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SELECTED MATHEMATICAL REVIEWS

concerning publications of
VAUGHN JONES

MR0766964 (86e:57006) 57M25; 46L10

Jones, Vaughan F. R.

A polynomial invariant for knots via von Neumann algebras.

American Mathematical Society. Bulletin. New Series **12** (1985), *no.* 1, 103–111.

This remarkable paper announces the discovery of a new Laurent polynomial with integer coefficients which is an invariant of the isotopy type of a tame oriented link in oriented S^3. The discovery of the new polynomial came out of investigations of a family of finite-dimensional von Neumann algebras, those investigations being quite unrelated to knot or link theory. The polynomial is easily seen to be distinct from the well-known Alexander polynomial, indeed as this review is being written it has been shown that both the Alexander and the new Jones polynomial are 1-variable specializations of a new 2-variable polynomial invariant of links found by P. Freyd et al. [see the following review; MR0776477].

The Jones polynomial can be computed from a representation of a link L as a closed braid, i.e. one begins with an element b in one of the Artin braid groups B_n and defines $L = b^\frown$ to be the link obtained from b by identifying the n free ends at the beginning of the braid with corresponding free ends at the end of the braid in a canonical fashion. Define the braid index of L to be the smallest integer n such that $L = b^\frown$ for some $b \in B_n$. By definition, it is a link type invariant; however, it has been uncomputable except in very special cases up to now. A sample of one of the many theorems announced in the paper under review goes as follows. The author introduces a family of representations $r_{n,k}\, B_n \to G_n$ of the groups B_n, where $n, k = 1, 2, 3, \cdots$. Theorem 8 asserts that if $b \in \ker r_{n,k}$ for some $k \geq 3$, then $L = b^\frown$ has braid index n.

The new polynomial is not a group invariant (it takes different values on the trefoil and its mirror image), nor is it an invariant of the complementary space (it takes different values on the so-called Whitehead link with k knots for each $k = 1, 2, 3, \cdots$, whereas these links all have homeomorphic complements). Its geometric meaning awaits explanation, at this writing, one year after the initial discovery.

For the benefit of the reader, we note a small error in Theorem 12. Relations I, II, III and the definition of g_i on page 105, line 2, imply that $t^{-1}g_i - tg_i^{-1} = (\sqrt{t} - 1/\sqrt{t})$. This in turn implies that the symbols $+$ and $-$ in Theorem 12 have been interchanged. Also, in the list of examples, the braid form of 8_{18} is incorrect. The correct entry is $(12^{-1})^4$.

J. S. Birman

From MathSciNet, July 2023

MR0791846 (86k:46091) 46L35; 46L10

Haagerup, Uffe

A new proof of the equivalence of injectivity and hyperfiniteness for factors on a separable Hilbert space.

Journal of Functional Analysis **62** (1985), *no.* 2, 160–201.

A von Neumann algebra M is said to be hyperfinite if there is an increasing sequence of finite-dimensional sub-*-algebras of M whose union is weakly dense. The algebra is said to be injective if, as a Banach subspace of the space of all bounded operators, it is complemented. In a wonderful paper [Ann. of Math. (2) **104** (1976), no. 1, 73–115; MR0454659], A. Connes proved that injective and hyperfinite are equivalent, along with several other conditions, among them the Effros–Lance condition of semidiscreteness, which means that the identity map on M can be approximated by completely positive maps onto finite-dimensional subspaces. Connes reduced the problem to the type II_1 case where he used a penetrating analysis of the automorphism group of a type II_1 factor. His argument is very subtle and of great technical difficulty. While there are many rich results of considerable independent interest, as a proof of the main result and its many corollaries, it remains enigmatic. It is important that more direct proofs be found.

The first new proof is due to the author. He never uses automorphisms. He assumes the implication (injective $\Rightarrow$ semidiscrete) and uses Choi and Effros' characterization of semidiscreteness (which involves matrix algebras rather than subspaces) to begin to create the appropriate finite-dimensional subalgebras. The author begins by giving a simple proof that injective implies hyperfinite in the properly infinite case as follows: The Choi–Effros result gives, for any unitaries $u_1, u_2, \cdots, u_k$ in M, a finite-dimensional subfactor F of M, a completely positive map $T\colon F \to M$ and y_i's in F with $T(y_i)$ close to u_i. A method, attributed to Kasparov, shows that T can be implemented by an isometry (since M is infinite), i.e. $T(x) = v^*xv$, $v^*v = 1$. The author then approximates v by a unitary w and we are done since w^*Fw approximately contains the u_i's.

The proof in the II_1 case is an adaptation of the above argument, though one encounters considerable technical problems. Kasparov's isometry trick cannot work so one must carry along a finite set $\{a_1, \cdots, a_d\} \subseteq M$ with $T(x) = \sum_{i=1}^d a_i^* x a_i$. The main technical result of the paper uses ultraproducts and an elegant probabilistic argument to create a single unitary w out of the a_i's with $w^* y_i w$ close to u_i as above. To grossly oversimplify, the argument asserts that if $x = \sum_{i=1}^d \omega_i a_i$ is a random linear combination of the a_i's with $|\omega_i| = 1$, then there is a nonzero probability that x is a unitary and does the job.

{Although the paper does have its tough parts, the argument is basically clear, and the average difficulty per line is a tiny fraction of that of Connes' paper.}

Vaughan Jones

From MathSciNet, July 2023

MR0908150 (89c:46092) 46L99; 20F36, 22D25, 46L35, 46L55, 57M25

Jones, V. F. R.

Hecke algebra representations of braid groups and link polynomials.

Annals of Mathematics. Second Series **126** (1987), *no.* 2, 335–388.

The work of the author to find invariants for subfactors of factors had a spectacular offspring in producing a completely new polynomial invariant V_L attached to a knot or an oriented link L [the author, Bull. Amer. Math. Soc. (N.S.) **12** (1985), no. 1, 103–111; MR0766964]. Though this Jones polynomial V_L is quite distinct from the Alexander polynomial Λ_L, both show a very tight relationship with skein theory. This suggests a two-variable polynomial P_L which subsumes both V_L and Δ_L. Indeed, at least five groups of authors have independently defined this P_L; see papers by P. Freydet al. [ibid. (N.S.) **12** (1985), no. 2, 239–246; MR0776477] as well as J. H. Przytyckiand P. Traczyk [Kobe J. Math. **4** (1988), no. 2, 115–139].

Here is Jones' account of P_L, defined via Hecke algebras of type A and Ocneanu traces. There are in fact two definitions of P_L: one involves closed braids à la Markov and the other braids with an even number of strings closed to give plats. Several examples are worked out, including knots and links which are closed 3 and 4 braids, and torus knots. The remarkable specializations Δ_L and V_L are discussed, as well as some results for which the operator algebra origin of all this is indeed useful, via positivity considerations.

There is a table which gives the braid index, a braid expression, amphicheirality information and the polynomials V_L for all unoriented prime knots up to 10 crossings.

The paper gives perspective to a large body of results: Bureau representation, structure of Hecke algebras, Ocneanu traces, Temperley-Lieb algebras, etc. It also suggests exciting new problems: Hecke algebras of other types, relation to works by Kazhdan and Lusztig, mapping class groups, etc.

Pierre de la Harpe

From MathSciNet, July 2023

MR0990772 (90h:57009) 57M25; 17B67, 57N10, 58D15, 58D30, 81E40

Witten, Edward

Quantum field theory and the Jones polynomial.

Communications in Mathematical Physics **121** (1989), *no.* 3, 351–399.

The author introduced the notion of a topological quantum field theory in a previous article [same journal **11** (1988), no. 3, 353–386; MR0953828] where he discussed the Donaldson invariants of 4-manifolds. The paper under review interprets the Jones invariants of links in the 3-sphere in terms of quantum field theory and at the same time introduces new invariants of links in arbitrary 3-manifolds. In particular, there are new invariants of closed 3-manifolds. Mathematicians should find this paper more accessible than the article cited above as the field theory here does not involve supersymmetry. We explained some general features of topological quantum field theory in our review of the previous article, so we proceed directly to the current paper.

Fix a compact Lie group G. In this paper the author deals only with simple groups, and to be definite we take $G = \mathrm{SU}(N)$; the general case is discussed further by R. Dijkgraafand the author ["Topological gauge theories and group cohomology", ibid., to appear], for example. Then if M is an oriented closed 3-manifold and A a connection on a (necessarily) trivial SU(N)-bundle over M, one defines the Chern-Simons invariant $\mathcal{L}(A)$. It takes real values, but changes by integers under gauge transformations. Also, the possible Chern-Simons invariants are parametrized by an integer k. (For a general compact group they are parametrized by $H^4(BG)$.) The variables N and k turn out to be simply related to the variables in the Jones polynomial. The Chern-Simons invariant is the Lagrangian of a classical field theory; the classical solutions are the flat connections. But it is the quantum theory which is of interest. The partition function, defined by integrating $\exp(2\pi i\mathcal{L}(A))$ over the space of connections, is proposed as a new invariant of the 3-manifold M. If C is an oriented loop in M, and R a representation of SU(N), then for each connection A we can evaluate the character of R on the holonomy of A around C; this is well-defined and gauge-invariant. When this is inserted into the path integral, repeatedly for a link with several components, one gets an invariant of a link in M. The author asserts that since there are no background geometric data (such as a metric) in the theory, these path integrals define topological invariants.

The author first addresses the issue of whether these Feynman path integrals make sense. The usual perturbative calculations of quantum field theory here become the large-k limit of the theory. This relates to previous work of A. S. Shvarts [Lett. Math. Phys. **2** (1978), no. 3, 247–252] and A. M. Polyakov [Modern. Phys. Lett. A **3** (1988), no. 3, 325–328; MR0927055]. The leading order behavior is thus computed in terms of the Chern-Simons invariant, Reidemeister torsion, certain combinations of η-invariants, and linking numbers. As expected, these are all topological invariants. This discussion points out one subtlety of the theory—the need to frame the 3-manifolds and the links in order to carry out the path integral. Much more striking is the agreement with the large-k behavior of the exact solutions computed later.

Next, the author considers the path integral on a 3-manifold of the form $\Sigma \times \mathbf{R}$, where Σ is an oriented closed surface. Using standard principles of quantum field theory which relate path integrals to canonical quantization and which prescribe the treatment of symmetries, he is led to the conclusion that the quantum Hilbert space attached to Σ is obtained by quantizing the moduli space of flat SU(N)-bundles over Σ, which is a symplectic manifold. Since this moduli space is compact, the quantum Hilbert space is finite-dimensional. What is the key to the entire paper comes with the realization that this is exactly the description given by G. Segal ["Two-dimensional conformal field theories and modular functors", *IAMP Proceedings* (Swansea, 1988), to appear] of the "space of conformal blocks" in the $(1+1)$-dimensional conformal field theory usually called the Wess-Zumino-Witten model. This space carries a (projective) representation of the mapping class group which has been extensively studied, for example by V. G. Kac and M. Wakimoto [Adv. Math. **70** (1988), no. 2, 156–236; MR0954660] in genus 1, and this eventually allows the author to make explicit computations. Similar remarks apply to punctured surfaces, which enter when there are links.

The final ingredient is a general feature of quantum field theories, which we might call the "gluing law". It allows one to calculate a path integral by chopping a manifold into smaller pieces. The author uses it to see how his invariants change under surgery.

At this point one has a concrete prescription for computing the invariants. This prescription is derived from the path integral, and in the author's presentation its validity depends on the path integral, but the algorithm itself is stated in terms of elementary computable formulæ. The author uses this prescription to derive the skein relation of knot theory, and so relate his invariants to the Jones knot polynomials. He also uses it to prove a conjecture of Verlinde (previously proved by G. Moore and N. Seiberg [Comm. Math. Phys. **123** (1989), no. 2, 177–254; MR1002038]). Other illustrations of the theory are also given.

The paper ends with a hint that not only the space of conformal blocks of a (1+1)-dimensional conformal field theory, but also the entire (1+1)-dimensional conformal field theory, can be derived from the Chern-Simons theory in 2+1 dimensions.

The author's paper catalyzed much activity by both mathematicians and physicists. By now mathematicians have verified much of what he asserts without using the path integral. The relationship to conformal field theory has been developed in more detail by many physicists. We have neither the space nor the license to do justice to these developments here.

Daniel S. Freed
From MathSciNet, July 2023

MR1078014 (92b:57008) 57M25; 14D20, 32G81, 58F06, 81S10, 81T40
Atiyah, Michael
The geometry and physics of knots. (English)
Lezioni Lincee. [Lincei Lectures].
Cambridge University Press, *Cambridge*, 1990, x+78 pp., $39.50,
ISBN 0-521-39521-6

This book, an expanded version of lectures given by the author in 1988, provides an invaluable commentary for mathematicians to Witten's important paper "Quantum field theory and the Jones polynomial" [Comm. Math. Phys. **121** (1989), no. 3, 351–399; MR0990772], in which the topological invariant introduced by V. F. R. Jones [Ann. of Math. (2) **126** (1987), no. 2, 335–388; MR0908150] for links in S^3 is generalized to links in arbitrary compact 3-manifolds. Witten's approach is based on path integrals, which are, as usual, not well defined, but which transform manageably under surgeries (Verlinde's fusion rules) and so can be treated as formal symbols. (This evasion of treating path integrals directly ignores the success of Witten's stationary phase approximation to the path integrals [cf. D. Freed and R. E. Gompf, "Computer calculation of Witten's 3-manifold invariant", Preprint; per revr.].) In contrast, the author concentrates mostly on the (mathematically rigorous) Hamiltonian aspects of topological quantum field theory, although he mentions that this approach lacks the striking physical motivation of the Lagrangian/path integral method and does not seem to elucidate the Witten-Freed-Gompf perturbative calcluations.

The author's task is to explain Witten's claim that for each "level" $k \in \mathbf{Z}$ there exists a topological quantum field theory Z_k which assigns a complex number $Z_k(M^3, L, \overline{\mu})$ to a closed 3-manifold M^3, an oriented link $L \subset M^3$, and irreducible special unitary representations $\overline{\mu} = (\mu_1, \cdots, \mu_r)$ associated to the components of the link, and which satisfies $Z_k(S^3, K, \mu_s) = V_K\big(\exp(2\pi i/(k+2))\big)$, where K is a knot, μ_s is the standard representation of SU(2), and V_K is the Jones polynomial. (In fact, Z_k also depends on a choice of framing for M and for L; cf. the discussion of Chapter 7 below.) Since the set of such values characterizes the Jones polynomial, Witten's theory generalizes the Jones polynomial to links in arbitrary 3-manifolds, and in particular produces new topological invariants of 3-manifolds by taking K to be the empty knot. It is important to note, however, that even if one avoids path integral questions by calculating $Z_k(M^3)$ via fusion rules, one is left with a difficult consistency check that the invariant produced is independent of the series of surgeries chosen; this has been shown recently for the large class of manifolds obtained from S^3 by plumbing on trees by P. Melvin, who has identified Witten's invariant with the topological invariant introduced by N. Reshetikhin and V. G. Turaev [Invent. Math. **103** (1991), no. 3, 547–597; MR1091619].

Witten's invariant for a closed 3-manifold M is defined as follows. Let $L(A) = (1/4\pi)\int_M \operatorname{Tr}(A \wedge dA + \frac{2}{3}A \wedge A \wedge A)$ be the Chern-Simons action for a connection A on a (necessarily) trivial $\mathrm{SU}(n)$-bundle E over M. For a knot K in M and a representation μ of $\mathrm{SU}(n)$, the Wilson line $W_K(A) = \operatorname{Tr}_\mu \operatorname{Mon}_K(A)$ is the μ-trace of the monodromy of A around K. Then formally

$$Z_k(M, K, \mu) = \int_{\mathcal{A}} \exp(ikL(A)) \cdot W_K(A)\ \mathcal{D}A$$

where the path integral is taken over the space of all connections $\mathcal{A}$ on E. More generally, for a 3-manifold M with boundary Σ containing a link L (marked by representations as above) hitting Σ in points $\{P_i\}$ (marked by representations $\partial\mu_i$ associated to the μ_i), Witten's theory assigns a vector $Z(M, L, \overline{\mu})$ in a finite-dimensional Hilbert space $\mathcal{H} = \mathcal{H}_{\Sigma,\{P_i\},\partial\mu_i}$. Moreover, this theory is exactly soluble in that under chopping a closed 3-manifold M into two manifolds $M_\pm$ with common boundary Σ (with opposite orientations $\Sigma_\pm$), the rules of topological quantum field theory force $Z_k(M, L, \overline{\mu}) = \langle Z(M_+, L \cap M_+, \overline{\mu}), Z(M_-, L \cap M_-, \overline{\mu})\rangle$, where $\langle\ ,\ \rangle$ is the natural pairing of $\mathcal{H}_{\Sigma_+}$ and $\mathcal{H}_{\Sigma_-}$. Reducing M to S^3 by a sequence of surgeries finally reduces the calculation to one on S^3, where explicit formulas are available. As mentioned above, the author's main focus is the construction of the spaces $\mathcal{H}$; the construction of the vector $Z(M, L, \overline{\mu})$ is only given by a formal expression, more fully discussed and calculated in Witten's paper.

The monograph is organized as follows. The first chapter presents a summary of the main properties of the Jones polynomial. In the second chapter, topological quantum field theory is presented in the axiomatic form developed by the author [Inst. Hautes Études Sci. Publ. Math. No. 68 (1989), 175–186; MR1001453]. For motivation (see the discussion of Chapter 5 below), there is also a review of the standard geometric quantization of a surface Σ: the linear symplectic space $H^1(\Sigma; \mathbf{R})$ (which consists of classical fields by Hodge theory) gives rise to a holomorphic line bundle L over Σ, once a complex structure σ is chosen on Σ, whose holomorphic sections form the desired quantized state space $\mathcal{H}_{\Sigma,\sigma}$. The Stone-von Neumann theorem guarantees that the projectivized state space is independent of choice of complex structure. This independence is reinterpreted by stating that the bundle over the Siegel upper half-space, the space of complex structures, whose fiber at the point σ is $\mathcal{H}_{\Sigma,\sigma}$, admits a projectively flat connection. Moreover, the group $\Lambda = H^1(\Sigma, \mathbf{Z})$ acts on $H^1(\Sigma; \mathbf{R})$, and quantizing the quotient $H^1(\Sigma, \mathrm{U}(1))$ should be the same as taking the Λ-invariant piece of the $\mathcal{H}_\sigma$. Forming the line bundle L (or L^k for the level k theory) more or less as before leads to a bundle over the Siegel half-space admitting a projectively flat connection.

In Chapter 3, the author begins the discussion of quantizing $H^1(\Sigma, G)$, where G is a compact simply connected Lie group. Note that this space parametrizes homomorphisms $\pi_1(\Sigma) \to G$, or equivalently the space of flat G-connections on

Σ. By Narasimhan and Seshadri, $H^1(\Sigma, \mathrm{SU}(n))$ is homeomorphic to an algebraic variety, the set of (equivalence classes) of semistable rank n holomorphic bundles over Σ, once a complex structure is chosen. There is a generalization of this result taking marked points on Σ into account. Chapter 4, which focuses on the relationship between the symplectic and algebraic geometric viewpoints, treats the case of a compact Lie group G acting on an algebraic variety X with ample line bundle. Following Kirwan, the author outlines the identification of the Mumford quotient of X by G^c with the symplectic quotient (reduced phase space) $X//G = \mu^{-1}(0)/G$, where $\mu\colon X \to \mathrm{Lie}(G)^*$ is the moment map. As before, one quantizes X to $\mathcal{H}_X$ using geometric quantization of the symplectic structure and a choice of Kähler structure (if one exists), and for nice actions the G-invariant part of the quantum Hilbert space for X will be the quantum Hilbert space of the Mumford quotient. In particular, if M_λ is a coadjoint orbit in $\mathrm{Lie}(G)^*$ associated to an irreducible representation λ of G with associated symplectic quotient $Y_\lambda = \mu^{-1}(M_\lambda)/G$, then the quantum Hilbert space $\mathcal{H}_{Y_\lambda} = \mathrm{Hom}_G(\lambda, \mathcal{H}_X)$, the λ-covariant part of $\mathcal{H}_X$.

In Chapter 5, the author constructs Witten's quantum Hilbert space for the surface Σ. For an unmarked surface, the "classical" symplectic space is the infinite-dimensional linear space of connections $\mathcal{A}$ on the given bundle over Σ, and since the gauge transformations $\mathcal{G}$ preserve the Chern-Simons action, the desired quantum Hilbert space should be the $\mathcal{G}$-invariant part of the quantum Hilbert space of $\mathcal{A}$. By analogy with the finite-dimensional situation, one expects that this quantum space should be the quantization of the symplectic quotient $\mathcal{A}//\mathcal{G}$. By work of the author and Bott, $\mathcal{A}$ has a moment map given by the curvature of the connection, so $\mathcal{A}//\mathcal{G}$ is the space of flat connections $H^1(\Sigma, G)$. As before, once a complex structure σ has been chosen for Σ, there is a moduli space M_σ of holomorphic G^c-bundles over Σ and a homeomorphism $H^1(\Sigma, G) \to M_\sigma$. Note that this has reduced the quantization of the infinite-dimensional space $\mathcal{A}$ to the quantization of the compact space M_σ; this ensures that the quantum space will be finite-dimensional. For $G = \mathrm{SU}(n)$, $\mathcal{A}$ may be identified with the space of rank n holomorphic bundles over Σ, which determines a Hermitian holomorphic line bundle L, Quillen's determinant line bundle, over $\mathcal{A}$. L descends to a bundle, also called L, over M_σ, and the holomorphic sections of L^k form the quantum space $\mathcal{H}_{\Sigma,\sigma}$. It must still be shown that the quantum spaces are projectively independent of the choice of σ.

The case of a surface with marked points is slightly more complicated. A point P on Σ determines an evaluation map $e_P\colon \mathcal{G} \to G$ and a dual map $\delta_P\colon \mathrm{Lie}(G)^* \to \mathrm{Lie}(\mathcal{G})^*$. Thus given points $\mathbf{P} = \{P_1, \cdots, P_r\}$ and representations $\lambda = \{\lambda_1, \cdots, \lambda_r\}$ with associated orbits M_{λ_i}, there is a G-orbit $M(\mathbf{P}, \mathbf{\Lambda}) = \sum \delta_{P_i}(M_{\lambda_i}) \subset \mathrm{Lie}(\mathcal{G})^*$ and the generalized symplectic quotient $[\mu^{-1}(M(\mathbf{P}, \mathbf{\Lambda}, k))]//\mathcal{G}$. Here the integer k denotes that the symplectic form on $\mathcal{A}$ has been multiplied by k. This quotient can be identified with connections which are flat outside $\mathbf{P}$ and which have δ-function curvatures at $\mathbf{P}$. Moreover, in polar coordinates at P_j the connection looks like $A_j d\theta$, where A_j is in the conjugacy class of the orbit $(1/k)M_{\lambda_j}$ and hence has monodromy around P_j a kth root of unity. As such, the generalized symplectic quotient is the moduli space of representations of $(\Sigma, \mathbf{P})$ discussed in Chapter 3. The quantum Hilbert space is again the space of holomorphic sections of a line bundle over this moduli space. This is the rigorous formulation of the expectation that the quantization of the marked surface should pick out the piece of the quantization of $\mathcal{A}$ (which is not defined) which transforms according to the representation $\bigoplus e^*_{P_j}(\lambda_j)$

of $\mathcal{G}$. Finally, it is crucial to note that if the P_j are replaced by deleted disks centered at P_j, the quantization procedure at the boundary circles becomes the conformal field theory quantization via loop group representations as in a paper by G. Segal [in *IXth International Congress of Mathematical Physics*, 22–37, Hilger, Bristol, 1989; MR1033753]. This gives the unexpected connection between topological quantum field theory in 2 + 1 dimensions and conformal field theory in 1 + 1 dimensions.

In Chapter 6, three approaches to the projective flatness of the bundle of quantum spaces are outlined. The first, most direct approach, developed by Hitchin and Axelrod-Della Pietra-Witten, initially treats $\mathcal{A}$ formally as a finite-dimensional linear space, for which the quantization and projective flatness are easily formulated. Since $\mathcal{A}$ is infinite-dimensional, the formulas involved in the projective flatness diverge and must therefore be suitably regularized. The second approach is to replace the marked points by disks as above, in which case Segal has shown the projective flatness using loop group theory in the article cited above. The third approach, not completely worked out at present, is based on N. J. Hitchin's classification of Higgs bundles [Proc. London Math. Soc. (3) **55** (1987), no. 1, 59–126; MR0887284]. The main point is that the moduli space M_σ (for $G = \mathrm{SU}(n)$) embeds into the moduli space for Higgs bundles in such a way that M_σ appears as a degeneration of a family of abelian varieties. Using this embedding, the author outlines how it should be possible to prove the projective flatness on M_σ by reducing to the known projective flatness in the abelian case.

Chapter 7 is devoted to a presentation of the formal Lagrangian/path integral approach to the Jones-Witten theory. Applying stationary phase approximation formally to the path integral defining $Z_k(M)$ (i.e. for K the empty knot) leads to an asymptotic expression in k as $k \to \infty$ involving determinants and signatures of elliptic operators associated to flat connections on the given bundle over M. This expression becomes meaningful after zeta- and eta-function regularization, both of which involve a choice of metric on M. The resulting complex number has an amplitude which is, as expected, a topological invariant (essentially the sum of the Ray-Singer analytic torsions of the elliptic complex associated to each flat connection). However, the phase is not independent of the metric. To rectify this, Witten adds an ad hoc counterterm, a metric analogue of the Chern-Simons functional, to his stationary phase formula. This produces a topological phase factor as demanded, but this invariant depends on the framing chosen for M. (This part of the theory is only sketched in the monograph. More details can be found in Witten's paper. For the canonical framing of twice the tangent bundle found in a paper by the author [Topology **29** (1990), no. 1, 1–7; MR1046621], the added phase factor vanishes, as shown by Freed and Gompf. Moreover, Freed and Gompf correct Witten's stationary phase approximation; there is an error due to the fact that equation (2) on page 62 of this book is only true mod 1.)

The last chapter of final comments gives a formal expression for the vector $Z(M) \in Z(\Sigma)$ for the case $\partial M = \Sigma$, outlines the appearance of the skein relation characterizing the Jones polynomial in Witten's theory, and mentions how Witten's invariant can be computed for closed 3-manifolds via a sequence of surgeries reducing the given manifold to S^3. A full discussion of these points could very well have doubled the size of the book, so the reader is left instead with a series of tantalizing comments. Again, Witten's paper should be consulted for details.

Steven Rosenberg

From MathSciNet, July 2023

MR1091619 (92b:57024) 57N10; 17B37, 57M25, 81R50

Reshetikhin, N.; Turaev, V. G.

Invariants of 3-manifolds via link polynomials and quantum groups.

Inventiones Mathematicae **103** (1991), *no.* 3, 547–597.

The authors construct new topological invariants of compact oriented 3-manifolds and of framed links in such manifolds. The invariant of (a link in) a closed oriented 3-manifold is a sequence of complex numbers parametrized by complex roots of 1. For a framed link in the three-sphere the terms in the sequence are equal to the values of a Jones polynomial of the link evaluated in the corresponding roots of 1. Thus, for links in the three-sphere, the invariants in this paper are essentially equivalent to the Jones polynomial.

In this context the Jones polynomial refers to the original one-variable Jones polynomial and its relatives obtained from the quantum group $\mathrm{SL}(2)_q$. The original Jones polynomial corresponds to the fundamental representation of the quantum group. The invariants in the paper are constructed by labelling each component of the link with a given representation of the quantum group. This gives a particular Jones polynomial corresponding to the labelling. The authors show that, by using values of q that are roots of unity, and by summing (with appropriate coefficients) the Jones polynomials of a link corresponding to such a coloring, an invariant of framed links is obtained that is also invariant under the Kirby moves. The Kirby moves are modifications of framed links that give homeomorphic 3-manifolds in the class of 3-manifolds obtained by surgery on framed links.

In this way, invariants of 3-manifolds are obtained via invariants of knots and links. Each three-manifold is presented as surgery on a framed link, and the invariant of that link, being invariant under Kirby moves, is an invariant of the 3-manifold.

The paper uses a number of techniques and formulations. First of all there is the notion of a ribbon Hopf algebra (of which the universal enveloping algebra for the quantum lie algebra for $\mathrm{SL}(2)_q$ is an example). In another paper [Comm. Math. Phys. **127** (1990), no. 1, 1–26; MR1036112] the authors showed that ribbon Hopf algebras are an appropriate category of Hopf algebras for formulating invariants of framed links. In this method the invariant is formulated as a functor from a category of diagrams (with tangles as morphisms) to a corresponding module category. The functor takes a closed link diagram to a morphism from the complex numbers to itself, hence to a number. The core of the algebraic part of the paper is a careful treatment of the quantum group for $\mathrm{SL}(2)_q$ at roots of unity, showing that the representation theory is appropriate for the solution to the problem of obtaining invariants of the Kirby moves.

This paper is important as the first construction of a nontrivial 3-manifold invariant via invariants of framed links and the Kirby moves. It is also important as an instantiation of the program of invariants of 3-manifolds initiated by E. Witten [ibid. **121** (1989), no. 3, 351–399; MR0990772]. Witten's program uses ideas from quantum field theory and conformal field theory. The present paper is more elementary, but presumably produces the same invariants as the Witten program. The relationships between the Reshetikhin-Turaev approach and the Witten program will become clear as soon as the relationships between quantum groups and conformal field theory are more fully understood.

Louis H. Kauffman
From MathSciNet, July 2023

MR1318886 (97d:57004) 57M25
Bar-Natan, Dror
On the Vassiliev knot invariants.
Topology. An International Journal of Mathematics **34** (1995), *no.* 2, 423–472.

This paper is one of the standard introductions to the subject of Vassiliev invariants. It synthesises the axiomatic approach to Vassiliev's work à la Birman and Lin with the theory of quantum knot invariants of Reshetikhin and Turaev.

J. S. Birman and X. S. Lin [Invent. Math. **111** (1993), no. 2, 225–270; MR1198809] characterized Vassiliev's invariants as finite-type invariants. A knot invariant which takes values in an abelian group can be extended to singular knots—that is, immersions of the circle in 3-space which have a finite number of transversal self-intersections. This extension is done by considering a singular knot as an alternating sum of proper knots obtained by resolving each of the double points in the two possible directions. A knot invariant is said to be of finite type or of type n if, for some $n \in \mathbf{N}$, it vanishes on knots with more than n double points.

The Reshetikhin-Turaev invariant τ_V [see, e.g., Comm. Math. Phys. **127** (1990), no. 1, 1–26; MR1036112] is a (framed) knot invariant which depends on a semisimple Lie algebra $\mathfrak{g}$ and a representation V, and it takes values in the complex power series in a formal parameter h. For example, the Jones polynomial with the substitution $q = e^h$ gives a power series in h which is actually the invariant coming from the defining representation of SU(2). The author was working on perturbative Chern-Simons theory and noticed connections with diagrams in the Vassiliev theory. This led to the realization that the coefficient of h^n in τ_V is a type-n invariant. This added impetus to both approaches.

The Vassiliev theory is underpinned by certain combinatorial objects. To a knot with n double points can be associated a chord diagram—that is, an oriented circle with chords marked on it—each chord corresponding to a double point, the chord ends signifying the points on the circle which meet at the double point. It is not difficult to show that the value that a type-n Vassiliev invariant takes on a singular knot with n double points depends only on the underlying chord diagram of the singular knot. Thus a type-n invariant naturally determines a function on the set of chord diagrams with n chords. This function can be thought of as the leading term or "symbol" of the invariant. From topological considerations each of these functions can be shown to satisfy two combinatorially defined relations—known as the four-term (4T) and one-term (1T) relations (the latter also referred to as "isolated chord" and as "framing independence"). Functions of this form satisfying 4T and 1T are called weight systems.

The invariants in this paper take values in a field $\mathbf{F} = \mathbf{Q}$, $\mathbf{R}$ or $\mathbf{C}$. Let $\mathcal{A}^r$ be the graded vector space over $\mathbf{F}$ which has as the homogeneous components of degree n linear combinations of chord diagrams with n chords, modulo the ideal generated by the 1T and 4T relations. Note that an element of $\mathcal{A}^r$ is not necessarily a finite linear combination of chord diagrams. Although this is a simple space to define, the 1T and 4T relations make it difficult to figure out even the dimensions of the homogeneous parts; up to now the dimensions are known only up to degree 9.

M. Kontsevich's theorem (also known as the fundamental theorem) [in *I. M. Gel′fand Seminar*, 137–150, Amer. Math. Soc., Providence, RI, 1993; MR1237836] says that over $\mathbf{F}$, the structure of the Vassiliev invariants is the same as the structure of $\mathcal{A}^r$; the first rigorous proof (due to Kontsevich) is presented in the paper under review. The theorem consists of the transcendental construction of a function $\tilde{Z}$, the Kontsevich integral, from the set of knots to $\mathcal{A}^r$, with the property that a knot with n double points is mapped to its underlying chord diagram plus higher order terms (i.e. diagrams with more than n chords). $\tilde{Z}$ is universal in the sense that all Vassiliev invariants factor through it. Dualizing this means that, modulo invariants of lower order, a Vassiliev invariant is determined by the weight system it defines.

Actually, the construction of $\tilde{Z}$ given in the paper under review, although the first rigorous one, is not very satisfactory as it is not easy to do any calculations with. Other constructions have since appeared but nothing entirely satisfactory—see a paper by the author and A. Stoimenow [in *Geometry and physics (Aarhus, 1995)*, 101–134, Dekker, New York, 1997] for an overview of some of these.

The set of knots admits the operation of connected sum, which makes it into a commutative monoid, and the set of $\mathbf{F}$-valued knot invariants has a commutative product coming from the multiplication in $\mathbf{F}$; similarly $\mathcal{A}^r$ can be given a natural Hopf algebra structure with the product and coproduct having simple combinatorial definitions, so that $\tilde{Z}$ respects these structures. There is a different normalization of $\tilde{Z}$ which is relevant for the Reshetikhin-Turaev invariants mentioned below.

The structure theory of Hopf algebras implies that $\mathcal{A}^r$ is actually a polynomial algebra generated by its so-called primitive elements. Again the structure of these primitive elements is not properly understood, but $\mathcal{A}^r$ has an alter ego in which they are easier to spot: it is sometimes more convenient to consider the space of Chinese character diagrams which have extra trivalent vertices and a relation STU replacing 4T. The author proves that this space is isomorphic to $\mathcal{A}^r$.

The quantum invariants are really naturally defined for framed knots, and one can repeat the above theory for framed knots. One obtains a Hopf algebra $\mathcal{A}$ which is chord diagrams modulo just the 4T relation (or Chinese character diagrams modulo just STU); and as an algebra it is generated by one more primitive element than $\mathcal{A}^r$—the diagram with one chord. There is also a natural Hopf algebra map $\mathcal{A}^r \to \mathcal{A}$ which corresponds to assigning the zero framing to a knot.

Recall that for V a representation of the semi-simple Lie algebra $\mathfrak{g}$, the coefficient, τ_V^n, of h^n in τ_V is a type-n invariant; so the τ_V^n define weak weight systems (i.e. they satisfy 4T but not 1T). These weak weight systems have a simple description: a chord diagram (or Chinese character diagram) gives a recipe for "glueing together" various tensors associated to $\mathfrak{g}$ to obtain an element of the center of the universal enveloping algebra of $\mathfrak{g}$; taking the trace of this in the representation V gives a number—this is the value of the weight system on the diagram.

For the classical Lie algebras with certain representations, the author gives computationally easier ways of evaluating these weight systems. This is possible because of the behaviour of these weight systems with respect to some operations in the representation ring of a fixed $\mathfrak{g}$. On the level of (framed) knots one can define various cabling operations. One is the ith disconnected cabling, which results in an i-component link consisting of i parallel (along the framing) copies of the original knot. Evaluating $\tau_{V\otimes W}$ on a knot gives the same result as evaluating τ on the second disconnected cabling with (in the language of quantum invariants) one component coloured by V and one by W—this generalizes in the obvious way to higher-order cabling. The disconnected cabling operation descends to a map on the chord diagram level which can be utilised to evaluate $\tau_{V\otimes W}$.

Also very interesting is the ith connected cabling operation on knots which results in a knot which wraps i times around the old knot. This operation is known to be related to the Adams operation in the representation ring of $\mathfrak{g}$. On the level of chord diagrams one obtains a family of combinatorially defined maps $\psi^i\colon \mathcal{A}\to\mathcal{A}$ satisfying $\psi^i\psi^j=\psi^{ij}$, and indeed these are adjoint to the Adams operations, $\widehat{\psi}^i$, in the representation ring of $\mathfrak{g}$, i.e. for a diagram D, $\tau^n_{\widehat{\psi}^i V}(D)=\tau^n_V(\psi^i D)$. Further, each homogeneous part of $\mathcal{A}$ splits naturally into a direct sum of simultaneous eigenspaces for these operations, and spanning sets for these eigenspaces are given by the Chinese characters which are introduced.

A question raised in this paper is, do all weight systems come from semisimple Lie algebras? This has been answered in the negative by P. Vogel [“Algebraic structures on modules of diagrams”, Preprint, Inst. Math. Jussieu, Paris; per revr.].

The author also presents another method for constructing weight systems which is at least as strong as the semisimple Lie algebra method. It involves the use of marked surfaces but why marked surfaces should lead to knot invariants is not clear. This construction is conjectured to give all weight systems.

Many of the subsidiary results in the paper are left as exercises for the reader, and several outstanding problems are listed. All in all, this paper is a good introduction to a subject in the intersection of many areas of current mathematical interest.

Simon Willerton

From MathSciNet, July 2023

MR1473221 (98h:46067) 46L37; 46-01, 46L10, 57M25

Jones, V.; Sunder, V. S.

Introduction to subfactors. (English)

London Mathematical Society Lecture Note Series, 234.
Cambridge University Press, *Cambridge*, 1997, xii+162 pp., \$37.95,
ISBN 0-521-58420-5

The theory of subfactors originates in an apparently rather special problem in functional analysis: to understand the structure of inclusions of von Neumann factors of type II_1. However, this problem turns out to be related to several other fields in mathematics and mathematical physics, particularly to algebraic topology, statistical mechanics and quantum field theories.

From a technical viewpoint, the theory took off with Jones’ seminal paper [Invent. Math. **72** (1983), no. 1, 1–25; MR0696688] from the early eighties, where the index of subfactors was defined, and its possible values determined. Specifically, the index $[M\,N]$ of an inclusion $N\subseteq M$ is defined as the Murray-von Neumann coupling constant [cf. F. J. Murray and J. von Neumann, Ann. Math. (2) **37** (1936), 116–229; JFM 62.0449.03] of N in the standard representation $L^2(M,\text{trace})$, and Jones proved (1) $[MN]\in\{4\cos(2\pi/n)\ n=3,4,\cdots\}\cup[4,\infty]$ with all values being realized for some inclusion $N\subseteq M$. The proof is essentially algebraic, after the relevant algebraic relations have been extracted from the subfactor; this amounts, roughly, to iterating a construction of E. Christensen [Math. Ann. **243** (1979), no. 1, 17–29; MR0543091], and then proving that the resulting sequence (e_j) of projection satisfies (2) $e_i e_j = e_j e_i$ for $|i-j| > 1$, (3) $e_j e_{j\pm 1} e_j = [MN]^{-1} e_j$ for $j > 1$.

Now, in addition to the use in proving (1), the above relations were recognized by Jones to be very similar to Artin's braid group relations. By an ingenious development of this connection, Jones succeeded to construct a new polynomial invariant for knots [cf. Bull. Amer. Math. Soc. (N.S.) **12** (1985), no. 1, 103–111; MR0766964; Ann. of Math. (2) **126** (1987), no. 2, 335–388; MR0908150]. Moreover, Jones, and others, found spectacular applications of the theory to mathematical physics [cf., e.g., V. F. R. Jones, in *Proceedings of the International Congress of Mathematicians, Vol. I, II (Kyoto, 1990)*, 121–138, Math. Soc. Japan, Tokyo, 1991; MR1159209]. In 1990, Jones was awarded the Fields medal for his outstanding cross-disciplinary work originating in von Neumann algebras.

After the explosive developments of the eighties, the theory of subfactors seems to have stabilized in the nineties, as a research field which is deeply rooted in von Neumann algebras, yet with a distinctive character coming from the ongoing interaction with the above-mentioned fields of application. The field has also seen its first generation of monographs, mainly aimed at introducing other researchers to the field.

The volume under review seems to be the first which is primarily meant as an introduction for the beginning graduate student in the field. In the words of the authors, "the aim of this book is to give an introduction to some of the beautiful ideas and results which have been developed, since the inception of the theory of subfactors, by such mathematicians as Adrian Ocneanu and Sorin Popa; an attempt has been made to keep the material as self-contained as possible; in fact, we feel it should be possible to use this monograph as the basis of a two-semester course for second-year graduate students with a minimal background in Hilbert space theory." In the opinion of the reviewer, this is to be understood as follows: a sound foundation in von Neumann algebras, up to type classification of factors, should either be assumed from the start, or developed prior to studying the book, in which Section 1.1 merely gives a brief overview of the required knowledge (including good references to existing textbooks, which may be useful for the instructor or the student). From this point on, the book really is reasonably self-contained, with clear albeit condensed style of proof suitable for the level of prospective students. The main exceptions are, unfortunately but in a sense also inevitably, some deep and technically very demanding theorems of the subjects, in particular the Ocneanu-Popa spanning theorem (5.6.3 in the book, cf. references given at the end of the book), but also the infinite-index criterion contained in the Pimsner-Popa inequality (5.1.3 in the book). Of course, these analytic sides of the theory may well be postponed to later study for the student who is really determined to work on subfactors.

In brief outline, the topics covered are: basics of von Neumann algebra and factors (partially without proofs), coupling constant and index, the basic construction, principal graphs, the bimodule pictures, Pimsner-Popa basis and index inequality, examples of commuting squares coming from braid groups and statistical mechancis, path algebras and Ocneanu compactness argument; and finally a chapter with detailed computation of invariants for important examples of subfactors (arising from the above class of commuting squares), mainly from works of the authors.

In conclusion, this handy volume (162 pp., including bibliographical notes and subject index) fills a real need for a discipline about to reach maturity. Its study may be well prepared by readings from the book by the second author [*An invitation to von Neumann algebras*, Springer, New York, 1987; MR0866671].

Carl Winsløw

From MathSciNet, July 2023

MR1729488 (2001c:46116) 46L37; 46L10, 81R50

Popa, Sorin

Some properties of the symmetric enveloping algebra of a subfactor, with applications to amenability and property T.

Documenta Mathematica **4** (1999), 665–744.

S. T. Popa [Math. Res. Lett. **1** (1994), no. 4, 409–425; MR1302385] introduced the symmetric enveloping inclusion $M \vee M^{\mathrm{op}} \subset M \underset{e_N}{\boxtimes} M^{\mathrm{op}}$ associated to an extremal inclusion of II_1 factors $N \subset M$ with finite Jones index. The algebra $M \underset{e_N}{\boxtimes} M^{\mathrm{op}}$ is constructed as follows: It can be shown that the C^*-algebra generated by M, JMJ and the Jones projection e_N (as a subalgebra of $B(L^2(M))$) has a unique trace. The symmetric enveloping II_1 factor $M \underset{e_N}{\boxtimes} M^{\mathrm{op}}$ is then obtained from this C^*-algebra via the GNS-representation with respect to this trace.

The first two chapters of the paper under review deal with the construction and the basic properties of the symmetric enveloping algebra. In particular, it is shown how the construction relates to the tower of factors associated to $N \subset M$, and various descriptions of the enveloping II_1 factor are given. If $N \subset M$ is hyperfinite with strongly amenable graph [see S. T. Popa, Acta Math. **172** (1994), no. 2, 163–255; MR1278111], then the inclusion $M \vee M^{\mathrm{op}} \subset M \underset{e_N}{\boxtimes} M^{\mathrm{op}}$ is isomorphic to Ocneanu's asymptotic inclusion $M \vee (M' \cap M_\infty) \subset M_\infty$ (see, e.g., D. E. Evans and Y. Kawahigashi's monograph [*Quantum symmetries on operator algebras*, Oxford Univ. Press, New York, 1998; MR1642584] for a discussion of the asymptotic inclusion), where M_∞ denotes the weak closure of the union of the factors in the Jones tower associated to $N \subset M$. Note that $M \vee M^{\mathrm{op}} \subset M \underset{e_N}{\boxtimes} M^{\mathrm{op}}$ has finite Jones index if and only if $N \subset M$ has finite depth.

Popa computes in Chapter 3 the symmetric enveloping inclusion associated to a locally trivial subfactor associated with finitely many automorphisms of a II_1 factor Q [see, e.g., S. T. Popa, op. cit.; MR1278111]. The symmetric enveloping algebra is in this case a (cocycle) crossed product by the (possibly infinite) group G generated by these automorphisms in the outer automorphism group of Q.

Chapter 4 discusses thinness and quasi-regularity properties of the symmetric enveloping II_1 factor (see also [L. M. Ge and S. T. Popa, Duke Math. J. **94** (1998), no. 1, 79–101; MR1635904] for other results on thinness). A II_1 factor M is said to be thin if there exist two hyperfinite subfactors R_1 and R_2 in M such that M is the $\|\cdot\|_2$-closure of R_1R_2. Now, if $N \subset M$ is an extremal inclusion of hyperfinite II_1 factors with finite Jones index, then $M \underset{e_N}{\boxtimes} M^{\mathrm{op}}$ is thin. Other results along this line are proved in Chapter 4. Moreover, the first two relative commutants of the basic construction for $M \vee M^{\mathrm{op}} \subset M \underset{e_N}{\boxtimes} M^{\mathrm{op}}$ are computed (in the finite depth case, this result follows from Ocneanu's description of the principal graphs of the asymptotic inclusion [see, e.g., D. E. Evans and Y. Kawahigashi, op. cit.]). The computation shows that $M \underset{e_N}{\boxtimes} M^{\mathrm{op}}$ is a generalized crossed product construction over $M \vee M^{\mathrm{op}}$, i.e. the quasi-normalizer of $M \vee M^{\mathrm{op}}$ generates all of $M \underset{e_N}{\boxtimes} M^{\mathrm{op}}$. Chapter 4 ends with an ergodicity theorem for higher relative commutants. Popa shows that if $N \subset M$ is an inclusion of II_1 factors with finite index and $\{M_j\}_{j\in\mathbf{Z}}$ is a tunnel-tower associated to $N \subset M$, then the Jones projections $\{e_j\}_{j\in\mathbf{Z}}$ have trivial relative commutant in $A_{-\infty,\infty} = \overline{\bigcup_{i,j} M_i' \cap M_j}^{w}$. In particular, $A_{-\infty,\infty}$ is always a factor! Moreover, if $N \subset M$ is in addition extremal, then a tunnel can be chosen such that the Jones projections $\{e_j\}_{j\in\mathbf{Z}}$ have trivial relative commutant in the symmetric enveloping II_1 factor $M \underset{e_N}{\boxtimes} M^{\mathrm{op}}$.

Popa shows in Chapter 5 that amenability of $N \subset M$ [S. T. Popa, op. cit.; MR1278111] is equivalent to the statement that $M \underset{e_N}{\boxtimes} M^{\mathrm{op}}$ is amenable relative to $M \vee M^{\mathrm{op}}$. In particular, the details of some of the results that he announced in [op. cit.; MR1302385] are provided here. Other conditions equivalent to amenability of the subfactor are given, for instance a Følner-type condition. As an application of these results, Popa shows that if a standard λ-lattice [see S. T. Popa, Invent. Math. **120** (1995), no. 3, 427–445; MR1334479] has an amenable sublattice, then it must be amenable as well. Furthermore, finite index sublattices of amenable standard λ-lattices must be amenable.

In Chapter 6 several additional characterizations of amenability are proved. For instance, amenability of the standard invariant of an extremal inclusion of subfactors is shown to be equivalent to a maximality property of the local Connes-Størmer entropies of the core of $N \subset M$. The simplest characterization of amenability of $N \subset M$ is probably the Kesten-type condition $\|\Gamma_{N,M}\|^2 = [M\colon N]$, where $\Gamma_{N,M}$ denotes the principal graph of $N \subset M$ [see, e.g., F. M. Goodman, P. de la Harpe and V. F. R. Jones, *Coxeter graphs and towers of algebras*, Springer, New York, 1989; MR0999799]. As a consequence of the results of this chapter, Popa shows that intermediate subfactors of amenable subfactors are themselves amenable, and reduced subfactors of amenable subfactors are amenable (a similar result in the finite depth case was proved in [D. H. Bisch, Pacific J. Math. **163** (1994), no. 2, 201–216; MR1262294]). Given a nondegenerate commuting square of finite index subfactors such that the indices of the horizontal inclusions are finite as well, Popa shows that (strong) amenability of the standard invariant of the top inclusion is equivalent to (strong) amenability of the standard invariant of the bottom inclusion.

Chapter 7 gives a characterization of amenability of a subfactor in terms of the associated enveloping II_1 factor. It is shown that an extremal inclusion $N \subset M$ is amenable if and only if $M \underset{e_N}{\boxtimes} M^{\mathrm{op}}$ is the hyperfinite II_1 factor. Other characterizations, for instance in terms of hypertraces or in terms of representations of $N \subset M$, are given as well. Furthermore, Popa shows the following surprising result: Given an extremal inclusion of hyperfinite II_1 factors $N \subset M$ with amenable standard invariant and an inclusion of factors $Q \subset P$ embedded in $N \subset M$ as commuting squares, then the standard invariant of $Q \subset P$ is amenable as well.

In Chapter 8 Popa proves that amenability of a subfactor $N \subset M$ is equivalent to an Effros-Lance type condition, namely the simplicity of certain natural C^*-algebras associated to the subfactor (for instance the simplicity of $C^*(M, e_N, JMJ)$ [see also E. G. Effros and E. C. Lance, Adv. Math. **25** (1977), no. 1, 1–34; MR0448092]).

Popa introduces in Chapter 9 a notion of property T for the standard invariant of a subfactor. Property T for von Neumann algebras was introduced by A. Connes [J. Operator Theory **4** (1980), no. 1, 151–153; MR0587372; A. Connes and V. F. R. Jones, Bull. London Math. Soc. **17** (1985), no. 1, 57–62; MR0766450] and a notion of relative property T was studied independently by Popa ["Correspondences", Preprint, Natl. Inst. Sci. Inf. (INCREST), Bucharest, 1986; per bibl.] and C. Anantharaman-Delaroche [Math. Japon. **32** (1987), no. 3, 337–355; MR0914742]. The standard invariant of an extremal subfactor $N \subset M$ is said to have property T if the symmetric enveloping II_1 factor $M \underset{e_N}{\boxtimes} M^{\mathrm{op}}$ has property T relative to the II_1 factor $M \vee M^{\mathrm{op}}$. Considerable effort goes into showing that this definition depends only on the standard invariant of $N \subset M$ and not on the particular choice of the extremal inclusion $N \subset M$. For the locally trivial subfactors associated with finitely many automorphisms, property T of the standard invariant is shown to be equivalent to Kazhdan's property T of the associated group G. For examples of irreducible subfactors with property T standard invariant, see [D. H. Bisch and S. T. Popa, Geom. Funct. Anal. **9** (1999), no. 2, 215–225; MR1692494]. Popa proves that if a sublattice of a standard λ-lattice has property T, then the standard λ-lattice itself must have property T. The converse is true if the sublattice has finite index. In particular, the Temperley-Lieb-Jones A_∞ sublattice of an infinite depth amenable standard λ-lattice does not have property T. The paper ends with a number of open questions regarding property T for standard lattices.

There are two appendices. The first appendix gives a simple proof of the main result in [S. T. Popa, Ann. Sci. École Norm. Sup. (4) **32** (1999), no. 6, 743–767; MR1717575] for inclusions of type II_1 factors with finite index. Based on an argument of Connes [Ann. of Math. (2) **104** (1976), no. 1, 73–115; MR0454659], the second appendix contains a proof of a perturbation result for square integrable operators in semifinite von Neumann algebras.

Dietmar H. Bisch
From MathSciNet, July 2023

MR2257402 (2007h:46077) 46L37

Grossman, Pinhas; Jones, Vaughan F. R.

Intermediate subfactors with no extra structure.

Journal of the American Mathematical Society **20** (2007), *no.* 1, 219–265.

The paper under review makes some important advances in the study of intermediate subfactors and of angles between subfactors, notably with a surprising rigidity result, in the spirit of V. F. R. Jones' original work [Invent. Math. **72** (1983), no. 1, 1–25; MR0696688].

The notion of an intermediate subfactor $N \subset P \subset M$ was first investigated by D. H. Bisch, with an abstract characterization of the corresponding Jones projection [Pacific J. Math. **163** (1994), no. 2, 201–216; MR1262294]. Then Bisch and Jones worked out the diagrammatics of the whole system of projections [Invent. Math. **128** (1997), no. 1, 89–157; MR1437496]. A finiteness result regarding intermediate subfactors was obtained by Y. Watatani [J. Funct. Anal. **140** (1996), no. 2, 312–334; MR1409040]. In general, the classification problem for a given inclusion $N \subset M$ is reputed to be difficult.

As for the notion of an angle between two subfactors $P, Q \subset M$, this was introduced by T. Sano and Watatani in [J. Operator Theory **32** (1994), no. 2, 209–241; MR1338739]. A finiteness result, leading to the conclusion that under suitable assumptions the angle should be "quantized", comes from a result of Jones and F. Xu [Internat. J. Math. **15** (2004), no. 7, 717–733; MR2085101]. Once again, the general questions here are reputed to be difficult.

The new idea in the paper under review is that a good framework for both problems is that of quadrilaterals of subfactors $N \subset P, Q \subset M$. There are two natural assumptions on such a quadrilateral, namely the finiteness of the index, $[M\colon N] < \infty$, and the irreducibility condition $N' \cap M = \mathbb{C}$.

The simplest case is the one in which there is "no extra structure". This means that all four subfactors $N \subset P$, $N \subset Q$, $P \subset M$, $Q \subset M$ have no extra structure, in the sense that they correspond to the Temperley-Lieb algebra. This assumption is very natural, in view of the above-mentioned work of both Bisch-Jones and Sano-Watatani. For instance, this condition makes the angle between P and Q a usual real number (in general, the angle appears as a somewhat abstract spectral theoretic quantity).

The main result is that under the above assumptions, one of the following happens: (1) the quadrilateral commutes, in the sense that the expectations onto P, Q commute; (2) $[M\colon N] = 6$, the angle is $\pi/3$, and the whole situation is described by an action of S_3; or (3) $[M\colon N] = 6 + 4\sqrt{2}$, the angle is $\arccos(\sqrt{2} - 1)$, and the quadrilateral comes from a GHJ subfactor associated to D_5.

The proof heavily relies on planar algebra methods, with technical ingredients from [V. F. R. Jones, in *Essays on geometry and related topics, Vol. 1, 2*, 401–463, Enseignement Math., Geneva, 2001; MR1929335] and [Z. A. Landau, Geom. Dedicata **95** (2002), 183–214; MR1950890].

Teodor Banica

From MathSciNet, July 2023

MR2805599 57M27; 18G40, 57R58

Kronheimer, P. B.; Mrowka, T. S.

Khovanov homology is an unknot-detector.

Publications Mathématiques. Institut de Hautes Études Scientifiques (2011), *no.* 113, 97–208.

In this paper the authors extend their study of singular instanton Floer homology of knots and links [J. Topol. **4** (2011), no. 4, 835–918; MR2860345], based on their earlier work on instantons on 4-manifolds with codimension-2 singularities [Topology **32** (1993), no. 4, 773–826; MR1241873; Topology **34** (1995), no. 1, 37–97; MR1308489].

The most striking application is their theorem which is stated in the title. M. G. Khovanov defined a bigraded cohomology theory Kh which categorifies the Jones polynomial in [Duke Math. J. **101** (2000), no. 3, 359–426; MR1740682], and introduced a related reduced version Khr in [Experiment. Math. **12** (2003), no. 3, 365–374; MR2034399]. The authors prove that the reduced Khovanov homology $\mathrm{Khr}(K)$ of a knot K is isomorphic to the reduced Khovanov homology of the unknot if and only if the knot is the unknot. Whether the Jones polynomial detects the unknot is still an open question.

Given a 3-manifold Y and a knot or link K in Y, the authors consider the resulting orbifold $\check{Y}$ with cone angle $\pi/2$ along K, and they consider an orbifold bundle $\check{P} \to \check{Y}$ determined by some 'singular bundle data' $\mathbf{P}$ defined in the article. This defines an ordinary PU(2) bundle $P \to Y \backslash K$. In this setup they consider a space of connections $C(Y, K, \mathbf{P})$ (as usual, with some L^2_k Sobolev completion of the actual connection space). With an orbifold metric on $\check{Y}$ one can define the Chern-Simons functional CS on this space of connections. Extending ideas of A. Floer [in *Geometry of low-dimensional manifolds, 1 (Durham, 1989)*, 97–114, London Math. Soc. Lecture Note Ser., 150, Cambridge Univ. Press, Cambridge, 1990; MR1171893], they study the Morse homology of this functional yielding an abelian group $\mathrm{I}(Y, K, \mathbf{P})$. This involves the discussion of moduli spaces of flow lines which correspond to moduli spaces of anti-selfdual connections on the 4-manifold $\mathbb{R} \times Y$, involving issues of compactness, transversality, and orientations of moduli spaces. Some new index computations are necessary due to the singularities.

The authors show that their construction defines a functor from a suitable cobordism category to the category of projective abelian groups (morphisms defined up to sign).

The extension of the authors' work [op. cit.] is that the embedded surfaces (appearing when studying the effect of cobordisms) now no longer need to be orientable. Also, the singular bundle data is more general than previously; in particular, the bundle P does not need to extend to all of Y. Finally, the condition for avoidance of reducible (singular) critical points of the Chern-Simons functional reads as follows: A surface Σ in Y is a non-integral surface if either

(1) Σ is disjoint from K and the second Stiefel-Whitney class $w_2(P)$ is nonzero on Σ, or
(2) Σ is transverse to K and $K \cdot \Sigma$ is odd.

The singular bundle data $\mathbf{P}$ is said to be non-integral if it admits a non-integral surface.

Given a knot K in the 3-sphere, there is no non-integral surface for any singular bundle data. This is circumvented by introducing a 2-component link $K^\natural$ which is K together with the boundary of a meridional disc centered in a marked point of K. The singular bundle data is then chosen so that $w_2(P)$ is Poincaré dual to an arc ω on the meridional disc joining the two components of $K^\natural$. Up to canonical isomorphism, this determines a non-integral singular bundle data $\mathbf{P}$ for $(S^3, K^\natural)$, and the resulting group $\mathrm{I}(S^3, K^\natural, \mathbf{P})$ is denoted $\mathrm{I}^\natural(K)$. More generally, the construction may also be applied to links with a marked point, and there is a second version $\mathrm{I}^\sharp(K)$ introduced where the above construction is applied to the distant union of K with an unknot U bearing a marked point.

That Khovanov homology detects the unknot is finally proved in the following two steps. In a first step, which uses a version of Floer's excision theorem [P. J. Braam and S. K. Donaldson, in *The Floer memorial volume*, 195–256, Progr. Math., 133, Birkhäuser, Basel, 1995; MR1362829], the authors prove that $\mathrm{I}^\natural(K) \otimes \mathbb{Q}$ is isomorphic to $\mathrm{KHI}(S^3, K; \mathbb{Q})$, a version of instanton Floer homology for closed 3-manifolds built from the knot complement that the authors introduced in [J. Differential Geom. **84** (2010), no. 2, 301–364; MR2652464]. They have proved that the rank of $\mathrm{KHI}(S^3, K; \mathbb{Q})$ is strictly bigger than 1 if and only if the knot K is not the unknot, so KHI detects the unknot.

Khovanov homology of a link K is constructed from a cube of resolutions of a diagram of K. At each vertex there is a group, and each edge yields a homomorphism coming from a natural pair of pants cobordism between the two resolutions at the endpoints of the edge. There is an associated bi-graded complex, and its homology is the Khovanov homology of K.

The instanton Floer homology $\mathrm{I}^\natural$ of a link K can also be computed from the resolution cube (Theorem 6.8 in the article). This uses an unoriented skein exact triangle sequence that the authors establish and an algebraic trick that was also used by P. S. Ozsváth and Z. Szabó in [Adv. Math. **194** (2005), no. 1, 1–33; MR2141852]. In the computation of $\mathrm{I}^\natural(K)$ via the resolution cube the differential is compatible with a natural filtration yielding an exact sequence that converges to $\mathrm{I}^\natural(K)$.

At the E_1 page the chain groups occurring for $I^\#$ at the edges are the same as the groups occurring in the unreduced Khovanov chain complex of the diagram. The authors study the differentials and deduce that the E_2 page is isomorphic to the unreduced Khovanov homology $\mathrm{Kh}(\overline{K})$ of the mirror image $\overline{K}$ of K. Via the long exact skein exact sequence they deduce the statement for the reduced theories. As a consequence, there is a spectral sequence from the reduced Khovanov homology of a link K converging to the instanton Floer homology $\mathrm{I}^\natural(K)$.

Together with the first step mentioned before, this shows that the rank of the reduced Khovanov homology $\mathrm{Khr}(K)$ of a knot K is equal to that of the unknot if and only if the knot K is the unknot.

Raphael Zentner
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Jones, V. F. R.

Planar algebras, I.

New Zealand Journal of Mathematics **52** (2021 [2021–2022]), 1–107.

The paper under review is an exact copy of V. F. R. Jones' original paper posted at arXiv in 1999 ["Planar algebras, I", preprint, `arXiv:math/9909027`]. The paper stayed in preprint form for more than 20 years, and it has now been published following Jones' untimely death.

Jones' 1983 paper "Index for subfactors" [Invent. Math. **72** (1983), no. 1, 1–25; MR0696688] started a revolution in the world of von Neumann algebras. The fact that the possible values of the index $[M : N]$ for an inclusion $N \subset M$ of II_1-factors are

$$\left\{4\cos^2\frac{\pi}{k} : k \in \mathbb{N},\ k \geq 3\right\} \cup [4, \infty] \tag{1}$$

was immediately remarkable and suggested lots of connections. Jones' proof that the numbers above are the allowable (and existing) values of the index used an idea called the *basic construction*, where, starting from the initial inclusion $N \subset M$ of II_1@-factors, one produces a tower of inclusions $N \subset M \subset M_1 \subset M_2 \subset \cdots$, where M_{k+1} is generated by M_k and the so-called *Jones projection* e_k. The projections $\{e_k\}$ satisfy the same relations as the generators of the Temperley-Lieb algebra, known in statistical mechanics. Via its relation with the braid group and more ingenuity by Jones, this suggested a new polynomial invariant for knots [V. F. R. Jones, Bull. Amer. Math. Soc. (N.S.) **12** (1985), no. 1, 103–111; MR0766964; Ann.

of Math. (2) **126** (1987), no. 2, 335–388; MR0908150]. Besides this most famous connection, there was a lot of interest in the index, and *subfactors* became an area on its own. Some details about this interest and developments are mentioned in the first paragraph of the present paper.

Even after S. T. Popa's definitive classification results in the early 1990s [Acta Math. **172** (1994), no. 2, 163–255; MR1278111; *Classification of subfactors and their endomorphisms*, CBMS Regional Conf. Ser. in Math., 86, Conf. Board Math. Sci., Washington, DC, 1995; MR1339767] there was a lot to do in subfactor theory. It was then that the "Planar algebras, I" paper opened a big new avenue of research. It features a novel approach of creating a graphic language to encode properties like those in the generators of the Temperley-Lieb algebra. Technically, Jones uses planar algebras to provide a new encoding of the *standard invariant*; this invariant, the "tower of relative commutants" was suggested by the ideas in his original index paper, and formalized and studied by Ocneanu, Popa, and others. The planar algebra approach allows Jones to prove many relations by graphical means and thus avoid complicated symbolic manipulations. Since "Planar algebras, I" appeared in 1999, subfactors have become a thriving area, with lots of young researchers getting very interesting new results. At the time of writing this review, Mathematical Reviews shows 94 papers with "Planar algebra(s)" in the title, and that is just the tip of the iceberg of all the research inspired by Jones' "Planar algebras, I" paper. It is great for the mathematical community that this great paper is finally in printed form.

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