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A NEW PERSPECTIVE ON THE SULLIVAN DICTIONARY VIA ASSOUAD TYPE DIMENSIONS AND SPECTRA

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Abstract. The Sullivan dictionary provides a beautiful correspondence between Kleinian groups acting on hyperbolic space and rational maps of the extended complex plane. We focus on the setting of geometrically finite Kleinian groups with parabolic elements and parabolic rational maps. In this context an especially direct correspondence exists concerning the dimension theory of the associated limit sets and Julia sets. In recent work we established formulae for the Assouad type dimensions and spectra for these fractal sets and certain conformal measures they support. This allows a rather more nuanced comparison of the two families in the context of dimension. In this expository article we discuss how these results provide new entries in the Sullivan dictionary, as well as revealing striking differences between the two families.

1. Introduction

Seminal work of Sullivan in the 1980s [39] resolved a long-standing problem in complex dynamics by proving that the Fatou set of a rational map has no wandering domains. This work served to establish remarkable connections between the dynamics of rational maps and the actions of Kleinian groups. This connection subsequently stimulated activity in both the complex dynamics and hyperbolic geometry communities and led to what is now known as the *Sullivan dictionary*; see, for example, [30]. The Sullivan dictionary provides a framework to study the relationships between Kleinian groups and rational maps; see Table 1. In many cases there are analogous results, even with similar proofs, albeit expressed in a different language; see [12, Table 1] and also [41] and references therein.

Both Kleinian groups and rational maps generate important examples of dynamically invariant fractal sets: *limit sets* in the Kleinian case, and *Julia sets* in the rational map case; see Figure 1. The Sullivan dictionary is very well suited to understanding the connections between these two families of fractals, and the correspondence is especially strong in the context of dimension theory: in both settings there is a *critical exponent* which, for certain classes of Kleinian groups and rational maps, describes all of the most commonly used notions of fractal dimension. For Kleinian groups the critical exponent is the Poincaré exponent, denoted by δ, and

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TABLE 1. Some well-known "entries" in the Sullivan dictionary in the setting of geometrically finite Kleinian groups and parabolic rational maps. See Section 2 for definitions and notation. In Section 3 we describe an expansion of this dictionary, including several new entries as well as some striking differences ("nonentries").

Kleinian groups	Rational maps
Kleinian group Γ	rational map T
Kleinian limit set $L(\Gamma)$	Julia set $J(T)$
Poincaré exponent δ	critical exponent h
Patterson–Sullivan measure μ	h-conformal measure m
$\dim_{\mathrm{H}} L(\Gamma) = \dim_{\mathrm{B}} L(\Gamma) = \delta$	$\dim_{\mathrm{H}} J(T) = \dim_{\mathrm{B}} J(T) = h$
$\dim_{\mathrm{H}} \mu = \delta$	$\dim_{\mathrm{H}} m = h$
finite set of inequivalent parabolic points	finite set of parabolic points $\boldsymbol{\Omega}$
rank of parabolic point $k(p)$	petal number of parabolic point $p(\omega)$
dimension bound $\delta > k_{\max}/2$	dimension bound $h > p_{\max}/(1 + p_{\max})$

for rational maps the critical exponent is the smallest zero of the topological pressure, denoted by h. For both nonelementary geometrically finite Kleinian groups and parabolic (or hyperbolic) rational maps, the critical exponent coincides with the Hausdorff, packing, and box dimensions of the associated fractal as well as the Hausdorff, packing, and entropy dimensions of the associated ergodic conformal measure of maximal dimension.

There has been a recent increase in interest in the *Assouad type dimensions*, and these dimensions (and associated dimension spectra) do not behave in such a straightforward manner in the presence of parabolicity. In particular, the critical exponent does *not necessarily* give the Assouad dimension of the associated fractals. As we shall see, by slightly expanding the family of dimensions considered, a much richer and more varied tapestry of results emerges. In this expository paper we discuss recent work from [19, 21, 22] and show how this can be used to provide a new perspective on the Sullivan dictionary.

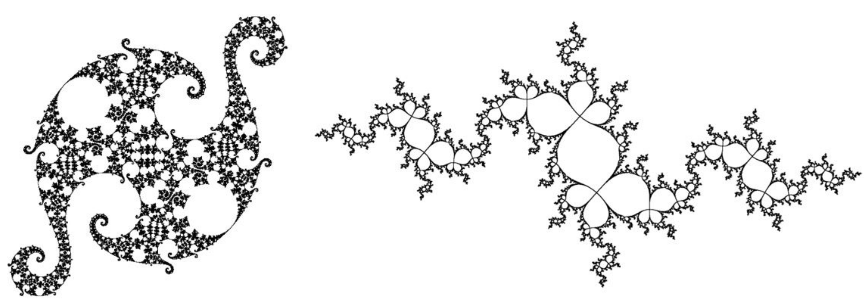

FIGURE 1. Left: an example of a Kleinian limit set. Here $d = 2$ and the boundary $\mathbb{S}^2$ has been identified with $\mathbb{R}^2 \cup \{\infty\}$. Parabolic points with rank 1 are easily identified. Right: an example of a parabolic Julia set. Parabolic points with petal number 4 are easily identified. See Section 2 for definitions and notation.

2. Definitions and background

2.1. **Dimensions of sets and measures and dimension interpolation.** We recall and motivate the key notions from fractal geometry and dimension theory which we use. For a more in-depth treatment see the books [6, 18] for background on Hausdorff and box dimensions, and [20] for Assouad type dimensions. We work with fractals in two distinct settings. Kleinian limit sets will be compact subsets of the d-dimensional sphere $\mathbb{S}^d$ which we view as a subset of $\mathbb{R}^{d+1}$. On the other hand, Julia sets will be compact subsets of the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. However, by a standard reduction we will assume that the Julia sets are compact subsets of the complex plane $\mathbb{C}$ which we identify with $\mathbb{R}^2$; see Section 2.3. Therefore, it is convenient to recall dimension theory for nonempty compact subsets of Euclidean space only.

Throughout this section, let $F \subseteq \mathbb{R}^d$ be nonempty and compact. Perhaps the most commonly used notion of fractal dimension is the Hausdorff dimension, but it will be especially important for us to consider several notions of dimension together. We write $\dim_{\mathrm{H}} F$, $\dim_{\mathrm{B}} F$, $\overline{\dim}_{\mathrm{B}} F$ and $\underline{\dim}_{\mathrm{B}} F$ for the *Hausdorff, box, upper box*, and *lower box dimensions* of F, respectively, but we refer the reader to [6, 18] for the precise definitions. We write

$$|F| = \sup_{x,y\in F} |x-y| \in [0,\infty)$$

to denote the diameter of F. Given $r > 0$, we write $N_r(F)$ for the smallest number of balls of radius r required to cover F. In the last ten years there has been an increase in interest in the Assouad dimension in the context of fractal geometry. This notion has been of central importance in other fields for much longer, however, and stems from work in embedding theory and conformal geometry; see [27, 33]. The *Assouad dimension* of F is defined by

$$\dim_{\mathrm{A}} F = \inf\Bigg\{ s \geqslant 0 \mid \exists C > 0 \ : \ \forall\, 0 < r < R \ : \ \forall x \in F : \\ N_r(B(x,R)\cap F) \leqslant C\left(\frac{R}{r}\right)^s \Bigg\}.$$

The lower dimension is the natural *dual* to the Assouad dimension, and it is particularly useful to consider these notions together. The *lower dimension* of F is defined by

$$\dim_{\mathrm{L}} F = \sup\Bigg\{ s \geqslant 0 \mid \exists C > 0 \ : \ \forall\, 0 < r < R \leqslant |F| \ : \ \forall x \in F : \\ N_r(B(x,R)\cap F) \geqslant C\left(\frac{R}{r}\right)^s \Bigg\}$$

provided $|F| > 0$, and otherwise it is 0. Importantly (and using that F is compact),

$$\dim_{\mathrm{L}} F \leqslant \dim_{\mathrm{H}} F \leqslant \overline{\dim}_{\mathrm{B}} F \leqslant \dim_{\mathrm{A}} F.$$

The Assouad and lower spectra were introduced much more recently in [23] and provide an *interpolation* between the box dimension and the Assouad and lower dimensions, respectively. The motivation for the introduction of these *dimension* *spectra* was to gain a more nuanced understanding of fractal sets than that provided by the dimensions considered in isolation. This is already proving a fruitful programme with applications emerging in a variety of settings including to problems in harmonic analysis; see work of Anderson, Hughes, Roos, and Seeger [2] and [34]. These spectra provide a parametrised family of dimensions by fixing the relationship between the two scales $r < R$ used to define the Assouad and lower dimensions. Studying the dependence on the parameter within this family thus yields finer and more nuanced information about the local structure of the set. For example, one may understand which scales "witness" the behaviour described by the Assouad and lower dimensions. For $\theta \in (0,1)$, the *Assouad spectrum* of F is given by

$$\dim_{\mathrm{A}}^{\theta} F = \inf\left\{ s \geqslant 0 \mid \exists C > 0 \ : \ \forall\, 0 < r < 1 \ : \ \forall x \in F : \right. \\ \left. N_r(B(x, r^\theta) \cap F) \leqslant C \left(\frac{r^\theta}{r}\right)^s \right\}.$$

The *lower spectrum* of F, denoted by $\dim_{\mathrm{L}} F$, is defined similarly by using the parameter to fix the relationship $R = r^\theta$ in the definition of the lower dimension. It was shown in [23] that

$$\begin{aligned} \overline{\dim}_{\mathrm{B}} F \leqslant \dim_{\mathrm{A}}^{\theta} F &\leqslant \min\left\{\dim_{\mathrm{A}} F,\ \frac{\overline{\dim}_{\mathrm{B}} F}{1-\theta}\right\}, \\ \dim_{\mathrm{L}} F \leqslant \dim_{\mathrm{L}}^{\theta} F &\leqslant \underline{\dim}_{\mathrm{B}} F. \end{aligned} \tag{2.1}$$

In particular, $\dim_{\mathrm{A}}^{\theta} F \to \overline{\dim}_{\mathrm{B}} F$ as $\theta \to 0$. The limit of $\dim_{\mathrm{A}}^{\theta} F$ exists and coincides with the quasi-Assouad dimension. The quasi-Assouad and Assouad dimensions do not necessarily coincide, but in many cases of interest they do. It is not necessarily true that $\dim_{\mathrm{L}}^{\theta} F \to \underline{\dim}_{\mathrm{B}} F$ as $\theta \to 0$, but it was proved in [20, Theorem 6.3.1] that this does hold provided F satisfies a strong form of dynamical invariance. Whilst the fractals we study are not quite covered by this result, we shall see that this interpolation holds nevertheless.

There is an analogous dimension theory of measures, and the interplay between the dimension theory of fractal sets and the measures they support is fundamental to fractal geometry, especially in the dimension theory of dynamical systems. For example, a problem of interest is to identify dynamical measures witnessing the dimension of the support, e.g., invariant measures of full Hausdorff dimension. Let ν be a locally finite Borel measure on $\mathbb{R}^d$, i.e., $\nu(B(x,r)) < \infty$ for all $x \in \mathbb{R}^d$ and $r > 0$. We write $\mathrm{supp}(\nu) = \{x \in \mathbb{R}^d \mid \nu(B(x,r)) > 0 \text{ for all } r > 0\}$ for the *support* of ν. We say that ν is *fully supported* on F if $\mathrm{supp}(\nu) = F$. We write $\dim_{\mathrm{H}} \nu$ for the (lower) Hausdorff dimension of ν and note that $\dim_{\mathrm{H}} \nu \leqslant \dim_{\mathrm{H}} \mathrm{supp}(\nu)$ and (using that F is compact)

$$\dim_{\mathrm{H}} F = \sup\{\dim_{\mathrm{H}} \nu \mid \mathrm{supp}(\nu) \subseteq F\};$$

see [29]. The *Assouad dimension* of ν with $\operatorname{supp}(\nu) = F$ is defined by

$$\dim_{\mathrm{A}} \nu = \inf \left\{ s \geqslant 0 \mid \exists C > 0 \ : \ \forall\, 0 < r < R < |F| \ : \ \forall x \in F : \right.$$
$$\left. \frac{\nu(B(x,R))}{\nu(B(x,r))} \leqslant C \left(\frac{R}{r}\right)^s \right\}$$

and, provided $|\operatorname{supp}(\nu)| = |F| > 0$, the *lower dimension* of ν is given by

$$\dim_{\mathrm{L}} \nu = \sup \left\{ s \geqslant 0 \mid \exists C > 0 \ : \ \forall\, 0 < r < R < |F| \ : \ \forall x \in F : \right.$$
$$\left. \frac{\nu(B(x,R))}{\nu(B(x,r))} \geqslant C \left(\frac{R}{r}\right)^s \right\},$$

and otherwise it is 0. By convention we assume that $\inf \emptyset = \infty$. The Assouad and lower dimensions of measures were introduced in [25], where they were referred to as the upper and lower regularity dimensions, respectively. It is well known (see [20, Lemma 4.1.2]) that, for a Borel probability measure ν supported on F,

$$\dim_{\mathrm{L}} \nu \leqslant \dim_{\mathrm{L}} F \leqslant \dim_{\mathrm{A}} F \leqslant \dim_{\mathrm{A}} \nu$$

and, furthermore, we have the stronger fact that

$$\dim_{\mathrm{A}} F = \inf \{\dim_{\mathrm{A}} \nu \mid \nu \text{ is a Borel probability measure fully supported on } F\}$$

and

$$\dim_{\mathrm{L}} F = \sup \{\dim_{\mathrm{L}} \nu \mid \nu \text{ is a Borel probability measure fully supported on } F\}.$$

For $\theta \in (0,1)$, the *Assouad spectrum* of ν, denoted by $\dim_{\mathrm{A}}^{\theta} \nu$, and the *lower spectrum* of ν, denoted by $\dim_{\mathrm{L}}^{\theta} \nu$, are defined similarly to the Assouad and lower dimensions but, again, using the parameter $\theta \in (0,1)$ to fix the relationship $R = r^{\theta}$.

It is known (see [17] for example) that

$$\dim_{\mathrm{L}} \nu \leqslant \dim_{\mathrm{L}}^{\theta} \nu \leqslant \dim_{\mathrm{A}}^{\theta} \nu \leqslant \dim_{\mathrm{A}} \nu$$

and, if ν is fully supported on F, then

$$\dim_{\mathrm{L}}^{\theta} \nu \leqslant \dim_{\mathrm{L}}^{\theta} F \leqslant \dim_{\mathrm{A}}^{\theta} F \leqslant \dim_{\mathrm{A}}^{\theta} \nu.$$

There are also upper and lower box dimensions for measures, recently introduced in [17]. We omit the formal definitions, referring the reader to [17, 20]. Following [17], it is useful to note that

$$\overline{\dim}_{\mathrm{B}} F = \inf \left\{\overline{\dim}_{\mathrm{B}} \nu \mid \nu \text{ is a finite Borel measure fully supported on } F\right\}$$

with an analogous result for the lower box dimension. Furthermore, it was shown that the upper box dimension of ν can be related to the Assouad spectrum of ν in a similar manner to sets, that is, for $\theta \in (0,1)$,

$$\overline{\dim}_{\mathrm{B}} \nu \leqslant \dim_{\mathrm{A}}^{\theta} \nu \leqslant \min \left\{ \dim_{\mathrm{A}} \nu, \frac{\overline{\dim}_{\mathrm{B}} \nu}{1-\theta} \right\},$$

and so $\overline{\dim}_{\mathrm{B}} \nu = \lim_{\theta \to 0} \dim_{\mathrm{A}}^{\theta} \nu$.

2.2. **Kleinian groups and limit sets.** For a more thorough study of hyperbolic geometry and Kleinian groups, we refer the reader to [4,28]. For $d \geqslant 1$, we model $(d+1)$-dimensional hyperbolic space using the Poincaré ball model

$$\mathbb{D}^{d+1} = \{z \in \mathbb{R}^{d+1} \mid |z| < 1\}$$

equipped with the hyperbolic metric $d_{\mathbb{H}}$, and we call the boundary

$$\mathbb{S}^{d} = \{z \in \mathbb{R}^{d+1} \mid |z| = 1\}$$

the *boundary at infinity* of the space $(\mathbb{D}^{d+1}, d_{\mathbb{H}})$. We denote by $\mathrm{Con}(d)$ the group of orientation-preserving isometries of $(\mathbb{D}^{d+1}, d_{\mathbb{H}})$. We say that a group is *Kleinian* if it is a discrete subgroup of $\mathrm{Con}(d)$ (such groups are often referred to as *Fuchsian* in the case when $d = 1$), and given a Kleinian group Γ, the *limit set* of Γ is defined to be $L(\Gamma) = \overline{\Gamma(\mathbf{0})} \setminus \Gamma(\mathbf{0})$ where $\mathbf{0} = (0, \ldots, 0) \in \mathbb{D}^{d+1}$. It is well known that $L(\Gamma)$ is a compact Γ-invariant subset of $\mathbb{S}^d$; see Figure 1. If $L(\Gamma)$ consists of zero, one or two points, it is said to be *elementary*, and otherwise it is *nonelementary.* In the nonelementary case, $L(\Gamma)$ is a perfect set, and often has a complicated fractal structure. We consider *geometrically finite* Kleinian groups. Roughly speaking, this means that there is a fundamental domain with finitely many sides, but we refer the reader to [8] for a precise definition. We define the *Poincaré exponent* of a Kleinian group Γ to be

$$\delta = \inf\left\{ s > 0 \mid \sum_{g \in \Gamma} e^{-s d_{\mathbb{H}}(\mathbf{0}, g(\mathbf{0}))} < \infty \right\}.$$

Due to work of Patterson and Sullivan [32,38], it is known that for a nonelementary geometrically finite Kleinian group Γ, the Hausdorff dimension of the limit set is equal to δ. Further, it was later proved by Stratmann and Urbański [35, Theorem 3] (see also Bishop and Jones [5, Corollary 1.5]) that the box and packing dimensions of the limit set are also equal to δ. Even in the nonelementary geometrically *infinite* case, δ is still an important quantity. In fact it always gives the Hausdorff dimension of the *radial* limit set, and therefore always provides a lower bound for the Hausdorff dimension of the limit set; see [5].

From now on we only discuss the nonelementary geometrically finite case. We write μ to denote the *Patterson–Sullivan measure*, which is a measure first constructed by Patterson in [32]. Strictly speaking, there is a family of (mutually equivalent) Patterson–Sullivan measures. However, we may fix one for simplicity (and hence talk about *the* Patterson–Sullivan measure since the dimension theory is the same for each measure). The geometry of Γ, $L(\Gamma)$, and μ are heavily related. For example, μ is a conformal Γ-ergodic Borel probability measure which is fully supported on $L(\Gamma)$. Moreover, μ is exact dimensional (with dimension δ) and therefore has Hausdorff, packing, and entropy dimension equal to δ. Exact dimensionality is a consequence of the global measure formula together with finer analysis of the parabolic fluctuations; see [37]. The limit set is Γ-invariant in the strong sense that $g(L(\Gamma)) = L(\Gamma)$ for all $g \in \Gamma$. However, μ is only quasi-invariant and $\mu \circ g$ is related to μ by a geometric transition rule; see [7, Chapter 14] for a more detailed exposition of this.

If Γ contains no parabolic elements, then

$$\dim_{\mathrm{A}} L(\Gamma) = \dim_{\mathrm{L}} L(\Gamma) = \dim_{\mathrm{A}} \mu = \dim_{\mathrm{L}} \mu = \dim_{\mathrm{B}} \mu = \delta;$$

see [19]. Therefore, we assume from now on that Γ contains at least one parabolic element.

Let $P \subseteq L(\Gamma)$ denote the countable set of parabolic fixed points. For $p \in P$, write $k(p)$ to denote the maximal rank of a free abelian subgroup of the stabiliser of p (in Γ) and call this the *rank* of p. We write

$$k_{\min} = \min\{k(p) \mid p \in P\},$$
$$k_{\max} = \max\{k(p) \mid p \in P\}.$$

It was proven in [38] that $\delta > k_{\max}/2$.

2.3. **Rational maps and Julia sets.** For a more detailed discussion of the dynamics of rational maps, see [11, 31]. Let $T : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ denote a rational map of degree at least 2, and write $J(T)$ to denote the *Julia set* of T, which is equal to the closure of the repelling periodic points of T; see Figure 1. The Julia set is closed and T-invariant. We may assume that $J(T)$ is a compact subset of $\mathbb{C}$ by a standard reduction. This is achieved by conjugating a point in the complement of the Julia set to the *point at infinity* and noting that the case when the Julia set is the whole of $\hat{\mathbb{C}}$ is trivial.

A periodic point $\xi \in \mathbb{C}$ with period p is said to be *rationally indifferent* (or *parabolic*) if $(T^p)'(\xi) = e^{2\pi i q}$ for some $q \in \mathbb{Q}$. We say that T and $J(T)$ are *parabolic* if $J(T)$ contains no critical points of T, but contains at least one parabolic point. Define h to be the smallest zero of the topological pressure $t \mapsto P(T, -t\log|T'|)$. In the parabolic setting, it was proven in [15] that $\dim_{\mathrm{H}} J(T) = h$. Furthermore, in [16] it was shown that the box and packing dimensions of $J(T)$ are equal to h. Due to work of Aaronson, Denker and Urbański [1, 14, 15] it is known that, for parabolic T, there exists a unique atomless h-conformal probability measure m supported on $J(T)$. It also again follows from the global measure formula together with finer analysis of the parabolic fluctuations (for example [36, Lemma 5.3 or Proposition 5.4]) that m is exact dimensional (with dimension h) and therefore the Hausdorff, packing, and entropy dimensions of m are also given by h.

If T contains no critical points nor parabolic points, then it is *hyperbolic* and, analogous to case of geometrically finite Kleinian groups with no parabolic elements,

$$\dim_{\mathrm{A}} J(T) = \dim_{\mathrm{L}} J(T) = \dim_{\mathrm{A}} m = \dim_{\mathrm{L}} m = \dim_{\mathrm{B}} m = h;$$

see [21]. Therefore, we assume from now on that T is parabolic.

Write $\mathbf{\Omega}$ to denote the finite set of parabolic points of T, and let

$$\mathbf{\Omega}_0 = \{\xi \in \mathbf{\Omega} \mid T(\xi) = \xi,\ T'(\xi) = 1\}.$$

As $J(T^n) = J(T)$ for every $n \in \mathbb{N}$, we may assume without loss of generality that $\mathbf{\Omega} = \mathbf{\Omega_0}$. Following [16, 36], for each $\omega \in \mathbf{\Omega}$, we can find a ball $U_\omega = B(\omega, r_\omega)$ with sufficiently small radius such that on $B(\omega, r_\omega)$ there exists a unique holomorphic inverse branch T_ω^{-1} of T such that $T_\omega^{-1}(\omega) = \omega$. For a parabolic point $\omega \in \mathbf{\Omega}$, the Taylor series of T about ω is of the form

$$z + a(z - \omega)^{p(\omega)+1} + \cdots .$$

We call $p(\omega)$ the *petal number* of ω, and we write

$$p_{\min} = \min\{p(\omega) \mid \omega \in \mathbf{\Omega}\},$$
$$p_{\max} = \max\{p(\omega) \mid \omega \in \mathbf{\Omega}\}.$$

It was proven in [1] that $h > p_{\max}/(1 + p_{\max})$.

3. A new perspective on the Sullivan dictionary

3.1. Recent results on Assouad type dimensions and spectra. In this section we state various recent results concerning geometrically finite Kleinian groups with parabolic elements and parabolic Julia sets. These results provide a new perspective on the Sullivan dictionary in the context of dimension theory. We will examine this new perspective more thoroughly in Sections 3.2 and 3.3. The Assouad and lower dimensions of limit sets of geometrically finite Kleinian groups and associated Patterson–Sullivan measures were found in [19]. The analogous results for parabolic Julia sets were proved in [21]. The results concerning Assouad type spectra were proved in [21, 22]. Throughout we fix $\theta \in (0,1)$.

3.1.1. *Patterson–Sullivan measure μ.*

$$\dim_{\mathrm{A}}\mu = \max\{2\delta - k_{\min}, k_{\max}\}$$
$$\dim_{\mathrm{B}}\mu = \max\{2\delta - k_{\min}, \delta\}$$
$$\dim_{\mathrm{L}}\mu = \min\{2\delta - k_{\max}, k_{\min}\}$$
$$\dim_{\mathrm{A}}^{\theta}\mu = \begin{cases} \delta + \min\left\{1, \frac{\theta}{1-\theta}\right\}(k_{\max} - \delta) & \delta < k_{\min} \\ 2\delta - k_{\min} + \min\left\{1, \frac{\theta}{1-\theta}\right\}(k_{\min} + k_{\max} - 2\delta) & k_{\min} \leqslant \delta < (k_{\min} + k_{\max})/2 \\ 2\delta - k_{\min} & \delta \geqslant (k_{\min} + k_{\max})/2 \end{cases}$$
$$\dim_{\mathrm{L}}^{\theta}\mu = \begin{cases} 2\delta - k_{\max} & \delta \leqslant (k_{\min} + k_{\max})/2 \\ 2\delta - k_{\max} - \min\left\{1, \frac{\theta}{1-\theta}\right\}(2\delta - k_{\min} - k_{\max}) & (k_{\min} + k_{\max})/2 < \delta \leqslant k_{\max} \\ \delta - \min\left\{1, \frac{\theta}{1-\theta}\right\}(\delta - k_{\min}) & \delta > k_{\max} \end{cases}$$

3.1.2. *Kleinian limit sets $L(\Gamma)$.*

$$\dim_{\mathrm{A}} L(\Gamma) = \max\{\delta, k_{\max}\}$$
$$\dim_{\mathrm{L}} L(\Gamma) = \min\{\delta, k_{\min}\}$$
$$\dim_{\mathrm{A}}^{\theta} L(\Gamma) = \begin{cases} \delta + \min\left\{1, \frac{\theta}{1-\theta}\right\}(k_{\max} - \delta) & \delta < k_{\max} \\ \delta & \delta \geqslant k_{\max} \end{cases}$$
$$\dim_{\mathrm{L}}^{\theta} L(\Gamma) = \begin{cases} \delta & \delta \leqslant k_{\min} \\ \delta - \min\left\{1, \frac{\theta}{1-\theta}\right\}(\delta - k_{\min}) & \delta > k_{\min} \end{cases}$$

3.1.3. *h-conformal measures m.*

$$\dim_{\mathrm{A}} m = \max\{1, h + (h-1)p_{\max}\}$$
$$\dim_{\mathrm{B}} m = \max\{h, h + (h-1)p_{\max}\}$$
$$\dim_{\mathrm{L}} m = \min\{1, h + (h-1)p_{\max}\}$$
$$\dim_{\mathrm{A}}^{\theta} m = \begin{cases} h + \min\left\{1, \frac{\theta\, p_{\max}}{1-\theta}\right\}(1-h) & h < 1 \\ h + (h-1)p_{\max} & h \geqslant 1 \end{cases}$$
$$\dim_{\mathrm{L}}^{\theta} m = \begin{cases} h + (h-1)p_{\max} & h < 1 \\ h + \min\left\{1, \frac{\theta\, p_{\max}}{1-\theta}\right\}(1-h) & h \geqslant 1 \end{cases}$$

3.1.4. *Julia sets $J(T)$.*

$$\begin{aligned}
\dim_{\mathrm{A}} J(T) &= \max\{1, h\}\\
\dim_{\mathrm{L}} J(T) &= \min\{1, h\}\\
\dim_{\mathrm{A}}^{\theta} J(T) &= \begin{cases} h + \min\left\{1, \frac{\theta\, p_{\max}}{1-\theta}\right\}(1-h) & h < 1\\ h & h \geqslant 1\end{cases}\\
\dim_{\mathrm{L}}^{\theta} J(T) &= \begin{cases} h & h < 1\\ h + \min\left\{1, \frac{\theta\, p_{\max}}{1-\theta}\right\}(1-h) & h \geqslant 1\end{cases}
\end{aligned}$$

3.2. **New entries in the Sullivan dictionary.** Given the array of results in Section 3.1, it is clear that there are some parallels between the Kleinian and Julia settings akin to the Sullivan dictionary. Here we take a closer look at some of these parallels.

(1) *Interpolation between dimensions.* In both settings, the Assouad spectrum always interpolates between the upper box and Assouad dimensions of the respective sets and measures regardless of what form it takes, that is, $\lim_{\theta\to 1} \dim_{\mathrm{A}}^{\theta} F = \dim_{\mathrm{A}} F$, where F can be replaced by μ, $L(\Gamma)$, m, or $J(T)$. Recall that this interpolation does not hold in general. Similar interpolation holds as $\theta \to 1$ for the lower dimensions and spectra.

(2) *Failure to witness the box dimension of measures.* For the measures μ and m, the lower spectrum does not generally tend to the box dimension as $\theta \to 0$. In fact, if the lower spectrum does tend to the box dimension as $\theta \to 0$, then it is constant and $\delta = k_{\min} = k_{\max}$ (in the Kleinian setting) and $h = 1$ (in the Julia setting).

(3) *General form of the spectra.* For F a given set or measure, consider

$$\rho = \inf\{\theta \in (0,1) \mid \dim_{\mathrm{A}}^{\theta} F = \dim_{\mathrm{A}} F\}.$$

This quantity will be referred to as the *phase transition parameter*. Following some algebraic manipulation, we find that, in the cases where the Assouad spectrum is not constantly equal to the Assouad dimension,

$$\dim_{\mathrm{A}}^{\theta} F = \min\left\{\dim_{\mathrm{B}} F + \frac{(1-\rho)\theta}{(1-\theta)\rho}(\dim_{\mathrm{A}} F - \dim_{\mathrm{B}} F), \dim_{\mathrm{A}} F\right\}, \tag{3.1}$$

where F can be replaced by μ, $L(\Gamma)$, m, or $J(T)$. This formula, and the fact that the Assouad spectrum can be expressed purely in terms of the phase transition ρ together with the box and Assouad dimensions has appeared in a variety of settings; see [20, Section 17.7] and the discussion therein. For example, this formula also holds for self-affine Bedford–McMullen carpets. The phase transition ρ often has a natural *geometric significance* for the objects involved and opens the door to a new "dictionary" extending beyond the setting discussed here. It is worth noting that (3.1) does not hold generally, even failing for simple examples, such as the elliptical spirals considered in [10].

(4) *The phase transition and the Hausdorff dimension bound.* There is a correspondence between the phase transition ρ and the general lower bounds for the Hausdorff dimension. Applying (2.1) shows that, for any nonempty bounded set F, $\rho \geqslant 1 - \overline{\dim}_{\mathrm{B}} F / \dim_{\mathrm{A}} F$. When the spectra are nonconstant, in the Kleinian setting we always have $\rho = 1/2$, and in the Julia setting we always have $\rho = 1/(1+p_{\max})$.

Combining this with the general Hausdorff dimension bounds $\delta > k_{\max}/2 = k_{\max}\rho$ and $h > p_{\max}/(1+p_{\max}) = p_{\max}\rho$ in both settings yields $\rho > 1 - \overline{\dim}_{\mathrm{B}} F/\dim_{\mathrm{A}} F$, showing that the upper bound from (2.1) is never achieved in either setting (but is sharp in the sense that examples can be constructed with Assouad spectrum arbitrarily close to the upper bound from (2.1)).

(5) *The realisation problem.* Given the interplay between dimensions of sets and dimensions of measures seen in Section 2.1, one may ask if it is possible to construct an (invariant or quasi-invariant) measure ν which *realises* the dimensions of an (invariant) set F, that is, $\dim \nu = \dim F$. One can ask this about a particular choice of dimension dim or if a single measure can be constructed to solve the problem for several notions of dimension simultaneously. We note that the measures μ and m *always* realise the Hausdorff dimensions of $L(\Gamma)$ and $J(T)$, respectively. As for the Assouad and lower dimensions, μ realises the Assouad dimension of $L(\Gamma)$ when $\delta \leqslant (k_{\min}+k_{\max})/2$ and realises the lower dimension when $\delta \geqslant (k_{\min}+k_{\max})/2$. Similarly, for m to realise the Assouad dimension of $J(T)$, we require $h \leqslant 1$, and for m to realise the lower dimension of $J(T)$, we require $h \geqslant 1$. A similar relationship holds for the box dimension too: in the Kleinian setting we require $\delta \leqslant k_{\min}$ and in the Julia setting we require $h \leqslant 1$.

(6) *A special case.* Finally, we observe that in the (very) special case $k_{\min} = k_{\max} = p_{\max} = 1$, the formulae for the Assouad type dimensions and spectra are identical in the Kleinian and Julia settings. Does this suggest that this special case is one where we can expect the Sullivan dictionary to yield a particularly strong correspondence in other settings?

3.3. **New nonentries in the Sullivan dictionary.** Here we discuss some notable differences between the Kleinian and Julia settings. These are especially interesting to us since the Sullivan dictionary previously provided a very strong parallel in the context of dimension theory.

(1) *Assouad dimension.* Our results show that Julia sets of parabolic rational maps can never have full Assouad dimension, that is, we always have $\dim_{\mathrm{A}} J(T) < 2$. This uses our result together with [1, Theorem 8.8] which proves that $h < 2$. This is in stark contrast to the situation for Kleinian limit sets where it is perfectly possible for the Assouad dimension to be full, that is, $\Gamma \in \mathrm{Con}(d)$ with $\dim_{\mathrm{A}} L(\Gamma) = d = \dim \mathbb{S}^d$ for any integer $d \geqslant 1$. This can even happen when the limit set is nowhere dense (that is, when $\delta < d$; see [40, Theorem D]). We note that $\dim_{\mathrm{A}} J(T) < 2$ also follows from [24, Theorem 1.4], where it was proved that parabolic Julia sets are porous, together with [26, Theorem 5.2], which shows that porous sets in $\mathbb{R}^d$ must have Assouad dimension strictly less than d. Our results can thus be viewed as a refinement of the observation that parabolic Julia sets are porous.

We note that Julia sets of general rational maps need not be porous, and may even have positive area (even withtin the quadratic family); see [3, 9]. We proved in [21] that Julia sets with Cremer fixed points have Assouad dimension 2 (and are therefore not porous).

(2) *Lower dimension.* Our results, together with the standard bound $h > p_{\max}/(1+p_{\max})$, show that $\dim_{\mathrm{L}} J(T) = \min\{1, h\} > p_{\max}/(1+p_{\max})$, that is, the lower dimension respects the general lower bound satisfied by the Hausdorff dimension. Again, this is in stark contrast to the situation for Kleinian

TABLE 2. Summarising the possible relationships between the lower, Hausdorff, and Assouad dimensions of geometrically finite Fuchsian limit sets, geometrically finite Kleinian limits sets, and parabolic Julia sets with the obvious labelling. The symbol ✓ means that the configuration is possible, and × means the configuration is impossible. In other situations it is interesting to add box dimension into this discussion, but here this always coincides with Hausdorff dimension and so we omit it.

Configuration	Fuchsian	Kleinian	Julia
$\mathrm{L} = \mathrm{H} = \mathrm{A}$	✓	✓	✓
$\mathrm{L} = \mathrm{H} < \mathrm{A}$	✓	✓	✓
$\mathrm{L} < \mathrm{H} = \mathrm{A}$	×	✓	✓
$\mathrm{L} < \mathrm{H} < \mathrm{A}$	×	✓	×

limit sets where the standard bound for Hausdorff dimension is $\delta > k_{\max}/2$ but $\dim_{\mathrm{L}} L(\Gamma) = \min\{k_{\min}, \delta\} \leqslant k_{\max}/2$ is possible, even in the $d = 2$ case.

(3) *Relationships between dimensions.* An interesting aspect of dimension theory is to consider what configurations are possible between the different notions of dimension in a particular setting. We refer the reader to [20, Section 17.5] for a more general discussion of this. Our results show that

$$\dim_{\mathrm{L}} J(T) < \dim_{\mathrm{H}} J(T) < \dim_{\mathrm{A}} J(T)$$

is impossible in the Julia setting but the analogous configuration *is* possible in the Kleinian setting, even in the $d = 2$ case; see Table 2.

(4) *Form of the spectra.* Turning our attention to measures, the Assouad and lower spectra of μ in the Kleinian setting can take three different forms, in comparison to the Julia setting where we only have two possibilities for m. Furthermore, in the Kleinian setting, both $k_{\min}$ and $k_{\max}$ appear in the formulae for the Assouad and lower spectra, sometimes simultaneously, but in the Julia setting only $p_{\max}$ appears.

(5) *The realisation problem for dimension spectra.* One can also extend the realisation problem to the Assouad and lower spectra: when does an (invariant) set support an (invariant or quasi-invariant) measure with equal Assouad or lower spectra? In the Kleinian setting, we have $\dim_{\mathrm{A}}^{\theta} \mu = \dim_{\mathrm{A}}^{\theta} L(\Gamma)$ when $\delta \leqslant k_{\min}$ and $\dim_{\mathrm{L}}^{\theta} \mu = \dim_{\mathrm{L}}^{\theta} L(\Gamma)$ when $\delta \geqslant k_{\max}$. This can leave a gap when $k_{\min} < \delta < k_{\max}$ where neither of the spectra are realised by the Patterson–Sullivan measure. This is in contrast to the Julia setting, where $\dim_{\mathrm{A}}^{\theta} m = \dim_{\mathrm{A}}^{\theta} J(T)$ when $h \leqslant 1$ and $\dim_{\mathrm{L}}^{\theta} m = \dim_{\mathrm{L}}^{\theta} J(T)$ when $h \geqslant 1$, and so at least one of the spectra is always realised by m.

(6) *Dimension spectra as a fingerprint.* Suppose it is *not* true that $k_{\min} = k_{\max} = p_{\max} = 1$. Then simply by looking at plots of the Assouad and lower spectra, one can determine whether the set in question is a Kleinian limit set or a Julia set. Whenever the Assouad spectrum is nonconstant in either the Kleinian or Julia setting, there is a unique phase transition at

$$\rho = \inf\{\theta \in (0, 1) \mid \dim_{\mathrm{A}}^{\theta} F = \dim_{\mathrm{A}} F\}.$$

However, $\rho = 1/2$ in the Kleinian setting and $\rho = 1/(1 + p_{\max})$ in the Julia setting. Note that in the Kleinian setting the phase transition is constant across all Kleinian limit sets, whereas in the Julia setting the phase transition depends on the rational map T. This allows one to distinguish between the Assouad spectrum of a Kleinian limit set and a Julia set just by looking at the phase transition, provided $p_{\max} \neq 1$. However, even if $p_{\max} = 1$, the spectra will still distinguish between the two settings provided we do not also have $k_{\min} = k_{\max} = 1$.

3.4. **Examples.** We plot the Assouad and lower spectra for some examples. In the Kleinian setting, we assume that $d = 2$ throughout for a more direct comparison with the Julia setting, and we plot the following cases: $\delta < k_{\min}$, $\delta > k_{\max}$, and $k_{\min} < \delta < k_{\max}$. In the Julia setting, we plot examples with $h < 1$ and $h > 1$. Figures 2, 3, and 4 are plots of the Assouad and lower spectra as functions of $\theta \in (0, 1)$. The spectra of μ and m are plotted with dashed lines, and the spectra of $L(\Gamma)$ and $J(T)$ by solid lines. The Assouad spectra are plotted in black and the lower spectra are plotted in grey.

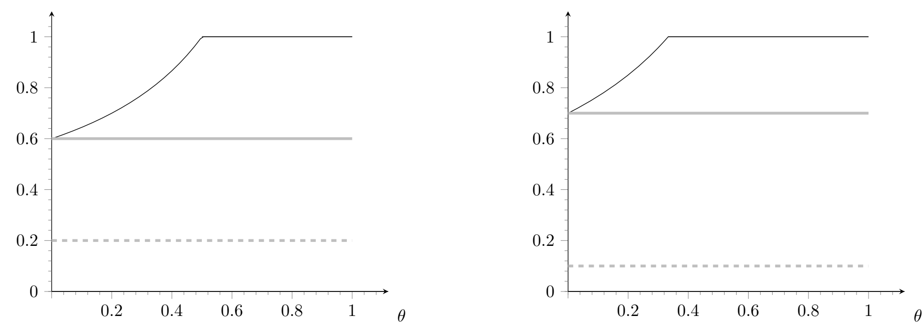

FIGURE 2. Left: a Kleinian limit set with $\delta = 0.6$ and $k_{\min} = k_{\max} = 1$. Right: a Julia set with $h = 0.7$ and $p_{\max} = 2$.

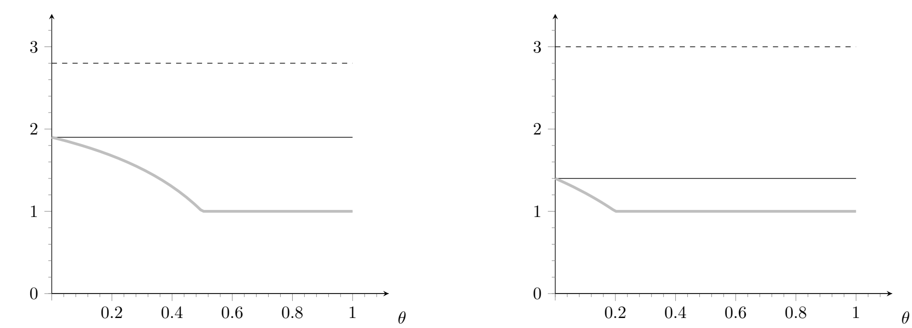

FIGURE 3. Left: a Kleinian limit set with $\delta = 1.9$ and $k_{\min} = k_{\max} = 1$. Right: a Julia set with $h = 1.4$ and $p_{\max} = 4$.

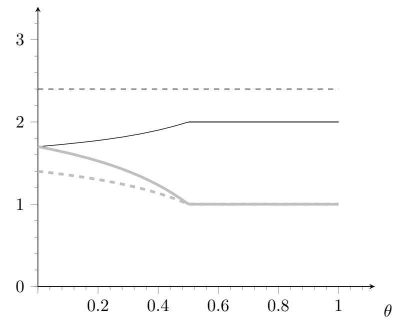

FIGURE 4. A Kleinian limit set with $\delta = 1.7$, $k_{\min} = 1$ and $k_{\max} = 2$. In the Julia setting we always have either $\dim_{\mathrm{A}}^{\theta} m = \dim_{\mathrm{A}}^{\theta} J(T)$ or $\dim_{\mathrm{L}}^{\theta} m = \dim_{\mathrm{L}}^{\theta} J(T)$, and so plots of this form are impossible in the Julia setting.

4. Future directions

This paper has focused on discussing geometrically finite Kleinian groups and parabolic (or hyperbolic) rational maps. We now have a fairly complete dimensional description of the Sullivan dictionary in these settings, at least from the point of view of the notions of dimension we discuss here. It would be interesting to move beyond these two settings, and many open questions remain.

In the Kleinian setting some geometrically infinite examples were discussed in [19] which demonstrate that the situation can be fairly wild. That said, one may consider certain classes, such as geometrically infinite but finitely generated Kleinian groups. Here it is not known if the box and Hausdorff dimensions of the limit set necessarily agree. One can ask how the Assouad dimension (and spectra) fit into this story. For example, is it true that, for a finitely generated Kleinian group, the Assouad dimension (and Assouad spectrum) can be characterised by parabolic points and a critical exponent? One could also move to nonproper settings and groups acting on infinite dimensional spaces; see [13].

In the rational maps setting even more questions arise. Most concretely, one can try to derive formulae for the dimensions and dimension spectra we consider here for rational maps whose Julia set contains critical points. One might expect most of the theory to go through in the nonrecurrent case, that is, when the presence of critical points is not so influential, but it is harder to predict what happens in the presence of recurrent critical points. In [21] we proved that the Assouad dimension of a Julia set with a Cremer fixed point is 2. It is a well-known (and very difficult) open problem to determine if the Hausdorff dimension of such Julia sets is also 2. An intermediate (and also open) question is to determine if the box dimension is 2. Since we are now equipped with a family of dimensions interpolating between the box and Assouad dimensions, we are led to a (continuous) hierarchy of open questions. For example, can it be shown that the Assouad spectrum approaches 2 as $\theta \to 1$, or perhaps equals 2 for some $\theta \in (0,1)$? Another promising direction is to consider transcendental dynamics. Here the dimension theory is often quite different from the rational maps case, and there appears to be very little known about the Assouad type dimensions.

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