BOOK REVIEWS

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Amenability of discrete groups by examples, by Kate Juschenko, Mathematical Surveys and Monographs, Vol. 266, American Mathematical Society, Providence, RI, 2020, xi+164 pp., ISBN 978-1-4704-7032-6

1. Amenable groups

The notion of amenability of groups was introduced by John von Neumann in his analysis [vN29] of the Hausdorff–Banach–Tarski paradox. Felix Hausdorff showed in [Hau14] that a sphere in $\mathbb{R}^3$ can be partitioned into three sets A, B, C such that A, B, C and $B \cup C$ are pairwise congruent, i.e., can be transformed into each other by rotations. This shows that there is no nonzero finitely additive measure on the sphere (and thus on $\mathbb{R}^3$) which is defined on all subsets and is invariant with respect to the group of rotations of the sphere (isometries of $\mathbb{R}^3$, respectively).

A more famous version is the Banach–Tarski paradox [BT24] stating that a ball in $\mathbb{R}^3$ can be partitioned into finitely many pieces, such that after moving the pieces by isometries of $\mathbb{R}^3$, one can obtain two balls of the same radius as the original one.

Both results are essentially about the group of rotations of the sphere. Namely, Hausdorff considers two rotations ϕ and ψ by angles π and $2\pi/3$, respectively, around two axes and shows that for all but countably many angles between the axes, the group generated by ϕ and ψ is isomorphic to the free product $C_2 * C_3$ of cyclic groups of order 2 and 3, i.e., there are no relations between ϕ and ψ except for the ones following from the obvious relations $\phi^2 = \psi^3 = 1$. Since fixed points of rotations are easy to control, one can first find a partition of the group satisfying the necessary paradoxical properties and then transfer this partition to the sphere.

Imitating the Banach–Tarski paradox, a *paradoxical decomposition* of a group G is a partition $G = A_1 \sqcup \cdots \sqcup A_n \sqcup B_1 \sqcup \cdots \sqcup B_m$ into disjoint subsets such that there exist elements $g_1, \ldots, g_n, h_1, \ldots, h_m \in G$ such that we get partitions $G = g_1A_1 \sqcup \cdots \sqcup g_nA_n = h_1B_1 \sqcup \cdots \sqcup h_mB_m$. Hausdorff–Banach–Tarski paradoxes are based on the fact that free groups of rank at least 2 admit a paradoxical decomposition.

An obvious obstacle to the existence of a paradoxical decomposition of a group G is the existence of a G-invariant finitely additive probability measure on G, i.e., a map $\mu : 2^G \longrightarrow [0,1]$ such that $\mu(G) = 1$, $\mu(A \sqcup B) = \mu(A) + \mu(B)$ for all $A, B \subset G$ such that $A \cap B = \emptyset$, and $\mu(gA) = A$ for all $g \in G$ and $A \subset G$. For example, finite groups obviously do not have paradoxical decompositions. John von Neumann noticed that existence of a paradoxical decomposition is equivalent to nonexistence of a G-invariant finitely additive probability measure.

J. von Neumann called groups with such an invariant measure "meßbar", i.e., "measurable". Later, Mahlon M. Day [Day57] introduced the English term "amenable", probably a pun based on the word "mean". It is natural to replace, in the definition of amenable groups, a finitely additive measure on G by its natural extension to a G-invariant positive linear functional on $\ell^\infty(G)$, which is then called an *invariant mean*. Groups with such means could be called "meanable" (*moyennable* in French, for example), and an anagram of this word is "amenable".

It is easy to see that existence of a paradoxical decomposition (i.e., nonamenability) is equivalent to the existence of a map $P : G \longrightarrow G$ such that $|P^{-1}(g)| = 2$ for every g, and the set $\{P(g)g^{-1} : g \in G\}$ is finite. In fact, an easy argument using the Cantor–Bernstein theorem implies that we can replace $|P^{-1}(g)| = 2$ by $|P^{-1}(g)| \geq 2$. Such a map is called a *Ponzi scheme* on the group: initially, every element has one coin, and then each element g gives their coin to their "neighbor" $P(g)$, so that each element has at least two coins. Finite groups, obviously, do not allow a Ponzi scheme. It is also easy to see that $\mathbb{Z}$ does not allow it, since if m is the maximum of $P(g) - g$ for $g \in \mathbb{Z}$, then coins from an interval $[-N, N]$ end up inside the interval $[-N - m, N + m]$, so if $(2N + 2m + 1) < 2(2N + 1)$, then it is impossible to have $|P^{-1}(g)| \geq 2$. But $(2N + 2m + 1)/(2N + 1)$ converges to 1 as $N \to \infty$.

A generalisation of this idea leads to the *Følner criterion* for amenability.

Theorem 1. *A group G is amenable if and only if for every $S \subset G$ and every $\epsilon > 0$ there exists a set F such that $\frac{|SF|}{|F|} \leq 1 + \epsilon$.*

If G is finitely generated, then it is enough to check the condition for S a generating set such that $S = S^{-1}$.

The "if" direction of Theorem 1 was explained above. The "only if" direction can be proven using a version of Hall's marriage theorem.

Sets F satisfying the conditions of the theorem (for given S and ϵ) are called *Følner sets* of the group. The condition $\frac{|SF|}{|F|} < 1 + \epsilon$ is interpreted as *almost invariance* of F. Namely, if we assume that $1 \in S$, then $SF \supset F$, so the condition means that only an ϵ-small portion of the elements of F are moved outside of F by elements of S. Averaging over F is then an *approximately invariant mean* on G. Passing to a weak limit (and using the axiom of choice in the form of weak-$*$ compactness) we can find then the invariant mean on G. This way Følner condition (and the closely related Reiter's condition) replaces a nonconstructive condition of existence of an invariant mean by a more manageable and potentially constructive condition.

Amenability of groups has many more equivalent definitions, related to a wide variety of useful applications of the concept in different areas of algebra and analysis. For example, many applications of amenability come from the following fixed point property.

Theorem 2. *A group G is amenable if and only if every action of G by affine transformations on a compact convex subset of a locally convex topological vector space has a fixed point.*

Amenability is an important property in the theory of random walks on groups. Let μ be a probability measure on G. Suppose that it is symmetric, i.e., $\mu(\{g\}) =$

$\mu(\{g^{-1}\})$ for every $g \in G$. Additionally, suppose that its support (i.e., the set of elements $g \in G$ such that $\mu(\{g\}) > 0$) generates G. Then the measure μ defines the associated *random walk*. It is the process $1, s_1, s_1s_2, s_1s_2s_3, \ldots$, where s_i are independent, μ-distributed random variables taking values in G. Denote by μ^{*n} the distribution of $g_n = s_1s_2\cdots s_n$, i.e., $\mu^{*n}(g)$ is the probability that the random walk is at position g after n steps.

The behaviour of the random walk is tightly related to the associated *Markov operator* on the Hilbert space $\ell^2(G)$ given by

$$M_\mu(f)(x) = \sum_{s\in G} f(sx)\mu(s).$$

In other words, $M_\mu(f)(x)$ is the expected value of $f(sx)$ when s is random μ-distributed. The condition that μ is symmetric implies that M_μ is a self-adjoint operator. The following characterization of amenability is due to H. Kesten; see [Kes59a].

Theorem 3. *Let G be a discrete group, and let μ be a symmetric measure on G whose support generates G. Then the following conditions are equivalent.*

(1) *G is amenable;*
(2) *the norm of the operator M_μ is equal to 1;*
(3) $\lim_{n\to\infty} \sqrt[2n]{\mu^{*2n}(1)} = 1$.

The third statement of Theorem 3 means that a group G is nonamenable if and only if the probability that the random walk $s_1\cdots s_n$ is equal to the identity decays exponentially fast with n. A tightly related criterion is the *cogrowth criterion* of R. Grigorchuk [Gri80a].

Theorem 4. *Let G be a finitely generated group, and let $\phi : F_k \longrightarrow G$ be an epimorphism from the free k-generated group to G. Let $\gamma(n)$ be the number of elements of the kernel of ϕ of length n. Then G is amenable if and only if* $\limsup_{n\to\infty} \sqrt[n]{\gamma(n)} = 2k - 1$.

Note that the total number of elements of length n in F_k is equal to $2k(2k-1)^{n-1}$, so amenability is basically equivalent to the kernel having the same exponential growth rate as the whole free group.

Another characterization of amenability in terms of a random walk is due to Kaimanovich and Vershik [KV83] and independently to Rosenblatt [Ros81], who showed that a group is amenable if and only if its *Poisson boundary* is trivial *for some* symmetric measure μ with support generating G. Triviality of the Poisson boundary is called the *Liouville property* of the random walk. The Poisson boundary is obtained by taking the quotient of the space of trajectories $(g_1, g_2, \ldots)$ of the random walk by the measurable hull of the equivalence relation identifying two trajectories (g_n) and (h_n) if there exist n, m such that $g_{n+t} = h_{m+t}$ for all $t \geq 0$. The Liouville property is equivalent to the condition that any bounded function $h : G \longrightarrow \mathbb{R}$ that satisfies $h(x) = \sum_{s\in G} f(sx)\mu(s)$ (is *μ-harmonic*) is constant. The Liouville property implies amenability in a constructive way, namely (under the aperiodicity assumption that $\mu(1_G) > 0$) it is equivalent to the fact that the sequence of convolutions μ^{*n} accumulates on an invariant mean in the weak-* topology.

Amenable groups are prominent in ergodic theory, as the class of groups for which many theories about $\mathbb{Z}$-actions can be generalized. Indeed many results in classical ergodic theory involve averaging over intervals in $\mathbb{Z}$ and have been extended to amenable groups by considering averaging over Følner sets. Another famous application of amenability to ergodic theory is the theorem of Ornstein and Weiss [OW80] that any two ergodic, free, probability measure preserving actions of infinite amenable groups are orbit-equivalent, namely there exists a measurable isomorphisms of the spaces which preserves the partitions into orbits.

Amenability also plays a role in the theory of operator algebras and has important generalizations there; see [Pat88]. A discrete group G is amenable if and only if all irreducible representations of G are contained in the left-regular representations. This is also equivalent to the condition that the *reduced* C^*-algebra of the group coincides with the *full* C^*-algebra. A generalization of amenability of groups to C^*-algebras is called *nuclearity* and it is a central property of the theory.

2. Attempts to understand amenability of groups

Already in his first paper on amenability, J. von Neumann proved several permanence properties of the class of amenable groups.

Theorem 5. *If a normal subgroup N of G and the quotient G/N are amenable, then G is also amenable.*

Any direct limit of amenable groups is an amenable group.

A subgroup of an amenable group is amenable. A quotient of an amenable group is amenable.

We have seen that $\mathbb{Z}$ has Følner sets and, hence, is amenable. Finite groups are also obviously amenable. Theorem 5 and the classical results on the structure of commutative groups imply that all commutative groups are amenable. Moreover, all solvable groups are amenable.

Let us continue using Theorem 5 in a systematic way. Denote by $\mathcal{E}_0$ the class of finite and commutative groups. If we have defined the class $\mathcal{E}_\alpha$ for an ordinal α, then define $\mathcal{E}_{\alpha+1}$ as the class of groups that can be obtained from the groups from $\mathcal{E}_\alpha$ using the constructions described in Theorem 5. Namely, extensions of elements of $\mathcal{E}_\alpha$ by elements of $\mathcal{E}_\alpha$ (see the first statement of Theorem 5), direct limits of elements of $\mathcal{E}_\alpha$, passing to subgroups and to quotients. In fact, it is enough to consider the first two constructions, since the classes $\mathcal{E}_\alpha$ will be invariant with respect to the other two. If α is a limit ordinal, then we define $\mathcal{E}_\alpha$ is the union of the classes $\mathcal{E}_\beta$ for $\beta < \alpha$.

The union of all classes $\mathcal{E}_\alpha$ (which will be equal to $\mathcal{E}_{\omega_1}$) is, by definition, the class of *elementary amenable groups*. This class was introduced and studied by M. Day in [Day57] and studied by C. Chou in [Cho80].

On the other hand, we know that the free group on two generators is not amenable. (For example, the map erasing the first letter in every reduced word is a Ponzi scheme.) Since subgroups of amenable groups are amenable, every group containing a free subgroup is nonamenable.

We get in this way two classes of groups: elementary amenable ("obviously amenable") and groups with free subgroups ("obviously nonamenable"). For a while these were the only two classes of groups for which amenability or nonamenability was known. It is not hard to see, however, that there are groups not belonging to either of these classes, as was already noticed by C. Chou. An early example of such a group is the *Thompson group*, introduced by R. Thompson in the 1970s;

see [CFP96, Tho80]. Its amenability is perhaps the most famous open problem about amenability.

One can show that a finitely generated infinite torsion group cannot be elementary amenable, and it obviously cannot contain a free subgroup. A finitely generated simple group cannot be elementary amenable, so if it has no free subgroups, then its amenability is a nontrivial problem.

First examples of nonamenable groups without free subgroups were constructed by A. Olshansky in [Ol80]. He proved nonamenability of some infinite finitely generated torsion groups using the co-growth criterion (Theorem 4). Later S. Adyan showed that his previous result—that *free Burnside* of large odd exponent are infinite—could be improved to yield nonamenability, also using the cogrowth criterion. It is interesting that H. Kesten mentioned the Burnside problem in relation with the spectral radius of the Markov operator in [Kes59b], but it seems that he conjectured that Burnside groups are amenable.

Groups of Burnside type were used as a starting point by A. Olshansky and M. Sapir in [OS02] to construct the first example of a *finitely presented* nonamenable group without free subgroups.

Recently, M. Ershov proved in [Ers11] that Golod–Shafarevich groups are nonamenable.

Existence of nonelementary amenable groups was known as the Day–von Neumann problem (though, J. von Neumann never formulated it).

C. Chou showed that growth (i.e., the number of elements that are the product of n or fewer generators and their inverses) of a finitely generated elementary amenable group is either exponential or polynomial. It follows that a group of *intermediate* growth (strictly between polynomial and exponential) is not elementary amenable. Any nonamenable group must have exponential growth (an easy consequence of paradoxical decomposition or Følner criterion).

The first examples of groups of intermediate growth—and thus the first example of nonelementary amenable groups—were constructed by R. Grigorchuk in [Gri80b, Gri83].

Groups of intermediate growth can be also considered “obviously amenable”, similarly to commutative groups. So, it is natural to change the definition of elementary amenable groups by defining $\mathcal{E}_0$ as the class of all groups of sub-exponential growth (including finite groups). Let us call this new class of groups *subexponentially amenable*.

The first example of an amenable group, which is not subexponentially amenable is the *iterated monodromy group of* $z^2 - 1$, also known as the *Basilica group* (the name coming from the name of the Julia set of $z^2 - 1$). This group was introduced for the first time by R. Grigorchuk and A.Żuk in [GŻ02], where also the question of its amenability was raised. Its amenability was proved by L. Bartholdi and B. Virág in [BV05] using a random walk. Their results were then generalized to the class of groups generated by *bounded automata* in [BKN10].

More recently a nonconstructive proof of the existence of nonelementary amenable groups was obtained by P. Wesolek and J. Williams using methods from descriptive set theory. They showed that elementary amenable groups form a nonBorel subset of the space of marked groups. Since amenable groups are easily shown to form a Borel subset, it follows that these two classes cannot coincide.

3. Amenable groups via actions

It is generally accepted by now that the class of amenable groups does not have a "purely algebraic" description (for example, in the spirit of the description of the class of elementary amenable groups), as it is an essentially analytic condition. This motivates exploring groups that are close to the boundary between the classes of amenable and nonamenable groups, and studying amenability of groups by examples, as in the title of the book of Kate Juschenko.

All recent examples of interesting groups from the point of view of amenability are built not using group-theoretic constructions, but as groups generated by transformations of topological spaces. Typically, the corresponding action of the group on the space is highly nonfree (has large stabilizers), so that it is usually relatively easy to understand the orbits of the action. A prototipical example of this situation is Thompson's group F mentioned above, which is defined as the group of all homeomorphisms of the interval $[0, 1]$ given by piecewise affine map of the form $x \mapsto 2^x + q$ for some dyadic rational q, with finitely many pieces whose endpoints are dyadic rationals. The orbit of each individual point is contained in its orbit under a subgroup of the affine group (which is an amenable group), yet the group has a much more complicated structure coming from the independence of its action on different points.

The partition of a space into orbits associated to a group action has a surprisingly rich structure. As an equivalence relation, it can be considered a *groupoid* whose elements are pairs (x, y) of points belonging to one orbit. Partially defined multiplication and inversion come from transitivity and symmetry of the equivalence relation. The orbit equivalence relation for an action of a countable group has a natural structure of a Borel subset of the direct square of the space. Equivalence relations on Borel and measure spaces were extensively studied in ergodic theory, logic, and operator algebras; see, for example, [KM04]. One of the famous results of this theory is the theorem of Ornstein and Weiss, mentioned above, giving a characterization of orbit equivalence relations coming from ergodic actions of amenable groups.

The notion of amenability of groups has a natural generalization to measured groupoids; see [AR00]. In particular, nonamenability of the orbit equivalence groupoid can be used to prove nonamenability of the acting group. This was used by N. Monod in [Mon13] to construct examples of a nonamenable group without a free subgroup with a much shorter and easier proof than the original Burnside type examples. These examples have a similar flavor as Thompson's group F, but are defined as groups of piecewise *projective* homeomorphisms of the real line. Y. Lodha and J. Moore constructed a finitely presented example of such a group in [LM16] as a subgroup of Monod's examples.

In the opposite direction (to prove amenability), this idea is generally harder to implement, as the amenability of the orbit equivalence relation in the measurable sense does not imply amenability of the acting group. For example, the amenability of the equivalence relation of the action of Thompson's group F does not say anything about amenability of the group itself. Nevertheless, all currently known examples of nonelementary amenable groups (which is the main subject of the book) are groups of homeomorphisms of topological spaces, whose amenability is proved using the orbits of the action. Essentially, the idea is that if the orbits are sufficiently *small* in a different sense, involving also the topology, then the group is amenable.

To formalise this idea, if a group acts by homeomorphisms on a topological space, then it is natural to consider a richer topological groupoid of *germs* of the action. The corresponding notion of orbit equivalence (i.e., of having isomorphic groupoids of germs) is defined as follows. We say that (G_1, X_1) and (G_2, X_2) are *continuously orbit equivalent* if there exists a homeomorphism $\phi : X_1 \longrightarrow X_2$ such that for every $g_1 \in G_1$ and $x \in X_1$ there exist $g_2 \in G_2$ and a neighborhood U of x such that $\phi(g_1(y)) = g_2(\phi(y))$ for every $y \in U$. In other words, ϕ conjugates the local action of G_1 on X_1 with the local action of G_2 on X_2.

For a given action (G, X), the *topological full group* $[[G]]$ is the maximal group of homeomorphisms of X such that the identity map is a topological orbit equivalence between (G, X) and $([[G]], X)$. In other words, it is the group of all homeomorphisms $\phi : X \longrightarrow X$ such that for every $x \in X$ there exists a neighborhood U of x and an element $g \in G$ such that $\phi|_U = g|_U$.

The first result on amenability, where the use of the groupoid of germs is explicit (though a different terminology was used in the original paper) is the theorem of K. Juschenko and N. Monod [JM13] on topological full groups of minimal homeomorphisms of the Cantor set.

Theorem 6. *Let $a : X \longrightarrow X$ be a homeomorphism of the Cantor set such that all orbits of the $\mathbb{Z}$-action generated by it are dense in X. Then the topological full group of the $\mathbb{Z}$-action is amenable.*

This theorem provides the first examples of amenable finitely generated infinite simple groups. It was proved by H. Matui [Mat06] that if the homeomorphism a is expansive, then the derived subgroup of the topological full group is simple and finitely generated.

One of main tools in the proof of Theorem 6 is the fact that there is a natural locally finite (i.e., a direct limit of finite groups) subgroup H of the topological full group $[[\mathbb{Z}]]$ such that for every element $g \in [[\mathbb{Z}]]$ the germs of g belong to the groupoid of germs of elements of H in all but finitely many *singular* points. Locally finite groups are amenable (see Theorem 5), and amenability of the topological full group can be concluded from the amenability of H by studying its action on the orbits of singular points, which coincide with the orbits of the original $\mathbb{Z}$-action. The crucial point is to leverage the very simple geometry of $\mathbb{Z}$ to show that this action satisfies a strong form of orbit-wise amenability, now called *extensive amenability* (this terminology was introduced only later in [JMMS18]). It was later observed in [JNS16] that the main geometric property needed to show extensive amenability is the *recurrence* of a simple random walk of $\mathbb{Z}$, and this method was extended to prove amenability of a wide variety of nonelementary amenable groups acting on compact spaces (including the examples from [BV05] and [BKN10]) using the groupoid of germs of their actions. Extensive amenability was later studied systematically in [JMMS18], where this method was also used to establish amenability of some subgroups of the group of interval exchanges (which can be seen as topological full groups of a class of $\mathbb{Z}^2$-actions on the Cantor set). It is now a widely open problem to find new criteria to establish the extensive amenability of an action beyond recurrence of the random walk on its orbits, which would potentially lead to many new applications.

4. Overview of the book

The focus of Kate Juschenko's book are examples of nonelementary amenable groups and techniques outlined in the previous section of this review. It is a good introduction to the cutting edge of the subject. The book starts with a self-contained introduction to the theory of amenability of discrete groups. The first chapter after the introduction (Chapter 2) starts with a discussion of the Hausdorff–Banach–Tarski paradox, the definition of amenability, and important criteria of amenability. Additionally, Appendix A contains more criteria of amenability (e.g., Kesten's criterion) with proofs.

Chapter 3 discusses the class of elementary amenable groups. An important part of this section are examples of groups that *do not* belong to this class. For example, it is shown that groups of Burnside type, groups of intermediate growth, and simple infinite finitely generated groups cannot be elementary amenable. The chapter is concluded with some examples of groups acting on rooted trees which are not elementary amenable,

The rest of the book is devoted to different classes of examples of nonelementary amenable groups. Chapter 4 is based on the paper [JM13] on amenability of topological full groups of minimal $\mathbb{Z}$-actions, see Theorem 6. It contains all the necessary background, e.g., the reconstruction theorem of T. Giordano, I. Putnam, and C. Skau [GPS99] and Boyle's flip conjugacy theorem [BT98]. Chapter 5 describes the techniques of extensively amenable actions from [JMMS18], generalizing the Juschenko–Monod theorem. Direct applications of the theory of extensively amenable actions are described in Chapter 6.

Chapter 7 describes the results of [Nek18], which provide another class of infinite simple finitely generated groups. They are first examples of simple groups of intermediate growth and are subgroups of the topological full groups of expansive $\mathbb{Z}$-actions.

The last three chapters, Chapters 8–10, describe the results of the paper [JNS16], where extensive amenability of actions is used to prove amenability of a wide class of groups acting on topological spaces. In particular, this includes groups generated by *bounded automata* (both acting on rooted trees and on spaces of paths in a Bratteli diagram).

The book has two appendices: a collection of different criteria of amenability (with proofs of their equivalence) and a collection of open problems.

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