The most interesting in this context in the *left-uniform structure* $\mathcal{U}_L$, which is the filter on $G\times G$ generated by entourages

$$E_V = \{(x,y)\in G\times G \mid x^{-1}y\in V\},$$

where V ranges over identity neighbourhoods in G. Observe that if $(X,+)$ is the additive topological group of a Banach space, this is simply the uniform structure given by the norm metric.

As always, with uniform spaces it is often useful to work with écarts generating the uniformity and, in the case of groups, one can even require these to be compatible with the algebraic structure. Indeed, an écart d is said to be *left-invariant* if

$$d(xy,xz)=d(y,z)$$

for all $x,y,z\in G$.

Theorem 2.1 (A. Weil [55])**.** *The left-uniform structure $\mathcal{U}_L$ on a topological group G is given by*

$$\mathcal{U}_L=\bigcup_d \mathcal{U}_d,$$

where the union is over all continuous left-invariant écarts d on G.

In fact, prior to this, independently G. Birkhoff [10] and S. Kakutani [30] showed that if a topological group G satisfies only a weak consequence of metrisability, namely, if it is first countable, then G in fact admits a *compatible* left-invariant metric d, i.e., inducing the topology of G. Moreover, in this case, by left-invariance, this metric will also be compatible with the left-uniform structure, that is, $\mathcal{U}_L=\mathcal{U}_d$.

In short, the following properties are equivalent for an arbitrary topological group G:

(1) $\mathcal{U}_L$ is metrisable;
(2) G admits a compatible left-invariant metric;
(3) G is first countable.

Apart from exceptional circumstances, one should not expect that a canonical metric, even up to rescaling, should exist on a metrisable topological group. Nevertheless, it is instructive to look at what such a metric should do for us. Because the general case is not much different, we shall not assume metrisability of G at the outset and hence deal with écarts rather than metrics.

First of all, an écart d should be continuous. This ensures that the induced topology τ_d is coarser than that τ_G of G itself. Secondly, to enforce compatibility with the algebraic structure, we should also require the écart to be left-invariant, which then guarantees that the uniform structure $\mathcal{U}_d$ is coarser than the left-uniform structure $\mathcal{U}_L$. Finally, in case G is metrisable, d can be assumed to be compatible with the topology, whereby actually $\mathcal{U}_L = \mathcal{U}_d$. By the results of Birkhoff, Kakutani, and Weil cited above, these requirements can always be fulfilled.

These were the general requirements. Now, how could we identify a canonical local Lipschitz structure on G? Because a local Lipschitz structure automatically gives us a metrisable uniform structure, we shall focus exclusively on metrics.

Definition 2.2 ([49])**.** A compatible left-invariant metric d on a topological group G is said to be *minimal* if, for every other compatible left-invariant metric ∂ on G,

$$(G,\partial) \xrightarrow{\mathsf{id}} (G,d)$$

is Lipschitz for short distances.

In fact, this definition relies on a quasi-ordering of metrics by setting $d \ll_L \partial$ if $(G,\partial) \xrightarrow{\mathsf{id}} (G,d)$ is Lipschitz for short distances. Then a minimal metric is just a minimum in this ordering.[3]

Clearly, any two minimal metrics are locally Lipschitz equivalent and thus define a local Lipschitz structure on G, which furthermore is compatible with the left-uniform structure on the group.

Definition 2.3. The *local Lipschitz structure* of a topological group (if it exists) is that given by any minimal metric.

Despite being conceptually clear, the definition of minimal metrics is unfortunately highly impredicative as it involves a quantification over objects of the same type, namely left-invariant metrics and hence functions on G. So is there a characterisation of minimal metrics that only quantifies over elements of G? For this, it will be slightly more convenient to consider length functions in place of metrics. So let us recall that a *length function*[4] on a (topological) group G is a map $\ell\colon G \to [0,\infty[$ so that

(1) $\ell(x) = 0 \Leftrightarrow x = 1_G$;
(2) $\ell(xy) \leqslant \ell(x) + \ell(y)$;
(3) $\ell(x^{-1}) = \ell(x)$.

[3]Despite the terminology, it is not clear if minimal elements in the quasi-ordering $\ll_L$ are automatically minimum too.

[4]Sometimes also called a *norm*.

Observe also that there is a bijective correspondence between length functions and left-invariant metrics on G given by $\ell(x) = d(x, 1)$ and $d(x, y) = \ell(x^{-1}y)$. This, of course, generalises the correspondence between the norm $\|\cdot\|$ on a Banach space and its associated norm distance. Note also that the metric d is continuous if and only if ℓ is continuous and that d is compatible with the topology on G if and only if ℓ is *compatible*, in the sense that the balls $B_\ell(\epsilon) = \{x \in G \mid \ell(x) < \epsilon\}$ form a neighbourhood basis at the identity in G. We thus say that a compatible length function ℓ on G is *minimal* if and only if the corresponding metric is minimal.

Theorem 2.4 ([49, Theorem 3])**.** *The following conditions are equivalent for a compatible length function ℓ on a topological group G:*

(1) *ℓ is minimal;*

(2) *for some identity neighbourhood U, constant K, and all $n \geqslant 1$ and $x \in G$,*

$$x, x^2, x^3, \ldots, x^n \in U \;\Rightarrow\; n \cdot \ell(x) \leqslant K \cdot \ell(x^n);$$

(3) *for some constants $\epsilon > 0$, K and all $n \geqslant 1$ and $x \in G$,*

$$n \cdot \ell(x) \leqslant \epsilon \;\Rightarrow\; n \cdot \ell(x) \leqslant K \cdot \ell(x^n).$$

Because $\ell(x^n) \leqslant n \cdot \ell(x)$ holds for all n and x, condition (2) is a linear growth condition on ℓ in an identity neighbourhood of the group. Thus, for example, the norm on a Banach space X is a minimal length function on the additive topological group $(X, +)$.

Condition (2) is obviously much simpler that the initial definition of minimality and also has the nontrivial consequence that the restriction of a minimal metric d on G to a subgroup $H \leqslant G$ with the induced topology is also minimal on H.

Example 2.5. Consider the Banach space $L^1([0, 1])$ of integrable real-valued functions on $[0, 1]$, and let

$$G = \{f \in L^1([0, 1]) \mid \mathsf{im}(f) \subseteq \mathbb{Z}\}.$$

Then G is a closed additive subgroup of $L^1([0, 1])$, whereby the norm is also minimal when restricted to G. Observe also that if $f \in G$, then $f_\lambda = f \cdot 1_{[0,\lambda]}$, $\lambda \in [0, 1]$, defines a continuous path from 0 to f in G. Thus, G is a connected abelian topological group with a minimal metric, in fact, a closed subgroup of a Banach space.

Whereas Theorem 2.4 provides us with a simple characterisation of minimal metrics, we do not have any informative reformulation of which groups admit a minimal metric and hence, equivalently, a local Lipschitz structure. The language of descriptive set theory allows us to make this question precise at least for the well-behaved class of *Polish* groups, that is, completely metrisable separable topological groups. Concretely, the class of Polish groups can be parametrised by a standard Borel space Gp, e.g., by letting Gp be the Effros–Borel space of closed subgroups of some injectively universal Polish group such as the homeomorphism group $\mathsf{Homeo}([0, 1]^{\mathbb{N}})$ of the Hilbert cube [54].

Problem 2.6. Is the class of Polish groups admitting a minimal metric a Borel set in the standard Borel space Gp of Polish groups?

A positive answer would show that one can characterise these groups without simply asking for an object of the same complicated type as a minimal metric itself. For locally compact second countable groups, the answer to Problem 2.6 is already known. Indeed, condition (3) of Theorem 2.4 formulated for the corresponding metric appears under the name *weak Gleason metric* in T. Tao's book [52]. Furthermore, by A. Gleason, D. Montgomery, H. Yamabe, and L. Zippin's solution to Hilbert's fifth problem along with Tao's exposition of this in [52] and Theorem 2.4, the following equivalent conditions for a locally compact second countable group emerge:

(1) G is locally euclidean;
(2) G has *no small subgroups*, that is, there is an identity neighbourhood in G not containing any nontrivial subgroup;
(3) G has a weak Gleason metric;
(4) G has a minimal metric;
(5) G is a Lie group.

In particular, a locally compact second countable group has a canonical local Lipschitz structure if and only if it is a Lie group.

Beyond locally compact groups, the problem of characterising those admitting minimal metrics remains largely open. The strongest positive result in this direction is the fact that connected Banach–Lie groups also admit minimal metrics. This is a recent result of H. Ando, M. Doucha, and Y. Matsuzawa [3] extending a previous, unpublished result due to C. Badea for the unitary groups of complex Banach algebras. We shall return to this in Section 2.3. However, as opposed to the case of finitely dimensional Lie groups, not every closed subgroup of a Banach–Lie group is Banach–Lie, whereas it will still have a minimal metric. In the light of the case of locally compact groups, it is therefore tempting to conjecture that the topological groups admitting a minimal metric are exactly the closed subgroups of Banach–Lie groups.

Working towards this conjecture, let us first verify that some amount of Lie group structure follows from having a minimal metric even in the general case.[5] For this, let us say that a topological group G has *ample square roots* if, for every identity neighbourhood W, the set of squares

$$\{g^2 \mid g \in W\}$$

is dense in some other identity neighbourhood. Recall also that a *one-parameter subgroup* of a topological group G is a continuous homomorphism $x\colon \mathbb{R} \to G$. We let $\mathfrak{g}$ denote the collection of all one-parameter subgroups of G and let

$$\mathfrak{g} \xrightarrow{\exp} G$$

be the map given by $\exp(x) = x(1)$. If $x \in \mathfrak{g}$ and $a \in \mathbb{R}$, let also $ax \in \mathfrak{g}$ be defined by $(ax)(t) = x(at)$. The following result (slightly simplifying a result of [49]) has its origins in work of C. Chevalley [11], Enflo [16], and Gleason [20].

Theorem 2.7. *Let G be a completely metrisable group with a minimal metric and ample square roots. Then there is neighbourhood basis at the identity consisting of sets W so that each $f \in W$ is of the form $f = \exp(x)$ for a unique $x \in \mathfrak{g}$ satisfying $x(t) \in W$ for all $t \in [-1, 1]$.*

[5]Because minimal metrics are exactly the weak Gleason metrics, it is possible to imitate the proof of Hilbert's fifth problem from [52] by successively adding the extra assumptions needed, which happen to be automatic in the case of locally compact groups. Although this procedure is rather straightforward, we keep matters simple and focus only on the abelian case.

Proof. For any identity neighbourhood V, let

$$\mathfrak{g}_V = \{x \in \mathfrak{g} \mid x(t) \in V \text{ for all } t \in [-1,1]\}.$$

By [49, Theorem 25], there are open identity neighbourhoods $U \supseteq O$ so that every $f \in O$ is of the form $f = \exp(x)$ for a unique $x \in \mathfrak{g}_U$. Moreover, by the last paragraph of the proof, U and thus also O can be supposed to be arbitrarily small. It follows that the collection

$$\mathcal{N} = \{\exp[\mathfrak{g}_V] \mid V \text{ is an identity neighbourhood and } V \subseteq O\}$$

is a neighbourhood basis at the identity.

Suppose now that V is a given identity neighbourhood. Then, for any $x \in \mathfrak{g}_V$ and $r \in [-1,1]$, we have $(rx)(t) = x(rt) \in V$ for all $t \in [-1,1]$, which shows that also $rx \in \mathfrak{g}_V$. We claim that

$$\mathfrak{g}_{\exp[\mathfrak{g}_V]} = \mathfrak{g}_V.$$

Indeed, $\exp[\mathfrak{g}_V] \subseteq V$ and so also $\mathfrak{g}_{\exp[\mathfrak{g}_V]} \subseteq \mathfrak{g}_V$. Conversely, if $x \in \mathfrak{g}_V$ and $r \in [-1,1]$, then $rx \in \mathfrak{g}_V$ and so $x(r) = \exp(rx) \in \exp[\mathfrak{g}_V]$, which shows that $x \in \mathfrak{g}_{\exp[\mathfrak{g}_V]}$ and hence that $\mathfrak{g}_{\exp[\mathfrak{g}_V]} \supseteq \mathfrak{g}_V$.

Suppose now that $W \in \mathcal{N}$ and write $W = \exp[\mathfrak{g}_V]$ for some identity neighbourhood $V \subseteq O$. Then

$$\exp[\mathfrak{g}_W] = \exp[\mathfrak{g}_{\exp[\mathfrak{g}_V]}] = \exp[\mathfrak{g}_V] = W.$$

Moreover, because $W = \exp[\mathfrak{g}_V] \subseteq V \subseteq O \subseteq U$, we see that $\mathfrak{g}_W \xrightarrow{\exp} W$ is also injective. □

The notation $\mathfrak{g}$ for the space of one-parameter subgroups of course indicates that we aim to make $\mathfrak{g}$ the Lie algebra associated with G. Whereas this may not pan out in general, let us first note that the the right derivative at 0 of the function $\ell(x(\cdot))$ defines an appropriate "norm" of $x \in \mathfrak{g}$. So, let G be a topological group with minimal length function ℓ, and let U and K be as in condition (2) of Theorem 2.4. Choose also $\epsilon > 0$ small enough so that $g \in U$ whenever $\ell(g) < \epsilon$.

Lemma 2.8. *Assume that $x \in \mathfrak{g}$, $x \neq \mathbf{0}$. Then*

$$\lim_{t\to 0} \frac{\ell(x(t))}{|t|} = \sup_{t\neq 0} \frac{\ell(x(t))}{|t|} \leqslant \frac{K\epsilon}{t_x},$$

where

$$t_x = \inf\big\{t \in \mathbb{R}_+ \mid \ell(x(t)) \geqslant \epsilon\big\}.$$

Proof. Because x is nonconstant, $\ell(x(t)) > \epsilon$ for some $t > 0$ and so

$$t_x = \inf\big\{t \in \mathbb{R}_+ \mid \ell(x(t)) \geqslant \epsilon\big\} > 0$$

and $\ell(x(t_x)) = \epsilon$. We first note that

$$\sup_{t\neq 0} \frac{\ell(x(t))}{|t|} \leqslant \frac{K\epsilon}{t_x}.$$

Indeed, because $\ell(x(-t)) = \ell(x(t)^{-1}) = \ell(x(t))$ and both x and ℓ are continuous, it suffices to verify that $\frac{\ell(x(rt_x))}{|rt_x|} \leqslant \frac{K\epsilon}{t_x}$ for all rational numbers $r > 0$. So suppose $p, q \geqslant 1$ are natural numbers and note that

$$\frac{\ell\big(x(\frac{pt_x}{q})\big)}{|\frac{pt_x}{q}|} = \frac{q\cdot\ell\big(x(\frac{t_x}{q})^p\big)}{pt_x} \leqslant \frac{qp\cdot\ell\big(x(\frac{t_x}{q})\big)}{pt_x} \leqslant \frac{K\cdot\ell\big(x(\frac{t_x}{q})^q\big)}{t_x} = \frac{K\cdot\ell\big(x(t_x)\big)}{t_x} = \frac{K\epsilon}{t_x}.$$

We now show that

$$\lim_{t\to 0}\frac{\ell(x(t))}{|t|}=\sup_{t\neq 0}\frac{\ell(x(t))}{|t|}. \tag{1}$$

Indeed, suppose $\frac{\ell(x(t))}{t}>\delta$ for some $t>0$ and $\delta>0$. By continuity of x and ℓ, we can find some $\eta>0$ so that

$$\frac{\ell(x(s))}{s}>\delta$$

whenever $|s-t|<\eta$. Now, if $0<r<\eta$, there is $n\geqslant 1$ so that $|nr-t|<\eta$, whereby

$$\delta<\frac{\ell(x(nr))}{nr}\leqslant\frac{n\cdot\ell(x(r))}{nr}=\frac{\ell(x(r))}{r},$$

showing that also $\liminf_{r\to 0}\frac{\ell(x(r))}{r}\geqslant\delta$. Thus, $\liminf_{t\to 0}\frac{\ell(x(t))}{|t|}\geqslant\sup_{t\neq 0}\frac{\ell(x(t))}{|t|}$, which proves (1). □

Suppose now further that G is abelian. Then, as is easily seen, $\mathfrak{g}$ becomes a normed vector space when equipped with the operations

$$(x+y)(t)=x(t)y(t),\qquad (ax)(t)=x(at)$$

and the norm

$$\|x\|=\lim_{t\to 0}\frac{\ell(x(t))}{|t|}=\sup_{t\neq 0}\frac{\ell(x(t))}{|t|}.$$

Observe also that the exponential map $\mathfrak{g}\xrightarrow{\exp}G$ is a continuous group homomorphism, in fact, $\ell(\exp(x))\leqslant\|x\|$.

Theorem 2.9. *Suppose G is a completely metrisable connected abelian topological group with minimal length function ℓ and ample square roots. Then $\mathfrak{g}$ is a Banach space and*

$$G\cong\mathfrak{g}/\Lambda,$$

where Λ is a discrete subgroup of $(\mathfrak{g},+)$. In particular, G is a Banach–Lie group.

Proof. Let us first note that the sets $\mathfrak{g}_W$, where W ranges over identity neighbourhoods in G, form a neighbourhood basis at $\mathbf{0}$. Indeed, that each $\mathfrak{g}_W$ is a norm neighbourhood of $\mathbf{0}$ follows from the fact that, for all $t\in[-1,1]$,

$$\ell(x(t))\leqslant\frac{\ell(x(t))}{|t|}\leqslant\|x\|.$$

Conversely, given $\delta>0$, we can find W small enough so that $\frac{K\epsilon}{t_x}<\delta$ for all $x\in\mathfrak{g}_W$, where $t_x=\inf\big\{t\in\mathbb{R}_+\mid \ell(x(t))\geqslant\epsilon\big\}$. By Lemma 2.8 it follows that $\|x\|<\delta$ for all $x\in\mathfrak{g}_W$.

As noted above, the exponential map $\mathfrak{g}\xrightarrow{\exp}G$ is a continuous group homomorphism. Because the sets $\mathfrak{g}_W$, where W ranges over identity neighbourhoods in G, form a neighbourhood basis at $\mathbf{0}$, Theorem 2.7 implies that exp restricts to a homeomorphism between two identity neighbourhoods $\mathfrak{g}_W$ and W. In particular, this shows that $\Lambda=\ker(\exp)$ is a discrete subgroup of $\mathfrak{g}$ and, because G is completely metrisable, that $\mathfrak{g}$ is complete, i.e., a Banach space. Moreover, the image of exp is an open subgroup of G, which is thus G itself, since G is connected. Finally, exp descends to an isomorphism between $\mathfrak{g}/\Lambda$ and G. □

2.2. **Coarse and quasimetric structure.** Of course having a minimal metric is already a restrictive condition among locally compact groups, and one should not expect it to be ubiquitous in other settings either. So let us instead turn our attention to quasimetric and coarse geometry.

Example 2.10 (Finitely generated groups)**.** The standard and indeed motivating example of a quasimetric geometry is that induced by the word metric

$$\rho_S(x,y) = \min(k \mid \exists s_1,\dots,s_k \in S^{\pm} \colon x = ys_1\cdots s_k)$$

on a group Γ generated by a finite generating subset $S \subseteq \Gamma$. The fundamental observation of geometric group theory is that this geometry is independent of the specific finite generating set S. Indeed, if T is another finite generating set, then there is a k so that each element of S can be written as a word of length at most k in T and so one sees that $\rho_T \leqslant k\cdot\rho_S$. By symmetry, it thus follows that the two metrics are bi-Lipschitz equivalent and hence define the same quasimetric and even Lipschitz structure.

Example 2.11 (Compactly generated groups)**.** A similar argument applies to compactly generated locally compact groups. Namely, if M and L are two symmetric compact generating sets containing 1 for a locally compact group G, then $M \subseteq M^2 \subseteq M^3 \subseteq \cdots$ is an exhaustive sequence of compact subsets and thus, by the Baire category theorem, some M^l has nonempty interior and therefore covers the compact set L by finitely many left-translates. It thus follows that $L \subseteq M^k$ for some $k \geqslant 1$ and therefore as in Example 2.10 the two word metrics ρ_M and ρ_L are Lipschitz equivalent. However, although left-invariant, the word metrics are no longer compatible with the topology on G unless G itself is discrete. But a simple argument using the construction of Birkhoff and Kakutani allows us to find a continuous left-invariant écart d representing the same quasi-isometry class as ρ_M and ρ_L.

The recent book by Y. de Cornulier and P. de la Harpe [12] provides a fuller picture of the geometric group theory of locally compact groups.

Example 2.12 (Fragmentation metrics on homeomorphism groups)**.** Fix a closed manifold M and let $\mathsf{Homeo}_0(M)$ be the identity component of the homeomorphism group equipped with the compact-open topology. We note that $\mathsf{Homeo}_0(M)$ consists of the isotopically trivial homeomorphisms. Fix also a covering $\mathcal{B}=\{B_1,B_2,\dots,B_n\}$ of M by embedded open balls and let $U_i \subseteq \mathsf{Homeo}_0(M)$ be the set of homeomorphisms g with $\mathsf{supp}(g) \subseteq B_i$. By results of R. D. Edwards and R. C. Kirby [13], every $g \in \mathsf{Homeo}_0(M)$ sufficiently close to the identity can be factored as $g = h_1\cdots h_n$ with $h_i \in U_i$. In other words,

$$U = U_1U_2\cdots U_n$$

is an identity neighbourhood in $\mathsf{Homeo}_0(M)$. Moreover, as $\mathsf{Homeo}_0(M)$ is connected, this means that U generates $\mathsf{Homeo}_0(M)$. The word metric ρ_U is called the *fragmentation metric* associated to the cover $\mathcal{B}$. Furthermore, as shown by E. Militon [40], any two such covers produce quasi-isometric fragmentation metrics and thus define a canonical quasimetric structure on $\mathsf{Homeo}_0(M)$.

Observe however that the definition on the fragmentation norm is not a priori intrinsic to the topological group, but rather depends on viewing $\mathsf{Homeo}_0(M)$ as a transformation group of the manifold M, that is, it depends on the group $\mathsf{Homeo}_0(M)$ along with its tautological action $\mathsf{Homeo}_0(M) \curvearrowright M$.

The above examples give us concrete quasimetric structures induced by word metrics, including on groups that don't have small generating sets in any reasonable topological sense. For other specific transformation groups there may be similar constructions, but is there a way to see these as instances of a general construction that applies to all groups? The correct way of doing this is to take as serious the idea that a coarse structure is somehow dual to uniform structure (without implying that there is an actual duality between these categories). We thus dualise Weil's Theorem 2.1 into a definition as follows.

Definition 2.13 ([50])**.** The *left-coarse structure* $\mathcal{E}_L$ on a topological group G is given by

$$\mathcal{E}_L = \bigcap_d \mathcal{E}_d,$$

where the intersection ranges over all continuous left-invariant écarts d on G.

As with uniform spaces, *metrisable* coarse spaces $(X, \mathcal{E})$ (that is, so that $\mathcal{E} = \mathcal{E}_d$ for some metric or, equivalently, for some écart d on the set X) are much simpler to understand than the general case. So let us call an écart d on X *coarsely proper* if it induces the coarse structure on X, i.e., if $\mathcal{E} = \mathcal{E}_d$.

Every coarse space $(X, \mathcal{E})$ has an associated *bornology* of bounded sets, i.e., an ideal $\mathcal{B}$ of subsets of X with $X = \bigcup_{B \in \mathcal{B}} B$. Namely, $B \subseteq X$ is said to be *coarsely bounded* if $B \times B \in \mathcal{E}$. A subset B of a topological group G is then coarsely bounded exactly when

$$\mathsf{diam}_d(B) < \infty$$

for every continuous left-invariant écart d on G. By left-invariance and continuity of the écarts defining $\mathcal{E}_L$, the bornology of coarsely bounded sets in G is furthermore stable under the operations

$$A \mapsto \mathsf{cl}(A), \quad A \mapsto A^{-1}, \quad (A, B) \mapsto A \cdot B.$$

Moreover, a continuous left-invariant écart d on G is coarsely proper provided the d-bounded sets are exactly the coarsely bounded sets of G.

Example 2.14 (Proper metrics)**.** Clearly every compact subset of a topological group is coarsely bounded. But conversely, by a theorem of R. A. Struble [51], every locally compact second countable group G admits a compatible left-invariant *proper* metric, i.e., so that closed sets of finite diameter are all compact. It follows that the coarsely bounded sets in G are exactly the relatively compact sets and hence that d is also coarsely proper. In particular, the coarse structure on a countable discrete group is that given by any left-invariant metric whose balls are finite.

As for minimal metrics, the characterisation of coarsely bounded sets involves quantification over a large sets of écarts, so one would like a simpler operative criterion for coarse boundedness and thus coarse properness too. If, for every identity neighbourhood V in G, there is a countable set $D \subseteq G$ so that G is generated by $V \cup D$, for example, if G is separable or connected, we do get a simpler description. Namely, a subset B of such a group G is coarsely bounded if and only if, for every identity neighbourhood V, there is a finite set $F \subseteq G$ and $k \geqslant 1$ so that

$$B \subseteq (FV)^k$$

(see [50, Proposition 2.15]). In particular, if G is Polish, the ideal of coarsely bounded sets is Borel in the Effros–Borel space $\mathcal{F}(G)$ of closed subsets of G.

Furthermore, we have an analogue of Birkhoff and Kakutani's characterisation of metrisable groups above.

Theorem 2.15 ([50, Theorem 2.38]). *The following conditions are equivalent for a Polish group G:*

(1) *the coarse structure $\mathcal{E}_L$ is metrisable;*
(2) *G admits a compatible left-invariant coarsely proper metric;*
(3) *G is* locally coarsely bounded*, i.e., has a coarsely bounded identity neighbourhood.*

As it is straightforward to see that no identity neighbourhood in the Polish group

$$\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \cdots$$

is coarsely bounded, this shows that not every Polish group has metrisable coarse structure. Nevertheless, most important transformation groups do and, in fact, they often admit a canonical compatible quasimetric structure.

Example 2.16 (Topological vector spaces). Topological vector spaces are a source of algebraically trivial, but occasionally topologically involved examples. So assume for simplicity that X is a real topological vector space. Let us first recall that a subset $A \subseteq X$ is said to be *Kolmogoroff bounded* if, for every 0-neighbourhood $U \subseteq X$, we have that

$$A \subseteq \delta U$$

for some $\delta > 0$. Usually, these are simply known as bounded sets, but we use the longer name here to distinguish them from sets that are coarsely bounded when viewed as subsets of the abelian topological group $(X, +)$.

By A. Kolmogoroff's normability criterion [33] and a simple argument using Lemma 2.8, we find that the following conditions are equivalent for X:

(1) $(X, +)$ admits a minimal metric;
(2) X is *normable*, that is, admits a topologically compatible vector space norm;
(3) X is locally *locally Kolmogoroff bounded*, i.e., has a Kolmogoroff bounded 0-neighbourhood, and is locally convex.

Because the topological group $(X, +)$ is connected, a subset $A \subseteq X$ is coarsely bounded if, for every 0-neighbourhood U, there is some n so that

$$A \subseteq \underbrace{U + \cdots + U}_{n \text{ times}}.$$

In general, coarse boundedness is strictly weaker than Kolmogoroff boundedness. For an extreme example, consider the vector space $L^0([0,1])$ of all measurable functions $[0,1] \xrightarrow{f} \mathbb{R}$, where we identify two functions if they agree almost everywhere with respect to Lebesgue measure λ. $L^0([0,1])$ becomes a Polish topological vector space when equipped with the topology of convergence in measure, where basic 0-neighbourhoods are of the form

$$U_\epsilon = \big\{ f \in L^0([0,1]) \mid \lambda\{\, |f(x)| \geqslant \epsilon \,\} < \epsilon \big\}$$

for $\epsilon > 0$. Note that, if $\frac{1}{n} < \epsilon$, then

$$L^0([0,1]) = \underbrace{U_\epsilon + \cdots + U_\epsilon}_{n \text{ times}},$$

which shows that $L^0([0,1])$ is itself coarsely bounded. On the other hand, if $\epsilon < 1$, then U_ϵ fails to be Kolmogoroff bounded in $L^0([0,1])$, so $L^0([0,1])$ is not even locally Kolmogoroff bounded.

For the two notions of local boundeness to be equivalent, we need furthermore to assume local pseudoconvexity. Here a set $A \subseteq X$ is *pseudoconvex* if $\lambda A \subseteq A$ for all $0 < \lambda \leqslant 1$ and also

$$A + A \subseteq \delta A$$

for some $\delta > 0$. Also X is *locally pseudoconvex* if it has a neighbourhood basis at 0 consisting of pseudoconvex sets.

One can then show that the following conditions are equivalent for a real topological vector space X:

(1) X is locally pseudoconvex and $(X, +)$ is locally coarsely bounded;
(2) X is locally Kolmogoroff bounded;
(3) X admits a compatible p-homogeneous length function ℓ for some $0 < p \leqslant 1$, i.e., so that $\ell(tx) = |t|^p \ell(x)$ for all $t \in \mathbb{R}$ and $x \in X$.

Here, the equivalence between (2) and (3) is given by the well-known Aoki–Rolewicz theorem, [4] and [47].

To address how a quasimetric structure compatible with the coarse structure may be defined, we first define a quasi-ordering of continuous left-invariant écarts on G by

$$\partial \lll d \Leftrightarrow (G,d) \xrightarrow{\mathsf{id}} (G,\partial) \text{ is bornologous.}$$

Then the coarsely proper écarts are simply the maximum elements of the ordering $\lll$. Refining $\lll$, we set

$$\partial \ll d \Leftrightarrow (G,d) \xrightarrow{\mathsf{id}} (G,\partial) \text{ is Lipschitz for large distances}$$

and say that a continuous left-invariant écart is *maximal* if maximum in this ordering. Since the sum of two écarts is still an écart, these are directed orderings and hence maximal elements are automatically maximum too. Also, as any two maximal écarts are obviously quasi-isometric, when they exist they induce an inherent quasimetric structure on G identifiable exclusively from the topological group structure. Moreover, because maximal écarts are also coarsely proper, the quasimetric structure is automatically compatible with the coarse structure.

As always, we are left with three main issues, namely, (i) find simpler operative charaterisations of maximal metrics, (ii) determine criteria for their existence, and (iii) analyse concrete groups.

Proposition 2.17 ([50, Proposition 2.72])**.** *The following are equivalent for a continuous left-invariant écart d on a topological group:*

(1) *d is maximal;*
(2) *d is coarsely proper and* large scale geodesic*, that is, for some constant K and all $x, y \in G$, there are $z_0 = x$, z_1, …, $z_n = y$ so that $d(z_{i-1}, z_i) \leqslant K$ and*

$$\sum_{i=1}^{n} d(z_{i-1}, z_i) \leqslant K \cdot d(x,y);$$

(3) *d is quasi-isometric to the word metric ρ_B given by a coarsely bounded generating set $B \subseteq G$.*

From condition (2), one easily gets that every outright geodesic metric is maximal and hence that the norm induces the quasimetric structure of the additive group $(X,+)$ of a Banach space. Since the norm metric is also minimal, we see that both the local Lipschitz and quasimetric structures on $(X,+)$ are what they should be, namely, those given by the norm.

One may also use condition (3) to give a simple criterion for when, e.g., Polish groups have maximal metrics and hence canonical quasimetric structure. But first a word of caution. Even for a Polish group, it is not true that the word metric ρ_B of every coarsely bounded generating set $B \subseteq G$ will induce the quasimetric structure. But, if B is either closed or if ρ_B is known to be quasimetric to a compatible metric on G, then it does.

Theorem 2.18 ([50, Theorem 2.73]). *A Polish group G admits a maximal metric and thus a quasimetric structure if and only if G is algebraically generated by a coarsely bounded subset $B \subseteq G$. Moreover, in this case, the word metric $\rho_{\overline{B}}$ associated to $\overline{B}$ induces the quasimetric structure.*

Because the Polish groups admitting maximal metrics are exactly the ones whose left coarse structure is generated by a single entourage, these are called *monogenic.*

Our examples before can now be seen as instances of this general setup and, in addition, many other groups have easily calculable quasimetric structure.

- Let Γ be a finitely generated group. Then the quasimetric structure of the discrete topological group Γ is simply that given by the word metrics.
- If G is a compactly generated locally compact second countable group, the quasimetric structure of G is that given by the word metric ρ_K, where K is any compact generating set.
- If M is a closed manifold, the quasimetric structure on $\mathsf{Homeo}_0(M)$ is that given by the fragmentation metric. In particular, the fragmentation metric is intrinsic to the topological group $\mathsf{Homeo}_0(M)$ without knowledge of its tautological action on M [37].
- Let T_n be the n-regular simplicial tree for $n = 2, 3, 4, \dots, \aleph_0$ and equip its automorphism group $\mathsf{Aut}(T_n)$ with the permutation group topology in which vertex stabilisers are declared to be open. Then, for any vertex $t \in T_n$, the orbit map
$$g \in \mathsf{Aut}(T_n) \mapsto g(t) \in T_n$$
is a quasi-isometry between $\mathsf{Aut}(T_n)$ and T_n (see [50, Example 6.34]).

Observe that, in the last example, when n is finite, $\mathsf{Aut}(T_n)$ is compactly generated locally compact. However, for $n = \aleph_0$, i.e., when the valency is denumerable, then $\mathsf{Aut}(T_n)$ is only Polish and thus cannot be compactly generated.

Of course, not every group has an inherent quasimetric structure, i.e., a maximal écart. For example, a countable, but not finitely generated, group will be such. It has a metrisable coarse structure, but any attempt at constructing a finer quasimetric structure will involve choices not dictated by the (topological) group structure.

With this framework in place, it is now possible develop substantial parts of geometric group theory in this larger setting; see [50] for an account. However, one must caution that there are dramatic changes from the theory of finitely generated or even locally compact groups to this more general setting. For example, if H

is a closed subgroup of G, then the inclusion mapping is automatically a uniform embedding and, if G and H are locally compact second countable, then it is also a coarse embedding. On the contrary, if G and H are no longer locally compact, H is in general not coarsely embedded in G and so, as opposed to minimal metrics, a coarsely proper metric on G need not restrict to a coarsely proper metric on H. This phenomenon is similar to the fact that a finitely generated subgroup of a finitely generated group may not be quasi-isometrically embedded and leads to substantial complications and new aspects of the theory that one must contend with.

Example 2.19 (Mapping class groups of infinite type surfaces)**.** A topic of high current interest in geometric topology is the study of mapping class groups of infinite type surfaces, that is, surfaces S for which the fundamental group is not finitely generated. Specifically, consider for simplicity a connected orientable surface S without boundary and define the mapping class group by

$$\mathsf{Map}(S) = \mathsf{Homeo}_+(S)/\mathsf{Homeo}_0(S),$$

where $\mathsf{Homeo}_+(S)$ is the group of orientation preserving homeomorphisms and $\mathsf{Homeo}_0(S)$ is again the component of the identity in $\mathsf{Homeo}_+(S)$. Thus, $\mathsf{Map}(S)$ is the group of isotopy classes of orientation preserving homeomorphisms of S. Because $\mathsf{Homeo}_+(S)$ is Polish and $\mathsf{Homeo}_0(S)$ a closed subgroup, also $\mathsf{Map}(S)$ is Polish, and the focus is therefore on the case when this group is uncountable. It turns out that a more combinatorial description of $\mathsf{Map}(S)$ is available. Namely, associated with the surface S, there is the so-called *curve graph* $C(S)$, whose vertices are the isotopy classes of essential simple closed curves in S and where two such classes are connected by an edge in $C(S)$ provided they admit disjoint realisations in S. It is easy to see that $\mathsf{Map}(S)$ acts by automorphisms on the graph $C(S)$, but, in fact, as shown independently by J. Bavard, S. Dowdall, and K. Rafi [8] and by J. Hernández Hernández, I. Morales, and F. Valdez [27], this action induces an isomorphism of topological groups

$$\mathsf{Map}(S) \cong \mathsf{Aut}(C(S)),$$

where the latter is equipped with the permutation group topology.

Furthermore, K. Mann and K. Rafi [36] have determined for which surfaces $\mathsf{Map}(S)$ is respectively locally coarsely bounded, monogenic, and coarsely bounded. This thus allows the partial transfer of geometric group theoretical methods into a previously unmapped territory. More information about these groups can be found in the survey paper by J. Aramayona and N. G. Vlamis [5].

One of the many beautiful results of M. Gromov's fundamental work on geometric group theory is the fact that quasi-isometric equivalence between finitely generated groups Γ and Λ is equivalent to the groups admitting a *topological coupling*, that is, a pair

$$\Gamma \curvearrowright X \curvearrowleft \Lambda$$

of commuting proper cocompact actions by homeomorphisms on a locally compact Hausdorff space X [24, Theorem 0.2.C_2']. On the one hand, this shows that one can pass from a weak metric equivalence between Γ and Λ to a more robust dynamical equivalence. On the other hand, it also provides the vantage point from which several other notions of couplings (e.g., measure theoretical) may be defined.

One direction of Gromov's theorem is rather straightforward and works in a wider generality. Namely, if $\Gamma \curvearrowright X \curvearrowleft \Lambda$ is a topological coupling, one may define a coarse equivalence $\Gamma \xrightarrow{\phi} \Lambda$ by simply requiring that, for some fixed $x \in X$ and compact set $K \subseteq X$ with $\Gamma \cdot K = X = K \cdot \Lambda$, we have

$$x \in gK\phi(g)^{-1}$$

for all $g \in \Gamma$.

For the other direction, one lets Γ and Λ act on the space Λ^Γ of functions from Γ to Λ by pre- and postcomposition with the left shifts of the groups on themselves. Clearly the actions commute, and one may simply take $X = \overline{\Gamma \cdot \phi \cdot \Lambda}$, which turns out to be locally compact.

If one tries to repeat this second construction for a coarse equivalence $G \xrightarrow{\phi} H$ between locally compact groups, one quickly realises that the action $G \curvearrowright X \subseteq H^G$ will not in general be continuous unless ϕ is uniformly continuous. Nevertheless, Gromov's theorem remains true for locally compact groups [6] and even in a much wider setting.

For this, let us say that a continuous action $G \curvearrowright X$ of a topological group on a locally compact Hausdorff space X is *coarsely proper* if, for every compact set $K \subseteq X$, the set

$$\{g \in G \mid K \cap g{\cdot}K \neq \emptyset\}$$

is coarsely bounded in G. Similarly, the action is *modest* if $\overline{B \cdot K}$ is compact for all coarsely bounded $B \subseteq G$ and compact $K \subseteq X$. In a second countable locally compact group, the coarsely bounded sets are relatively compact and hence all its actions are automatically modest. However, this is not the case for more general groups. Also, in locally compact second countable groups, coarse properness is just properness.

To better understand the condition of coarse properness, let us just note that a modest continuous action $G \curvearrowright X$ is coarsely proper exactly when the sets of the form

$$E_K = \{(g, f) \in G \times G \mid gK \cap fK \neq \emptyset\}$$

form a basis for the coarse structure on G as K varies over compact subsets of the locally compact Hausdorff space X. Now, as it turns out, not every group admits a coarsely proper modest cocompact action $G \curvearrowright X$ on a locally compact Hausdorff space. In fact, a Polish group G admits such an action exactly when G is coarsely equivalent to a proper metric space [50, Theorem 5.14]. Such G are said to have *bounded geometry* as it can be seen to be equivalent to G having bounded geometry as a coarse space in the sense of Roe [46].

Theorem 2.20 ([50, Theorem 5.31])**.** *Two Polish groups G and H of bounded geometry are coarsely equivalent exactly when they admit a* coarse coupling*, i.e., a pair of commuting, coarsely proper, modest, cocompact, continuous actions on a locally compact Hausdorff space,*

$$\Gamma \curvearrowright X \curvearrowleft \Lambda.$$

For a prototypical example of this setup, consider the group $\mathsf{Homeo}_{\mathbb{Z}}(\mathbb{R})$ of all lifts of orientation preserving homeomorphisms h of the circle S^1 to homeomorphisms $\tilde{h}$ of $\mathbb{R}$. Then $\mathsf{Homeo}_{\mathbb{Z}}(\mathbb{R})$ is given as a central extension

$$0 \to \mathbb{Z} \to \mathsf{Homeo}_{\mathbb{Z}}(\mathbb{R}) \to \mathsf{Homeo}_{+}(S^1) \to \mathsf{id}.$$

Alternatively, $\mathsf{Homeo}_{\mathbb{Z}}(\mathbb{R})$ is the group of homeomorphisms of $\mathbb{R}$ commuting with integral translations. Then $\mathsf{Homeo}_{\mathbb{Z}}(\mathbb{R})$ is a nonlocally compact Polish group coarsely equivalent with $\mathbb{Z}$ and, in fact, the canonical actions

$$\mathbb{Z} \curvearrowright \mathbb{R} \curvearrowleft \mathsf{Homeo}_{\mathbb{Z}}(\mathbb{R})$$

amount to a topological coupling of these groups.

It is worthwhile to consider the import of the large scale geometry on topological groups. Wherein lies its utility? Observe first that the coarse structure of a topological group provides a nontrivial invariant. That is, isomorphic topological groups are of course coarsely equivalent or even quasi-isometric (provided they have a well-defined quasimetric structure). This perspective may be useful for specific classes of groups or just to tell apart a few particular groups, but is unlikely to be of much practical value in classifying large groups such as $\mathsf{Homeo}_0(M)$ for compact manifolds M. A good example is Banach spaces that turn out to be incredibly difficult to tell apart up to isomorphism and a fortiori up to quasi-isometry. In fact, I would venture to postulate that, except for classes of spaces explicitly given by a set of parameters such as ℓ^p, $p \in [1, \infty]$, or $C(\alpha)$ for α a countable ordinal, there is no nontrivial family of Banach spaces that is classified up to isomorphism.

Thus rather than focusing on the entire coarse space $(G, \mathcal{E}_L)$ as an invariant for the topological group G, one should concentrate on the geometry imparted to G. For this reason, it is much easier to work with groups with metrisable coarse structure or even those admitting maximal metrics. This allows one, on the one hand, to gauge the relative *sizes* of group elements, namely, their distances to the identity, and, on the other hand, detect nontrivial geometric features of the group, for example, hyperbolicity and asymptotic dimension. One may also obtain nontrivial geometric bounds by investigating the properties of the Banach spaces into which the group coarsely embeds.

2.3. **Lipschitz geometry.** Having introduced the uniform and coarse structure and also discussed the conditions under which these can be further improved to provide locally Lipschitz and quasimetric structures, the last issue at hand is to determine when locally Lipschitz and quasimetric structure can be integrated. That is, suppose a topological group G has both a locally Lipschitz and a quasimetric structure. When are these two reducts of the same Lipschitz structure on G?

Proposition 2.21. *Suppose G has a minimal metric d and a maximal metric D (both compatible and left-invariant). Then G has a metric ∂ that is simultaneously minimal and maximal, and any two such metrics will be Lipschitz equivalent.*

Proof. Suppose first that ∂_1 and ∂_2 are both simultaneously minimal and maximal. Then, since ∂_1 is maximal,

$$(G, \partial_1) \xrightarrow{\mathsf{id}} (G, \partial_2)$$

is Lipschitz for large distances and, since ∂_2 is minimal, it is also Lipschitz for short distances. It therefore follows that the map is Lipschitz. By symmetry, we see that the two metrics are Lipschitz equivalent.

To construct ∂ from d and D, we observe first that, since D is maximal, G must be generated by a coarsely bounded set $B \subseteq G$. Then let $r > 0$ be large enough so that B is contained in the open D-ball V of radius r centred at the identity. Then

D is quasi-isometric with ρ_V and the formula

$$\partial(x,y) = \inf\Big(\sum_{i=1}^{n} d(v_i, 1_G) \mid x = yv_1 \cdots v_n \text{ and } v_i \in V\Big)$$

defines a compatible left-invariant metric on G that is quasi-isometric to ρ_V and hence also to D. Moreover, if U is an identity neighbourhood so that $U^2 \subseteq V$, then d and ∂ agree on U and hence ∂ is also minimal. Thus, ∂ is both minimal and maximal. □

By Proposition 2.21, a Lipschitz structure on G, if it exists, is simply that given by any compatible left-invariant metric that is simultaneously maximal and minimal. Moreover, the existence of this is equivalent to the conjunction of existence of locally Lipschitz and quasimetric structure.

The prime example of such groups is the class of compactly generated (locally compact) Lie groups. However, by a recent result of Ando, Doucha, and Matsuzawa [3], this even applies to Banach–Lie groups.

Theorem 2.22 ([3, Theorem A])**.** *Let G be a connected Banach-Lie group with Banach-Lie algebra $\mathfrak{g}$ and define the* exponential length *function* el_G *by the formula*

$$\mathsf{el}_G(g) = \inf\Big\{\sum_{i=1}^{n}\|X_i\| \mid n \geqslant 1,\ X_i \in \mathfrak{g},\ g = \exp(X_1)\cdots\exp(X_n)\Big\}.$$

Then the associated left-invariant metric $d(g,f) = \mathsf{el}_G(g^{-1}f)$ is a compatible metric on G that is simultaneously maximal and minimal and therefore defines the canonical global Lipschitz geometric structure on G.

It is worth noting that the metric d associated with exponential length function coincides with well-known Finsler distance (see [3, Remark 3.11]). This again means that the Finsler distance (at least up to bi-Lipschitz equivalence) is implicitly given by the abstract topological group and that the geometric group theory of G is simply the study of G with its Finsler distance.

It is now time to turn to Figure 3, the general picture of geometric categories we have constructed so far.

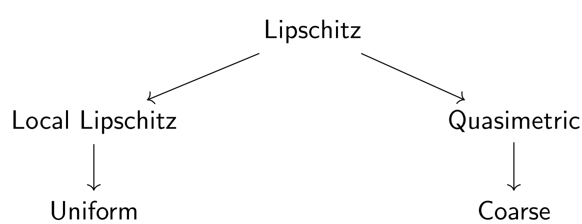

FIGURE 3. Geometric structures on topological groups

To sum up, every topological group G has canonical uniform and coarse structures $\mathcal{U}_L$ and $\mathcal{E}_L$. These may or may not be metrisable, depending on whether G is first countable, respectively, whether G is locally coarsely bounded (for Polish G). A local Lipschitz structure is then that given by a minimal metric, while a quasimetric structure is that given by a maximal metric, if such exist. However, when G has both, these are integrated into a single metric that is both maximal and minimal and defines the inherent Lipschitz geometry of G.

Not surprisingly, there is a tight relationship between the various geometric categories and, as for Banach spaces, there are rigidity phenomena of morphisms too. For example, by the proof of Lemma 1.13, we see that, if $G \xrightarrow{\phi} H$ is a bornologous map between topological groups with maximal metrics, then ϕ is automatically Lipschitz for large distances. In particular, every coarse equivalence between G and H is also a quasi-isometry.

Similarly, if $G \xrightarrow{\phi} H$ is a uniformly continuous map between topological groups and G has no proper open subgroups, then ϕ is bornologous. Thus, uniformly homeomorphic groups without proper open subgroups are also coarsely equivalent. As, for example, $\mathbb{Z}$ and $\mathbb{Z}^2$ are uniformly but not coarsely equivalent, this evidently fails in general.

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