BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 61, Number 1, January 2024, Pages 73–102
https://doi.org/10.1090/bull/1803
Article electronically published on July 31, 2023

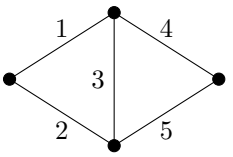
ESSENCE OF INDEPENDENCE: HODGE THEORY OF MATROIDS SINCE JUNE HUH

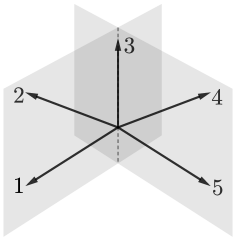
CHRISTOPHER EUR

ABSTRACT. Matroids are combinatorial abstractions of independence, a ubiquitous notion that pervades many branches of mathematics. June Huh and his collaborators recently made spectacular breakthroughs by developing a Hodge theory of matroids that resolved several long-standing conjectures in matroid theory. We survey the main results in this development and ideas behind them.

1. INTRODUCTION

The notion of "independence" resides everywhere, for example in graphs, vector configurations, field extensions, hyperplane arrangements, matchings, and discrete optimizations. Matroid theory captures the combinatorial essence of independence shared in these structures. For example, let us consider G (in Figure 1) with edges labeled $\{1, \dots, 5\}$ and the set of vectors $\{v_1, \dots, v_5\}$.





$$\begin{matrix} v_1 & v_2 & v_3 & v_4 & v_5 \end{matrix}$$
$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

FIGURE 1

We observe a common combinatorial structure: a subset of edges in G is acyclic if and only if the corresponding subset of vectors is linearly independent. This combinatorial structure is encoded as a matroid, introduced by Whitney [Whi32].

Definition 1.1. A **matroid** $\mathrm{M} = (E, \mathcal{I})$ consists of a finite set $E = \{1, \dots, n\}$, called its **ground set**, and a nonempty collection $\mathcal{I}$ of subsets of E, called the **independent sets** of M, such that

- if $I \in \mathcal{I}$ and $J \subseteq I$, then $J \in \mathcal{I}$, and
- if $I, J \in \mathcal{I}$ and $|I| < |J|$, then there exists an element $j \in J \setminus I$ such that $I \cup \{j\} \in \mathcal{I}$.

The definition implies that every maximal independent set of M has the same cardinality r, which we call the **rank** of M.

Received by the editors May 12, 2023.
2020 *Mathematics Subject Classification.* 05B35, 05E14, 14F43, 14C17.

Graphs and vector spaces give prototypical examples of matroids:

- When E is identified with the set of edges of a finite graph G, setting
$$\mathcal{I} = \{I \subseteq E : \text{the subset } I \text{ of edges in } G \text{ is acyclic}\}$$
defines a matroid $\mathrm{M} = (E, \mathcal{I})$. Matroids arising in this way are called **graphical matroids**.
- When E is identified with a finite set of vectors spanning a vector space V, setting
$$\mathcal{I} = \{I \subseteq E : \text{the subset } I \text{ of vectors in } V \text{ is linearly independent}\}$$
defines a matroid $\mathrm{M} = (E, \mathcal{I})$. Matroids arising in this way are called **realizable matroids**.

We see that the graph G and the set of vectors in Figure 1 define the same matroid.

1.1. **Combinatorial sequences from a matroid.** Several long-standing conjectures in matroid theory, recently resolved by June Huh and his collaborators, concern the behavior of sequences of invariants of a matroid. For a sequence $(a_0, a_1, \ldots, a_m)$ of nonnegative real numbers, we say that it

- is *unimodal* if there exists $0 \leq k \leq m$ such that
$$a_0 \leq a_1 \leq \cdots \leq a_k \geq a_{k+1} \geq \cdots \geq a_m;$$
- is *log-concave* if $a_i^2 \geq a_{i-1}a_{i+1}$ for all $1 \leq i \leq m-1$;
- has *no internal zeros* if $a_i a_j \neq 0$ implies $a_k \neq 0$ for all $0 \leq i \leq k \leq j \leq 0$; and
- is *top-heavy* if $a_i \leq a_{d-i}$ for all $0 \leq i \leq \frac{d}{2}$, where d is the largest index such that $a_d \neq 0$.

Note that a log-concave sequence is unimodal if and only if it has no internal zeroes. For a survey of unimodality and log-concavity in combinatorics, see [Sta89, Bre94].

We consider the following sequences of invariants of a rank r matroid M. We describe some of them only for a graphical or a realizable matroid, postponing their descriptions for arbitrary matroids to Section 2.

(a) For $0 \leq i \leq r$, let I_i be the number of independent sets of M of cardinality i. In other words, the sequence $(I_0, \ldots, I_r)$ is the f-vector of the simplicial complex whose faces are the independent sets of M.

(b) We may also consider the h-vector. That is, for $0 \leq i \leq r$, let I'_i be defined by the identity $\sum_{i=0}^{r} I'_i q^{r-i} = \sum_{i=0}^{r} I_i (q-1)^{r-i}$, where I_i is as in (a).

(c) Suppose M is the graphical matroid of a finite connected nontrivial graph G. The **chromatic polynomial** $\chi_G(q)$ of G is defined as
$$\chi_G(q) = \text{the number of proper colorings of } G \text{ with at most } q \text{ colors},$$
where a coloring of the vertices is *proper* if no edge has both vertices the same color. It is polynomial in q of degree $r+1$ and is always divisible by $q(q-1)$. Let $(w_0, \ldots, w_r)$ be the absolute values of the coefficients of $\frac{1}{q}\chi_G(q)$, starting from the highest degree term.

(d) Continuing the assumption that M is the graphical matroid of G, we define $(w'_0, \ldots, w'_{r-1})$ as the absolute values of the coefficients of $\frac{1}{q(q+1)}\chi_G(q+1)$.

(e) Suppose M is the realizable matroid of a set of vectors $\{v_1, \ldots, v_n\}$ spanning a vector space V. Let $(W_0, \ldots, W_r)$ be a sequence defined by setting for each $0 \leq i \leq r$,

$$W_i = \text{the number of } i\text{-dimensional linear subspaces } V' \text{ in } V \text{ such that } V' \text{ is the span of a subset of the vectors } \{v_1, \ldots, v_n\}.$$

We leave it as an exercise to check that for the matroid associated to the graph or the vector configuration in Figure 1, we have

$$(I_0, I_1, I_2, I_3) = (1, 5, 10, 8), \qquad (w_0, w_1, w_2, w_3) = (1, 5, 8, 4),$$
$$(I'_0, I'_1, I'_2, I'_3) = (1, 2, 3, 2), \qquad (w'_0, w'_1, w'_2) = (1, 2, 1),$$
$$(W_0, W_1, W_2, W_3) = (1, 5, 6, 1).$$

Notice in this example that every sequence is unimodal, log-concave, and top-heavy. Several conjectures from the 1970s posited that these sequences are unimodal, log-concave, or top-heavy for an arbitrary matroid. We describe these conjectures and their history more fully in Section 2.2.

1.2. **An approach from algebraic geometry.** After decades of little progress, a breakthrough occurred when many of these conjectures were resolved for realizable matroids using algebraic geometry: Huh and Katz [Huh12, HK12] showed that the sequence (c) (from Section 1.1) is log-concave (with no internal zeros); Huh [Huh15] showed that (d) is log-concave (with no internal zeros); and Huh and Wang [HW17] showed that (e) is top-heavy. These developments were particularly significant in light of the following phenomena in matroid theory.

The geometry of realizable matroids often inspires purely combinatorial constructions for all matroids. Certain geometric properties, a priori applicable only to realizable matroids, persist to all matroids through these purely combinatorial constructions. This is surprising because almost all matroids are not realizable [Nel18], but such a creative tension between geometry and combinatorics is a recurring theme in matroid theory.

A recent spectacular example of this phenomenon is the development of the Hodge theory of matroids by June Huh and his collaborators [AHK18, ADH22, BHM$^+$22, BHM$^+$], which successfully resolved conjectures about log-concavity or top-heaviness of the sequences (a)–(e) for arbitrary (not necessarily realizable) matroids. They established that matroids satisfy combinatorial analogues of certain Hodge-theoretic properties in algebraic geometry, known sometimes as the *Kähler package*:

Definition 1.2. Let $A^\bullet = \bigoplus_{i=0}^d A^i$ be a finite-dimensional graded real vector space with a symmetric bilinear form $P : A^\bullet \times A^{d-\bullet} \to \mathbb{R}$, and let $\mathcal{K}$ be a convex subset of graded linear operators $L : A^\bullet \to A^{\bullet+1}$ satisfying $P(Lx, y) = P(x, Ly)$ for all $x, y \in A^\bullet$. The triple $(A^\bullet, P, \mathcal{K})$ is said to satisfy the **Kähler package** if the following three properties hold for all nonnegative integers $i \leq \frac{d}{2}$:

(PD) *Poincaré duality.* The pairing $P : A^i \times A^{d-i} \to \mathbb{R}$ is nondegenerate.

(HL) *Hard Lefschetz property in degree i.* For any $L_1, \ldots, L_{d-2i} \in \mathcal{K}$, the linear map

$$A^i \to A^{d-i} \quad \text{given by} \quad x \mapsto L_1 \cdots L_{d-2i} x$$

is an isomorphism.

(HR) *Hodge–Riemann relations in degree i.* For any $L_0, L_1, \ldots, L_{d-2i} \in \mathcal{K}$, the symmetric bilinear pairing

$$A^i \times A^i \to \mathbb{R} \quad \text{given by} \quad (x, y) \mapsto (-1)^i P(x, L_1 \cdots L_{d-2i} y)$$

is positive definite when restricted to the kernel of the map $A^i \to A^{d-i+1}$ given by $x \mapsto L_0 L_1 \cdots L_{d-2i} x$.

Classical Hodge theory tells us that these properties are satisfied when $A^\bullet$ is the cohomology ring of real (p,p)-forms on a complex projective manifold, P is the Poincaré duality pairing, and $\mathcal{K}$ consists of multiplication by ample divisor classes; see [Huy05] or [Voi02].

The geometry behind realizable matroids led to purely combinatorial constructions for various *cohomologies* of a matroid. These constructions include the Chow ring of a matroid [FY04, AHK18], the conormal Chow ring of a matroid [ADH22], and the intersection cohomology of a matroid [BHM$^+$]. For a matroid realizable over $\mathbb{C}$, all three satisfy the Kähler package due to classical algebraic geometry. The incredible result of June Huh and his collaborators—Karim Adiprasito, Federico Ardila, Tom Braden, Graham Denham, Eric Katz, Jacob Matherne, Nick Proudfoot, Botong Wang—is that the Kähler package continues to hold for these cohomologies of a matroid even when the matroid is not realizable.

We survey this remarkable development in matroid theory and its connection to algebraic geometry in four parts. In Section 2, we give a brief introduction to matroids, and describe the long-standing conjectures resolved by the Hodge theory of matroids. In Section 3, we explain how the conjectures in the case of realizable matroids were resolved using algebraic geometry. In Section 4, we discuss the Kähler package for Chow rings of fans and matroids and how the validity of (HR) implies the conjectures about log-concavity. In Section 5, we discuss the intersection cohomology of a matroid and explain its implication to top-heaviness.

Several interesting topics had to be omitted, even though they are closely related to the topics discussed here. A partial list includes the following:

- The Kazhdan–Lusztig theory of matroids [EPW16], which was an inspiration behind the construction of intersection cohomology of a matroid. We point to [Pro18] for a survey of Kazhdan–Lusztig–Stanley polynomials in a more general context.
- The study of matroids through the polyhedral geometry of their base polytopes, their subdivisions, and the geometry of the Grassmannian [GGMS87, Laf03]. We point to the survey [Ard22, Section 4] and references therein.
- Other approaches to the cohomology of a matroid in the broader context of tropical geometry, for instance [IKMZ19, AP].

We hope that this survey will spark the reader's general interest in this active field of the study of matroids from an algebro-geometric perspective.

Notation 1.3. Throughout, let $E = \{1, \ldots, n\}$ be a finite set of cardinality n. For a subset $S \subseteq E$, we denote by $\mathbf{e}_S = \sum_{i \in S} \mathbf{e}_i$ the sum of standard basis vectors in $\Bbbk^E$, where the field $\Bbbk$ will be clear in context. An algebraic variety is reduced and irreducible (over an algebraically closed field).

2. Background in matroid theory

Here we give a minimal introduction to matroids. In addition to standard references on matroids, such as [Wel76, Oxl11], we point to [Ard22, Bak18, Huh18, Kat16] for surveys tailored towards studying matroids from an algebro-geometric viewpoint.

2.1. **Constructions.** Since subsets of independent sets are independent, we may specify a matroid by its maximal independent sets, called the **bases** of the matroid.

Example 2.1. For an integer $0 \leq r \leq n$, the **uniform matroid** of rank r on E is the matroid $\mathrm{U}_{r,n}$ whose bases are all subsets of cardinality r. When $n = r$, we say that $\mathrm{U}_{n,n}$ is the **Boolean matroid** on E. The Boolean matroid $\mathrm{U}_{0,0}$, i.e., when $E = \varnothing$ so $n = r = 0$, is called the **trivial matroid**. Any uniform matroid $\mathrm{U}_{r,n}$ is realizable over any infinite field $\Bbbk$ as a general collection of n vectors in $V = \Bbbk^r$.

Example 2.2. We may visualize a collection of vectors in a four-dimensional vector space as a collection of points in the projective 3-space $\mathbb{P}^3$. For example, the five column vectors of the matrix shown in Figure 2

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

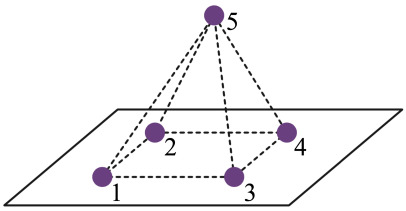


Figure 2

can be visualized as the purple points, four of which lie in a common projective plane. The bases of this matroid are $\{1235, 1245, 1345, 2345\}$.

We define the **dual matroid** $\mathrm{M}^\perp$ of a matroid M on ground set E by declaring

$$\text{the set of bases of } \mathrm{M}^\perp = \{E \setminus B : B \text{ a basis of } \mathrm{M}\}.$$

For example, we have $\mathrm{U}_{r,n}^\perp = \mathrm{U}_{n-r,n}$. For the matroid in Example 2.2, its dual has the set of bases $\{1, 2, 3, 4\}$. Many notions in matroid theory come in pairs via matroid duality. For instance, an element $e \in E$ is a **loop** of M if it is in no bases, and is a **coloop** if it is in every basis of M. The matroid in Example 2.2 has no loops and has a coloop 5, or equivalently, its dual matroid has no coloops and has a loop 5.

Another useful way of describing a matroid is by its rank function. For a matroid $\mathrm{M} = (E, \mathcal{I})$, its **rank function** $\mathrm{rk}_\mathrm{M} : 2^E \to \mathbb{Z}$ is defined by

$$\mathrm{rk}_\mathrm{M}(S) = \max\{|I| : I \in \mathcal{I} \text{ and } I \subseteq S\} \quad \text{for every subset } S \subseteq E.$$

In particular, an independent set of a matroid M is a subset $I \subseteq E$ whose rank $\mathrm{rk}_\mathrm{M}(I)$ equals its cardinality $|I|$. That is, independent sets are the minimal subsets of E with respect to a given rank. Considering the maximal subsets leads to the notion of flats of a matroid.

Definition 2.3. A **flat** of a matroid M on E is a subset of E that is maximal for its rank. That is, a subset $F \subseteq E$ is a flat of M if $\mathrm{rk}_{\mathrm{M}}(F \cup \{e\}) > \mathrm{rk}_{\mathrm{M}}(F)$ for all $e \in E \setminus F$.

The set of flats of a matroid M forms a poset under inclusion. This poset is a *lattice* with meet and join defined by

$$F \wedge F' = F \cap F' \quad \text{and} \quad F \vee F' = \text{the smallest flat containing } F \cup F'.$$

For example, the lattice of flats of the matroid in Example 2.2 is shown in Figure 3.

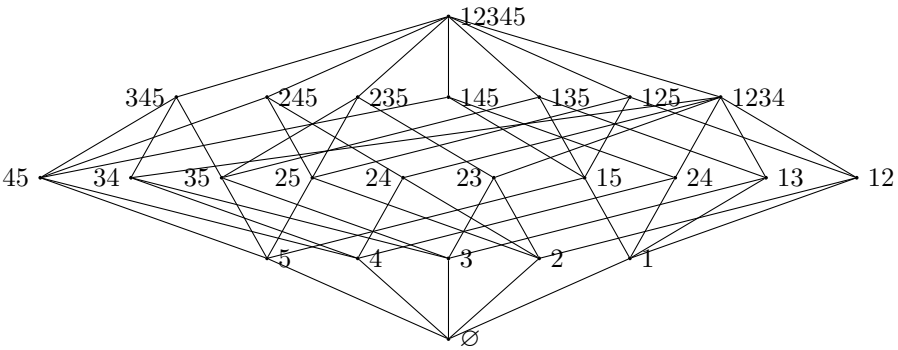


FIGURE 3

Exercise 2.4.

(1) Show that every subset is contained in a unique flat of the same rank. In particular, the join of flats is well defined.
(2) How can one recover the bases of a matroid from the lattice of its flats?
(3) Compute the lattice of flats of the matroid of the graph in Figure 1.

We record some linear algebraic interpretations of the notions introduced for a matroid M realized by a set of vectors $\{v_1, \ldots, v_n\}$ spanning a vector space V:

- The bases of M are subsets $B \subseteq E$ such that $\{v_i : i \in B\}$ is a basis of V.
- An element $e \in E$ is a loop if and only if $v_e = 0$.
- We have $\mathrm{rk}_{\mathrm{M}}(S) = \dim \mathrm{span}\{v_i : i \in S\}$ for any subset $S \subseteq E$.
- The flats of M are subsets $F \subseteq E$ such that $F = V' \cap \{v_1, \ldots, v_n\}$ for some linear subspace $V' \subseteq V$. That is, the flats correspond to the different spans of subsets of the vectors.

Per the last bullet point, we now define the sequence (e) for arbitrary matroids.

Definition 2.5. For a matroid M of rank r, the **Whitney numbers of the second kind** $(W_0, W_1, \ldots, W_r)$ are defined by

$$W_i = \text{the number of flats of M with rank } i.$$

Exercise 2.6. Show that if M is the graphical matroid of the complete graph on N vertices, then its flats of rank i correspond to partitions of N into $N-i$ (nonempty) parts. In particular, the numbers $(W_0, W_1, \ldots, W_{N-1})$ in this case are known as the Stirling numbers of the second kind.

Matroid duality also admits a linear algebraic interpretation, given in Exercise 2.7.

Exercise 2.7. Let $\Bbbk^E \twoheadrightarrow V$ be the map given by $\mathbf{e}_i \mapsto v_i$, and let K be its kernel. The short exact sequence $0 \to K \to \Bbbk^E \to V \to 0$ dualizes to $0 \to V^\vee \to \Bbbk^E \to K^\vee \to 0$. Show that the surjection $\Bbbk^E \twoheadrightarrow K^\vee$ realizes the dual matroid $\mathrm{M}^\perp$.

An important pair of operations for forming new matroids from a given matroid is the restriction and contraction.

Definition 2.8. For a matroid M on ground set E, and a subset $A \subseteq E$, we define two matroids $\mathrm{M}|A$ and M/A on ground sets A and $E \setminus A$, respectively, by specifying their rank functions:

$$\begin{aligned} \mathrm{rk}_{\mathrm{M}|A}(S) &= \mathrm{rk}_{\mathrm{M}}(S) && \text{for all } S \subseteq A; \\ \mathrm{rk}_{\mathrm{M}/A}(S) &= \mathrm{rk}_{\mathrm{M}}(S \cup A) - \mathrm{rk}_{\mathrm{M}}(A) && \text{for all } S \subseteq E \setminus A. \end{aligned}$$

The matroid $\mathrm{M}|A$ is called the **restriction** of M to A, and the matroid M/A is called the **contraction** of M by A. The **deletion** $\mathrm{M} \setminus A$ of M by A is the restriction $\mathrm{M}|(E \setminus A)$.

These operations behave particularly well for a flat F of a matroid M:

- the set of flats of $\mathrm{M}|F$ is $\{F' : F' \text{ a flat of M contained in } F\}$;
- the set of flats of M/F is $\{F' \setminus F : F' \text{ a flat of M containing } F\}$.

These operations have a graphical and linear algebraic interpretations as well: for the graphical matroid of a graph G, deletion corresponds to deleting the corresponding edges, and contraction corresponds to contracting the corresponding edges. When a matroid M is realized by a set of vectors $\{v_i : i \in E\}$, the restriction $\mathrm{M}|A$ is realized by the subset of vectors $\{v_i : i \in A\}$. The contraction M/A is realized by the images of the vectors $\{v_i : i \in E \setminus A\}$ under the quotient by the span of $\{v_i : i \in A\}$.

Exercise 2.9. Show that deletion and contraction are dual notions, that is, we have $(\mathrm{M} \setminus A)^\perp = \mathrm{M}^\perp/A$.

2.2. **Invariants.** Introduced for graphs by Tutte [Tut67] and extended to matroids by Crapo [Cra69], the Tutte polynomial is among the most famous invariants of a matroid. For the proofs of the statements here, as well as a fuller treatment of Tutte polynomials, see [BO92].

Definition 2.10. Tutte polynomial of a matroid M of rank r on ground set E is the bivariate polynomial defined by

$$\mathrm{T}_{\mathrm{M}}(x, y) = \sum_{S \subseteq E} (x-1)^{r - \mathrm{rk}_{\mathrm{M}}(S)} (y-1)^{|S| - \mathrm{rk}_{\mathrm{M}}(S)}.$$

The Tutte polynomial is the universal deletion-contraction invariant:

Theorem 2.11. *The Tutte polynomial can be defined recursively by*

$$\mathrm{T}_{\mathrm{M}}(x, y) = \begin{cases} x\mathrm{T}_{\mathrm{M}/e}(x, y) & \text{if } e \in E \text{ a coloop in M}, \\ y\mathrm{T}_{\mathrm{M}\setminus e}(x, y) & \text{if } e \in E \text{ a loop in M}, \\ \mathrm{T}_{\mathrm{M}\setminus e}(x, y) + \mathrm{T}_{\mathrm{M}/e}(x, y) & \text{if } e \in E \text{ neither loop nor coloop} \end{cases}$$

with $\mathrm{T}_{\mathrm{U}_{0,0}}(x,y)=1$. *If* f *is an invariant of matroids with values in a (commutative unital) ring* R *such that* $f(\mathrm{U}_{0,0})=1$ *and there exists* $x_0,y_0,a,b\in R$ *satisfying*

$$f(\mathrm{M})=\begin{cases} x_0 f(\mathrm{M}/e) & \text{if } e\in E \text{ a coloop in } \mathrm{M},\\ y_0 f(\mathrm{M}\setminus e) & \text{if } e\in E \text{ a loop in } \mathrm{M},\\ af(\mathrm{M}\setminus e)+bf(\mathrm{M}/e) & \text{if } e\in E \text{ neither loop nor coloop}\end{cases}$$

for all matroids M *and an element* e, *then we have*

$$f(\mathrm{M})=a^{|E|-\mathrm{rk}_{\mathrm{M}}(E)}b^{\mathrm{rk}_{\mathrm{M}}(E)}\mathrm{T}_{\mathrm{M}}\left(\frac{x_0}{b},\frac{y_0}{a}\right).$$

The theorem implies the following basic properties of the Tutte polynomial:

- The Tutte polynomial $T_{\mathrm{M}}(x,y)$ of a matroid M has nonnegative coefficients.
- The constant term of $T_{\mathrm{M}}(x,y)$ is zero unless M is the trivial matroid.
- For the dual matroid $\mathrm{M}^{\perp}$, we have $T_{\mathrm{M}^{\perp}}(x,y)=T_{\mathrm{M}}(y,x)$.

Univariate specializations of the Tutte polynomial lead to many interesting combinatorial sequences. For example, we deduce from the definition that for a matroid M of rank r,

$$T_{\mathrm{M}}(q+1,1)=\sum_{i=0}^{r} I_i q^{r-i},$$

where we recall from (a) that I_i is the number of independent sets of M of cardinality i. Consequently, we have that the sequence $(I'_0,\dots,I'_r)$ of (b) is given by

$$T_{\mathrm{M}}(q,1)=\sum_{i=0}^{r} I'_i q^{r-i}.$$

Another important specialization is the **characteristic polynomial** χ_{M} of a matroid M of rank r, defined as

$$\chi_{\mathrm{M}}(q)=(-1)^r T_{\mathrm{M}}(1-q,0).$$

The basic properties of the Tutte polynomial listed above imply that the coefficients of $\chi_{\mathrm{M}}(q)$ have alternating signs, and that $\chi_{\mathrm{M}}(q)$ is divisible by $(q-1)$ unless M is a trivial matroid. Thus, one often divides out $(q-1)$ to define the **reduced characteristic polynomial** $\overline{\chi}_{\mathrm{M}}(q)=\chi_{\mathrm{M}}(q)/(q-1)$. It follows from Theorem 2.11 that $\chi_{\mathrm{M}}=0$ if M has a loop.

The characteristic polynomial of a graphical matroid essentially equals the chromatic polynomial of the graph in the following way.

Exercise 2.12. Show that if M is the graphical matroid of a finite graph G, and G has c connected components, then $q^c\chi_{\mathrm{M}}(q)=\chi_G(q)$. (*Hint*: Appeal to Theorem 2.11 by showing that both satisfy an identical deletion-contraction relation).

We may now define the sequences (c) and (d), which we previously only defined for graphical matroids, for arbitrary nontrivial matroids: they are the coefficients of $T_{\mathrm{M}}(1+q,0)$ and $\frac{1}{q}T_{\mathrm{M}}(q,0)$, respectively.

Summarizing, we have the following sequences for a matroid M of rank r.

(a) The coefficients $(I_0,\dots,I_r)$ of $T_{\mathrm{M}}(q+1,1)=\sum_{i=0}^{r} I_i q^{r-i}$.
(b) The coefficients $(I'_0,\dots,I'_r)$ of $T_{\mathrm{M}}(q,1)=\sum_{i=0}^{r} I'_i q^{r-i}$.
(c) The coefficients $(w_0,\dots,w_r)$ of $T_{\mathrm{M}}(q+1,0)=\sum_{i=0}^{r} w_i q^{r-i}$.
(d) The coefficients $(w'_0,\dots,w'_{r-1})$ of $\frac{1}{q}T_{\mathrm{M}}(q,0)=\sum_{i=0}^{r} I_i q^{r-1-i}$.
(e) The Whitney numbers of the second kind $(W_0,\dots,W_r)$ of M.

Theorem 2.13 ([AHK18, ADH22, BHM$^+$]). *Let* M *be a matroid of rank* r.

(a) *The sequence* $(I_0, \dots, I_r)$ *is unimodal, log-concave, and top-heavy.*
(b) *The sequence* $(I'_0, \dots, I'_r)$ *is unimodal, log-concave, and top-heavy.*
(c) *The sequence* $(w_0, \dots, w_r)$ *is unimodal, log-concave, and top-heavy.*
(d) *The sequence* $(w'_0, \dots, w'_{r-1})$ *is unimodal, log-concave, and top-heavy.*
(e) *The sequence* $(W_0, \dots, W_r)$ *satisfies* $W_i \leq W_j$ *for all* $0 \leq i \leq j \leq r - i$. *In particular, it is top-heavy.*

The statements of the theorem were long-standing conjectures in matroid theory. The unimodality and log-concavity conjectures are due to Welsh [Wel71] and Mason [Mas72] for (a), Dawson [Daw84] for (b), Read [Rea68] and Hoggar [Hog74] for (c) of graphical matroids, Heron [Her72], Rota [Rot71], and Welsh [Wel76] for (c), and Brylawski [Bry82] for (d). Hibi [Hib92] and Swartz [Swa03] posed the top-heaviness of (b) and (d) (respectively).[1] Dowling and Wilson [DW74, DW75] conjectured the top-heaviness of (e), generalizing a theorem of de Bruijn and Erdős [dBE48] on point-line incidences in projective planes. There are two notable conjectures on (e) that remain open: Rota [Rot71] conjectured its unimodality, and Mason [Mas72] its log-concavity.

Remark 2.14. One may ask if there is a log-concavity statement for the whole Tutte polynomial of a matroid that explains the log-concavity of the four specializations in Theorem 2.13. This was achieved by Berget, Spink, Tseng, and the author [BEST] who showed that the 4-variable transformation

$$(x+y)^{-1}(y+z)^r(x+w)^{|E|-r}\mathrm{T}_{\mathrm{M}}\left(\frac{x+y}{y+z}, \frac{x+y}{x+w}\right)$$

of the Tutte polynomial of a matroid M satisfies a multivariate version of log-concavity. We note that without such a transformation, the coefficients of $\mathrm{T}_{\mathrm{M}}(x, y)$ can fail to be unimodal [Sch93].

In this survey, we will explain how the sequences (c) and (d) are shown to be log-concave with no internal zeros,[2] and how the sequence (e) is shown to be top-heavy. Since $\chi_{\mathrm{M}} = 0$ if M has a loop, and deleting loops of a matroid does not change the lattice of its flats, we assume the following:

Assumption 2.15. From now on, a matroid is loopless unless specified otherwise.

3. The realizable case

We explain how the statements in Theorem 2.13 can be deduced using algebraic geometry when the matroid in question is realizable. We assume familiarity with algebraic geometry; those who prefer purely combinatorial treatments may skip this section. For simplicity, we consider matroids realizable over $\mathbb{C}$. For other fields, one can run nearly identical arguments using the Chow cohomology ring [Ful98] or

[1] A slightly different terminology of *flawlessness* appears in [Hib92] and related works [Hib89, JKL18]. A nonnegative sequence $(a_0, \dots, a_m)$ is flawless if it is top-heavy and additionally satisfies $a_0 \leq \cdots \leq a_{\lfloor d/2 \rfloor}$, where d is the largest index such that $a_d \neq 0$. Note that unimodal and top-heavy sequences are flawless.

[2] As observed by Lenz [Len13], a result of Brylawski [Bry77, Theorem 4.2] implies that the statements for (a) and (b) follow from those for (c) and (d). The top-heaviness of (c) and (d) follows from their unimodality due to [JKL18, Theorem 1.2].

the ℓ-adic (intersection) cohomology in place of singular (intersection) cohomology $(I)H^\bullet(-)$ with rational coefficients.

Throughout this section, let M be a (nontrivial) matroid of rank r realized by a set of vectors $\{v_i : i \in E\}$ spanning a vector space $V \simeq \mathbb{C}^r$. The corresponding surjection $\mathbb{C}^E \twoheadrightarrow V$ dualizes to give an r-dimensional linear subspace $V^\vee \subseteq \mathbb{C}^E$.

Notation 3.1. Let us denote $L = V^\vee$ to avoid repeated use of the superscript $\vee$.

The set of independent sets of M can then be described also as the collection

$$\mathcal{I} = \{I \subseteq E : \text{the composition } L \hookrightarrow \Bbbk^E \twoheadrightarrow \Bbbk^I \text{ is surjective}\}.$$

We will often projectivize and work with $\mathbb{P}L \subseteq \mathbb{P}^{n-1}$.

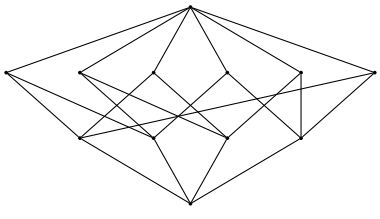
3.1. **Hyperplane arrangements.** We first discuss some structures of the matroid M in terms of its realization as a subspace $L \subseteq \mathbb{C}^E$. For each $i \in E$, let H_i be the ith coordinate hyperplane of $\mathbb{C}^E$. Our assumption that M is loopless implies that L is not contained in any H_i. We thus have a **hyperplane arrangement** $\mathcal{A}$ on L consisting of the hyperplanes $\{L \cap H_i : i \in E\}$. Dualizing the correspondence between the flats of M and the spans of subsets of the vectors $\{v_i : i \in E\}$, we obtain a correspondence

$$\{\text{flats of M}\} \longleftrightarrow \{\text{subspaces of } L \text{ arising as intersections of hyperplanes in } \mathcal{A}\}$$
$$F \longleftrightarrow L_F = L \cap \bigcap_{i \in F} H_i.$$

Note that the correspondence is order-reversing and, in particular, a flat of rank $r - i$ maps to the linear subspace L_F of dimension i.

Example 3.2. The columns of the matrix in Figure 4 realizes the matroid $\mathrm{U}_{3,4}$. Equivalently, the embedded subspace $L \subseteq \mathbb{C}^4$, where $L = \{x_1 + x_2 + x_3 + x_4 = 0\}$ is the row-span of the matrix, realizes the matroid $\mathrm{U}_{3,4}$.

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$



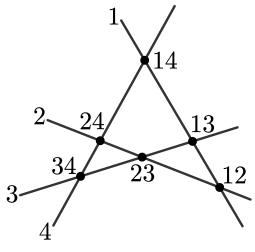


FIGURE 4

Next to the matrix, we have depicted the lattice of flats of M and the projectivization of the hyperplane arrangement $\mathcal{A}$ in $\mathbb{P}L \simeq \mathbb{P}^2$.

We denote the complement of the hyperplane arrangement by

$$\mathring{L} = L \setminus \bigcup \mathcal{A} = L \cap (\mathbb{C}^*)^E, \quad \text{and likewise,} \quad \mathbb{P}\mathring{L} = \mathbb{P}L \cap \big((\mathbb{C}^*)^E/\mathbb{C}^*\big).$$

Exercise 3.3. Suppose $i \in E$ is not a coloop, and let F be the smallest flat containing i. Show that the subspace L_F is a realization of the contraction M/F, and that the hyperplane arrangement complement $L \setminus \bigcup(\mathcal{A} \setminus \{L_F\})$ is a realization of the deletion $\mathrm{M} \setminus F$. What happens when i is a coloop?

The geometric study of (complements of) hyperplane arrangements and its interaction with matroid theory is a rich and on-going research field; some references include [OT92, Dim17]. Here, we only note the following classical fact [OS80], which states that the characteristic polynomial records the dimensions of the cohomologies of the arrangement complement.

Theorem 3.4. *Let $L \subseteq \mathbb{C}^E$ realize a matroid M of rank r. Then, we have*

$$\mathrm{T}_{\mathrm{M}}(1+q,0) = \sum_{i=0}^{r} \dim H^i(\mathring{L}) q^{r-i}$$

or, equivalently, by applying the Künnuth formula to $\mathbb{P}\mathring{L} \times \mathbb{C}^* \simeq \mathring{L}$,

$$\frac{1}{1+q}\mathrm{T}_{\mathrm{M}}(1+q,0) = \sum_{i=0}^{r-1} \dim H^i(\mathbb{P}\mathring{L}) q^{r-1-i}.$$

3.2. **Log-concavity via intersection degrees.** We now explain how the log-concavity of the sequences (c) and (d) can be shown using algebraic geometry. We start by describing a general strategy for log-concavity.

Let X be a smooth projective $\mathbb{C}$-variety X of dimension d. When considered as a $2d$-dimensional real compact manifold, Poincaré duality for the cohomology ring $H^\bullet(X)$ provides the isomorphism $\int_X : H^{2d}(X) \to \mathbb{Z}$, called the degree map. Recall that a divisor D on X is ample (resp., semi-ample) if the line bundle $\mathcal{O}_X(mD)$ is very ample (resp., globally generated) for some integer $m \gg 0$. The general strategy arises from the following Khovanskii–Teissier inequalities (see [Laz04a, Section 1.6] for a history and a fuller treatment).

Proposition 3.5. *Let $\alpha, \beta \in H^2(X)$ be the cohomology classes of two semi-ample (or more generally, nef) divisors on a smooth projective $\mathbb{C}$-variety X of dimension d. Then,*

the sequence $(a_0, \dots, a_d)$ of intersection degrees of α and β, i.e., $a_i = \displaystyle\int_X \alpha^{d-i}\beta^i$,

is log-concave with no internal zeros.

Sketch of the proof. By continuity, one can assume α, β to be ample, and then one reduces to the case of surfaces via the Bertini theorem. Then, one of the equivalent forms of the Hodge index theorem for surfaces [Har77, Exercise V.1.9] exactly yields the desired log-concavity. We note that the Hodge index theorem for surfaces stated as [Har77, Theorem V.1.9] is exactly the validity of the Hodge–Riemann relations for surfaces. □

Thus, starting from the realization $L \subseteq \mathbb{C}^E$, one may seek a smooth projective $\mathbb{C}$-variety equipped with two semi-ample divisor classes α, β such that their intersection degrees yield the appropriate combinatorial sequence. We explain how this is done for the sequences (c) and (d).

3.2.1. *Log-concavity of* (c). We show the slightly stronger statement that the closely related sequence $(\overline{w}_0, \dots, \overline{w}_{r-1})$ defined by

$$\frac{1}{1+q}\mathrm{T}_{\mathrm{M}}(1+q,0) = \sum_{i=0}^{r-1} \overline{w}_i q^{r-1-i}$$

is log-concave with no internal zeros. Theorem 3.4 states that this sequence is exactly the Betti numbers of the arrangement complement $\mathbb{P}\mathring{L}$. The sought-after projective variety is the wonderful compactification of a hyperplane arrangement complement introduced in [DCP95].

Definition 3.6. The **wonderful compactification** W_L is the variety obtained by blowing up $\mathbb{P}L$ at all the points $\{\mathbb{P}L_F : \mathrm{rk_M}(F) = r-1\}$, then by blowing-up all strict transforms of the lines $\{\mathbb{P}L_F : \mathrm{rk_M}(F) = r-2\}$, and so forth. Let $\pi_L : W_L \to \mathbb{P}L$ be the blow-down map.

By construction π_L is isomorphism on the open loci $\mathbb{P}\mathring{L}$. The boundary $\partial W_L = W_L \setminus \mathbb{P}\mathring{L}$ is a simple-normal-crossing divisor on W_L [DCP95]. For the sought-after divisor classes α, β on W_L, the wonderful compactification for the Boolean matroid plays a special role.

When $\mathrm{M} = \mathrm{U}_{n,n}$, that is, when $L = \mathbb{C}^E$, the wonderful compactification is known as the **permutohedral variety** $X_{A_{n-1}}$. Explicitly, it is obtained from $\mathbb{P}^{n-1}$ by blowing up all n coordinate points of $\mathbb{P}^{n-1}$, then blowing up all strict transforms of $\binom{n}{2}$ coordinate lines of $\mathbb{P}^{n-1}$, and so forth. Let $\pi_1 : X_{A_{n-1}} \to \mathbb{P}^{n-1}$ be the blow-down map. Blowing down the exceptional divisors in a *reverse manner* (see [Huh18] for a detailed description via toric geometry), one obtains a different blow-down map $\pi_2 : X_{A_{n-1}} \to \mathbb{P}^{n-1}$. The resulting birational transformation is the Cremona transformation, $\mathrm{crem} : \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{n-1}$ given by $[x_1, \ldots, x_n] \mapsto [\frac{1}{x_1}, \ldots, \frac{1}{x_n}]$ in the projective coordinates.

Returning to the case where M is not necessarily Boolean, we note the following: because the arrangement $\mathcal{A}$ on $\mathbb{P}L$ is the restriction to $\mathbb{P}L$ of the coordinate hyperplane arrangement on $\mathbb{P}^{n-1}$, the universal property of blowups implies that W_L is the strict transform of $\mathbb{P}L \subseteq \mathbb{P}^{n-1}$ under the blow-up π_1. Summarizing, we have a commuting diagram.

$$(\dagger) \qquad \begin{array}{ccccc} W_L & \hookrightarrow & X_{A_{n-1}} & & \\ \downarrow{\scriptstyle \pi_L} & & \downarrow{\scriptstyle \pi_1} & \searrow^{\pi_2} & \\ \mathbb{P}L & \hookrightarrow & \mathbb{P}^{n-1} & \xrightarrow{\mathrm{crem}} & \mathbb{P}^{n-1} \end{array}$$

Let h be the hyperplane class of $\mathbb{P}^{n-1}$, and define divisor classes α and β on W_L to be the restrictions of $\pi_1^* h$ and $\pi_2^* h$, respectively. Huh and Katz [HK12] showed that

$$\frac{1}{1+q} T_{\mathrm{M}}(1+q, 0) = \sum_{i=0}^{r-1} \left(\int_{W_L} \alpha^{r-1-i} \beta^i \right) q^{r-1-i}.$$

Since α and β are both hyperplane class pullbacks, they are globally generated, so Proposition 3.5 implies the desired log-concavity.

How might one think to do this, at least in hindsight? Two key steps are as follows.

(1) The commuting diagram (†) shows that α is also the pullback to W_L of the hyperplane class in $\mathbb{P}L$, so we may loosely interpret multiplication by α as restriction to a general hyperplane H in $\mathbb{P}L$. As a linear subvariety $H \subset \mathbb{P}^{n-1}$, this hyperplane H is again a realization of a matroid, known as the **truncation matroid** $\mathrm{tr}(\mathrm{M})$ of M. With well-known properties of characteristic polynomials [Zas87], it is straightforward to verify that $\frac{1}{1+q} \mathrm{T}_{\mathrm{tr}(\mathrm{M})}(1+q, 0)$ is obtained from $\frac{1}{1+q} \mathrm{T}_{\mathrm{M}}(1+q, 0)$ by erasing the

constant term and then dividing by q. That is, the sequence $(\overline{w}_0, \ldots, \overline{w}_{r-2})$ for tr(M) is obtained from that of M by simply removing the last entry.

(2) With the previous step, we now only need compare $\int_{W_L} \beta^{r-1}$ with the constant term $\mathrm{T_M}(1,0)$. Let $\mathbb{P}L^{-1}$ be the closure of the image of $\mathbb{P}\mathring{L}$ under the Cremona transformation, often known as the **reciprocal linear space**. By the construction of β, the degree of $\mathbb{P}L^{-1}$ as a subvariety of $\mathbb{P}^{n-1}$ equals $\int_{W_L} \beta^{r-1}$. On the other hand, the degree of $\mathbb{P}L^{-1}$ also equals $\mathrm{T_M}(1,0)$. This last key fact was proven in several contexts [Ter02, PS06, Huh12, HK12]. A topological approach in [Huh12] is as follows: A result of Dimca and Papadima [DP03] from (complex) Morse theory related the Euler characteristic of a hypersurface complement to the degree of the gradient map. The hypersurface $\{x_1x_2\cdots x_n = 0\} \subset \mathbb{P}^{n-1}$ is the coordinate hyperplane arrangement that restricts to the hyperplane arrangement $\mathcal{A}$ on $\mathbb{P}L$, and the gradient map of $x_1x_2\cdots x_n$ is exactly the Cremona map. Combining these facts with Theorem 3.4, one can deduce $\deg \mathbb{P}L^{-1} = \mathrm{T_M}(1,0)$.

Exercise 3.7. Let L be as in Example 3.2. Verify that $\mathrm{T_M}(1,0) = 3$, and verify that $\mathbb{P}L^{-1}$ is a cubic surface known as the Cayley nodal cubic. This cubic surface has four singular points, with a line through each pair of points. Explain where these come from in terms of the wonderful compactification W_L. (*Bonus*: This cubic surface has three more lines, for the total of nine. Where do they come from?)

3.2.2. *Log-concavity of* (d). The sought-after projective variety is the **variety of critical points** $\mathfrak{X}$, formally introduced in [CDFV11] but implicit in previous works related to Varchenko's problem on critical points of master functions on an affine hyperplane arrangement [Var95]. Here, in order to build upon our previous discussion in Section 3.2.1, we follow [BEST, Section 8] to describe a smooth birational model of $\mathfrak{X}$ in terms of the wonderful compactification W_L, although it differs slightly from the original description in [Huh15].

Consider the embedding $W_L \hookrightarrow X_{A_{n-1}}$ in the diagram (†). Let $\mathcal{N} = \mathcal{N}_{W_L/X_{A_{n-1}}}$ be the normal bundle, and let $\mathfrak{X}_L = \mathbb{P}_{W_L}(\mathcal{N}^\vee)$ be the projectivization[3] of the conormal bundle with the projection map $p : \mathfrak{X}_L \to W_L$. Recall the blow-down map $\pi_L : W_L \to \mathbb{P}L$.

The sought-after divisor classes on $\mathfrak{X}_L$ are as follows. Let γ be the pullback of the hyperplane class in $\mathbb{P}L$ via the composition $\mathfrak{X}_L \xrightarrow{p} W_L \xrightarrow{\pi_L} \mathbb{P}L$. Let $\delta = c_1(\mathcal{O}(1))$ be the first Chern class of the line bundle $\mathcal{O}(1)$ from the construction of $\mathfrak{X}_L$ as a projectivization of a vector bundle, which turns out to be semi-ample. One can then translate [DGS12, Theorem 1.1] to the statement that

$$\frac{1}{q}\mathrm{T_M}(q,0) = \sum_{i=0}^{r-1} \left(\int_{\mathfrak{X}_L} \gamma^{r-1-i}\delta^{n-r-1+i} \right) q^{r-1-i}.$$

Proposition 3.5 now implies the desired log-concavity.

How might one think to do this, at least in hindsight? For the original formulation of $\mathfrak{X}$, maximum likelihood problems in algebraic statistics provided a motivation; see [Huh13, HS14]. For the related construction $\mathfrak{X}_L$ here, we highlight

[3]Our convention for the projectivization of a vector bundle $\mathcal{E}$ on a variety X is that $\mathbb{P}_X(\mathcal{E}) = \mathrm{Proj}_X \mathrm{Sym}^\bullet(\mathcal{E}^\vee)$, which agrees with [EH16] but is the opposite of [Har77, Laz04b].

some key steps. In either case, one uses properties of log-tangent bundles and their characteristic classes; see [Alu05] for an introduction to these tools.

(1) By Theorem 3.4, the constant term equals

$$\left(\frac{1}{q}\mathrm{T}_{\mathrm{M}}(q,0)\right)\Big|_{q=0} = \left(\frac{1}{1+q}\mathrm{T}_{\mathrm{M}}(1+q,0)\right)\Big|_{q=-1} = (-1)^{r-1}\chi_{\mathrm{top}}(\mathbb{P}\mathring{L}),$$

the signed Euler characteristic of $\mathbb{P}\mathring{L}$. Euler characteristics satisfy the *scissors relation* that $\chi_{\mathrm{top}}(X) = \chi_{\mathrm{top}}(X \setminus Z) + \chi_{\mathrm{top}}(Z)$ for a closed embedding $Z \hookrightarrow X$ of $\mathbb{C}$-varieties. More generally, Chern–Schwartz–MacPherson (CSM) classes [Mac74] of $\mathbb{C}$-varieties are homological objects that respect such scissors relations. One then uses Exercise 3.3 to show that the degrees of the CSM classes of $\mathbb{P}\mathring{L}$ satisfy the same deletion-contraction relation satisfied by the coefficients of $\frac{1}{-q}\mathrm{T}_M(-q,0)$, so that Theorem 2.11 implies that they are the same.

(2) Having related the CSM classes to $\frac{1}{-q}\mathrm{T}_M(-q,0)$, we now relate powers of δ with the CSM classes of $\mathbb{P}\mathring{L}$. The varieties W_L and $X_{A_{n-1}}$ have simple-normal-crossing boundaries ∂W_L and $\partial X_{A_{n-1}} = X_{A_{n-1}} \setminus ((\mathbb{C}^*)^E/\mathbb{C}^*)$. Under the embedding $W_L \hookrightarrow X_{A_{n-1}}$, one can show that $\partial W_L = W_L \cap \partial X_{A_{n-1}}$ scheme-theoretically. Consequently, one has a short exact sequence (see for instance [EHL, Section 9])

$$0 \to \mathcal{T}_{W_L}(-\log \partial W_L) \to \mathcal{T}_{X_{A_{n-1}}}(-\log \partial X_{A_{n-1}})|_{W_L} \to \mathcal{N} \to 0.$$

Because $X_{A_{n-1}}$ is a toric variety with the dense open torus $(\mathbb{C}^*)^E/\mathbb{C}^*$, the log-tangent bundle $T_{X_{A_{n-1}}}(-\log \partial X_{A_{n-1}})$ is trivial [CLS11, Chapter 8]. Thus, the normal bundle $\mathcal{N}$ is globally generated so that δ is semi-ample. Moreover, the Segre classes of $\mathcal{N}$, which are given by powers of $-\delta$, equal the Chern classes of $\mathcal{T}_{W_L}(-\log \partial W_L)$, which are the CSM classes of the complement $W_L \setminus \partial W_L = \mathbb{P}\mathring{L}$ [Alu99].

3.3. **The top-heaviness via intersection cohomology.** We start with a general strategy for establishing top-heaviness, which first appeared in [BE09] to establish top-heaviness for Bruhat intervals. Suppose we have a (not necessarily smooth) projective $\mathbb{C}$-variety X with an **affine stratification**. There is a finite collection $\{U_j\}_{j\in J}$ of locally closed subvarieties of X, called the **strata**, such that X is the disjoint union of $\{U_j\}_{j\in J}$, the closure $\overline{U_j}$ of any strata is again a union of strata, and each strata is isomorphic to $\mathbb{C}^m$ for some m.

Theorem 3.8 ([BE09, Theorem 3.1])**.** *For $0 \le i \le d = \dim X$, let b_i be the number of strata of dimension i. Then we have $b_i \le b_j$ for all $0 \le i \le j \le d-i$.*

Sketch of the proof. Let us assume we have shown that $\dim H^{2i}(X) = b_i$. If X were smooth, the hard Lefschetz theorem implies that $(b_0, \dots, b_d)$ is in fact unimodal and symmetric, so we would be done. Since X may not be smooth, we need to consider the intersection cohomology $IH^\bullet(X)$, for which the hard Lefschetz theorem still holds [GM83, BBD82]. There is a natural graded map $H^\bullet(X) \to IH^\bullet(X)$, fitting

into a commutative diagram

$$\begin{array}{ccc} H^{2i}(X) & \longrightarrow & IH^{2i}(X) \\ {\scriptstyle \cdot c_1(\mathcal{L})^{j-i}}\downarrow & & \downarrow{\scriptstyle \cdot c_1(\mathcal{L})^{j-i}} \\ H^{2j}(X) & \longrightarrow & IH^{2j}(X) \end{array}$$

for any ample line bundle $\mathcal{L}$ on X, where the injectivity of the right vertical map follows from the validity of the hard Lefschetz property for intersection cohomology. Thus, if $H^\bullet(X) \to IH^\bullet(X)$ is injective, then the left vertical map is necessarily injective, so we can conclude the desired $b_i \leq b_j$.

The proof of $\dim H^{2i}(X) = b_i$ and the injection $H^\bullet(X) \hookrightarrow IH^\bullet(X)$ both follow from combining a standard long exact sequence of cohomologies and the result of Weber [Web04] (see also [BE09, Theorem 2.1]), which identified the kernel of $H^k(X) \to IH^k(X)$ in terms of the weight filtration on $H^k(X)$ given by the mixed Hodge structure [Del71]. □

Björner and Ekedahl [BE09] used Theorem 3.8 on Schubert varieties of a generalized flag variety to deduce top-heaviness for Bruhat intervals of a finite crystallographic Coxeter group. In our case, for a realization $L \subseteq \mathbb{C}^E$ of a matroid M, we consider its **matroid Schubert variety** Y_L defined as

$$Y_L = \text{the closure of } L \text{ inside } (\mathbb{P}^1)^E,$$

where $\mathbb{C}^E \subset (\mathbb{P}^1)^E$ via the identification $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$. Note that the identification $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ induces an affine stratification of $(\mathbb{P}^1)^E$ with strata $\{\mathbb{C}^S \times \{\infty\}^{E \setminus S} : S \subseteq E\}$. One shows that this stratification restricts to give an affine stratification of Y_L by using the computation of (the Gröbner basis for) the defining ideal of $Y_L \subseteq (\mathbb{P}^1)^E$ given by Ardila and Boocher [AB16].

Theorem 3.9 ([HW17, Theorem 14], [PXY18, Lemmas 7.5 and 7.6])**.** *The matroid Schubert variety Y_L admits an affine stratification by the strata $\{U^F : F \text{ a flat of } \mathrm{M}\}$ defined by*

$$U^F = Y_L \cap \left(\mathbb{C}^F \times \{\infty\}^{E \setminus F}\right).$$

For each flat F, the strata U^F is isomorphic to the image of the composition $L \twoheadrightarrow \mathbb{C}^E \twoheadrightarrow \mathbb{C}^F$, which has dimension $\mathrm{rk}_{\mathrm{M}}(F)$.

The top-heaviness of the sequence (e) now follows from Theorems 3.9 and 3.8.

Exercise 3.10. Verify Theorem 3.9 for a realization of the uniform matroid $\mathrm{U}_{2,3}$.

For Schubert varieties in generalized flag varieties, their intersection cohomology is closely related to Kazhdan–Lusztig theory. The terminology "matroid Schubert variety" was chosen because of the analogous relation between the intersection cohomology Y_L and Kazhdan–Lusztig theory of matroid developed in [EPW16, PXY18].

4. Tropical Hodge theory

We will begin by explaining how polyhedral fans and matroids give rise to cohomology rings. We then discuss two fundamental theorems for these cohomology rings concerning the validity of the Kähler package (Definition 1.2). We then explain how the log-concavity statements for arbitrary matroids can be deduced from

these fundamental theorems. We will assume some familiarity with the basics of polyhedral geometry. For a brief introduction see [Ful93, Chapter 1.2], and see [Zie95] for a fuller treatment.

4.1. **Chow rings of fans and matroids.** Let N be a lattice, i.e., a finitely generated free abelian group $\mathbb{Z}^m$. Let $N^\vee$ denote its dual lattice. We write $N_\mathbb{R} = N \otimes \mathbb{R}$. Recall that a fan Σ in $N_\mathbb{R}$ is *rational* if each ray ρ in Σ equals $\mathbb{R}_{\geq 0}u$ for some $u \in N$, *simplicial* if every k-dimensional cone in Σ is generated by k rays, and *pure-dimensional* if every maximal cone has the same dimension. For each ray ρ, let $u_\rho \in N$ be the *primitive ray generator*, i.e., the element such that $\rho \cap N = \mathbb{Z}_{\geq 0}u_\rho$. Let $\Sigma(1)$ denote the set of rays of Σ. The *support* of Σ is denoted $|\Sigma|$. A fan Σ in $N_\mathbb{R}$ is *complete* if $|\Sigma| = N_\mathbb{R}$.

Assumption 4.1. All fans we treat will be rational, simplicial, and pure-dimensional, but not necessarily complete.

Definition 4.2. The **Chow (cohomology) ring** (with real coefficients) of a fan Σ in $N_\mathbb{R}$ is the graded $\mathbb{R}$-algebra

$$A^\bullet(\Sigma) = \frac{\mathbb{R}[x_\rho : \rho \in \Sigma(1)]}{I_\Sigma + J_\Sigma},$$

where I_Σ and J_Σ are the ideals

$$I_\Sigma = \Big\langle \prod_{\rho \in S} x_\rho : S \subseteq \Sigma(1) \text{ do not form a cone in } \Sigma \Big\rangle \quad \text{and}$$
$$J_\Sigma = \Big\langle \sum_{\rho \in \Sigma(1)} m(u_\rho) x_\rho : m \in N^\vee \Big\rangle.$$

It is an exercise to show that the kth graded piece $A^k(\Sigma)$ of $A^\bullet(\Sigma)$ is generated by square-free monomials of degree k in the variables. In particular, $A^k(\Sigma) = 0$ for all k greater than the dimension d of Σ, and $A^\bullet(\Sigma) = \bigoplus_{i=0}^d A^i(\Sigma)$ is a finite-dimensional graded real vector space.

Exercise 4.3. For two fans Σ and Σ', show that $A^\bullet(\Sigma \times \Sigma') = A^\bullet(\Sigma) \otimes A^\bullet(\Sigma')$.

Borrowing language from algebraic geometry, let us call a linear combination of the variables x_ρ a **divisor** and its image in $A^1(\Sigma)$ its **divisor class** on Σ. Because Σ is simplicial, a divisor $D = \sum_{\rho \in \Sigma(1)} c_\rho x_\rho$ determines a piecewise-linear function φ_D on $|\Sigma|$ by assigning the value c_ρ to each primitive ray generator u_ρ.

Definition 4.4. A divisor D on a complete fan Σ is **ample** if the piecewise-linear function φ_D is strictly convex, i.e., $\varphi_D(u) + \varphi_D(v) < \varphi_D(u+v)$ for all $u, v \in N_\mathbb{R}$ not in the same cone of Σ. It is **nef** if only the weak inequality $\leq$ is satisfied.

For a not necessarily complete fan Σ, a divisor D is **ample** (resp., **nef**) if φ_D is the restriction of the piecewise-linear function of an ample (resp., nef) divisor on a complete fan containing Σ as a subfan. We denote by $\mathcal{K}(\Sigma) \subset A^1(\Sigma)$ the convex set of the divisor classes of ample divisors on Σ.[4] We often consider $\mathcal{K}(\Sigma)$ as a set of graded linear maps $A^\bullet(\Sigma) \to A^{\bullet+1}(\Sigma)$ given by multiplication.

[4]Technically, our definition of ample/nef divisors on noncomplete fans differs from that of [ADH22], which makes our $\mathcal{K}(\Sigma)$ a subset of the one in [ADH22], but will suffice for our discussion.

In terms of toric geometry, the ring $A^\bullet(\Sigma)$ is the Chow cohomology ring of the toric variety X_Σ associated to the fan Σ [Dan78, Bri96]. When Σ is the normal fan of a simple polytope, the toric variety X_Σ is a projective variety with mild (i.e., orbifold) singularities, whose ample cone is $\mathcal{K}(\Sigma)$. Classical results in algebraic geometry then imply the validity of the Kähler package for the ring $A^\bullet(\Sigma)$. Stanley used this to resolve McMullen's g-conjecture on the number of faces of a simple polytope [Sta80]. Afterwards, McMullen [McM93] gave a purely combinatorial proof of the Kähler package that works even for nonrational fans (with no associated toric variety in the background). In our case, the fans will not in general be complete, so a priori there is no reason to expect any validity of the Kähler package, since the associated toric varieties are not in general compact. The miraculous result is that certain fans from matroids turn out to enjoy the Kähler package.

For a matroid M on E of rank r, we construct a fan introduced and studied in [Stu02, AK06, Spe08] as a tropical geometric analogue of linear spaces. The fan will be in the real vector space over the lattice $\mathbb{Z}^E/\mathbb{Z}\mathbf{e}_E$. For a subset $S \subseteq E$, let us denote by $\overline{\mathbf{e}}_S$ the image of $\mathbf{e}_S = \sum_{i\in S} \mathbf{e}_i \in \mathbb{R}^E$ under the quotient map $\mathbb{R}^E \to \mathbb{R}^E/\mathbb{R}\mathbf{e}_E$.

Definition 4.5. The **Bergman fan** Σ_{M} of a rank r matroid M is a pure $(r-1)$-dimensional fan in $\mathbb{R}^E/\mathbb{R}\mathbf{e}_E$ consisting of the maximal cones $\mathbb{R}_{\geq 0}\{\overline{\mathbf{e}}_{F_1}, \ldots, \overline{\mathbf{e}}_{F_{r-1}}\}$, one for each maximal chain $\varnothing \subsetneq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_{r-1} \subsetneq E$ of nonempty proper flats of M.

Example 4.6. The Bergman fans of $\mathrm{U}_{2,3}$ and $\mathrm{U}_{3,4}$ are depicted in Figure 5.

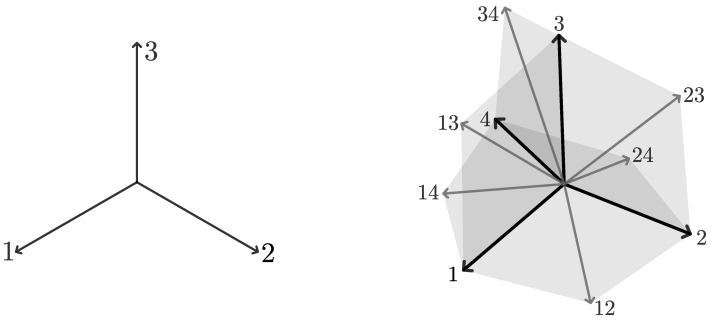


FIGURE 5

Example 4.7. For the Boolean matroid $\mathrm{U}_{n,n}$, note that the maximal cones of its Bergman fan $\Sigma_{\mathrm{U}_{n,n}}$ correspond to permutations of the ground set E. The fan $\Sigma_{\mathrm{U}_{n,n}}$ is known as the **permutohedral fan** or as the **braid arrangement**, denoted $\Sigma_{A_{n-1}}$. Note that by construction the Bergman fan of any matroid M (on ground set E) is a subfan of $\Sigma_{A_{n-1}}$. See [AA] for a survey of remarkable combinatorial properties of the permutohedral fan.

Definition 4.8. The **Chow ring** $A^\bullet(\mathrm{M})$ **of a matroid** M is the Chow ring $A^\bullet(\Sigma_{\mathrm{M}})$ of its Bergman fan. Explicitly, it is

$$A^\bullet(\mathrm{M}) = \bigoplus_{i=0}^{r-1} A^i(\mathrm{M}) = \frac{\mathbb{R}[x_F : F \text{ a nonempty proper flat of M}]}{I_{\mathrm{M}} + J_{\mathrm{M}}},$$

where I_{M} and J_{M} are the ideals

$$I_{\mathrm{M}} = \Big\langle x_F x_{F'} : F \not\subseteq F' \text{ and } F \not\supseteq F' \Big\rangle \quad \text{and}$$

$$J_{\mathrm{M}} = \Big\langle \sum_{F \ni i} x_F - \sum_{G \ni j} x_G : i, j \in E \Big\rangle, \quad \begin{array}{l}\text{where the two sums are over all flats} \\ \text{containing } i \text{ and } j, \text{ respectively.}\end{array}$$

Remark 4.9. When M has a realization $L \subseteq \mathbb{C}^E$, the wonderful compactification W_L defined in Section 3.2.1 is the closure of $\mathbb{P}\overset{\circ}{L}$ inside the toric variety $X_{\Sigma_{\mathrm{M}}}$. The resulting pull-back map of cohomologies is an isomorphism between the Chow ring $A^\bullet(\mathrm{M}) = A^\bullet(X_{\Sigma_{\mathrm{M}}})$ of the toric variety of Σ_{M} and the cohomology ring of W_L [FY04, DCP95]. This *Chow equivalence* is informed by the theory of *tropical compactifications* [Tev07]; see [MS15, Chapter 6] for an introduction. While the Kähler package for the Chow ring in this realizable case thus follows from classical Hodge theory, [AHK18, Theorem 5.12] states that the existence of such Chow equivalence is equivalent to the realizability of the matroid.

We note another fan associated to a matroid whose Chow ring will be used for log-concavity statements.

Definition 4.10. The **conormal fan** $\Sigma_{\mathrm{M},\mathrm{M}^\perp}$ of a (loopless and coloopless) matroid M is a fan in $\mathbb{R}^E/\mathbb{R}\mathbf{e}_E \times \mathbb{R}^E/\mathbb{R}\mathbf{e}_E$ whose support equals the support of the product $\Sigma_{\mathrm{M}} \times \Sigma_{\mathrm{M}^\perp}$.

We omit the precise definition, which involves the intricate and interesting combinatorics of the *bipermutohedron* and *biflags* introduced in [ADH22]. See [ADH22, Section 2.8] for its origin story.

Remark 4.11. Just as the Chow ring $A^\bullet(\mathrm{M})$ of a matroid M is modeled after the wonderful compactification, the *conormal Chow ring* of a matroid is modeled after the geometry described in Section 3.2.2. For instance, Exercise 2.7 suggests how the product $\Sigma_{\mathrm{M}} \times \Sigma_{\mathrm{M}^\perp}$ can serve as a polyhedral model of the projectivized conormal bundle $\mathfrak{X}_L = \mathbb{P}_{W_L}(\mathcal{N}^\vee)$.

4.2. **Fundamental theorems of tropical Hodge theory.** We now discuss two fundamental theorems concerning the Kähler package for Chow rings of not necessarily complete fans. To state them, we need a few more terminologies.

For a cone σ of a fan Σ in $N_{\mathbb{R}}$, the **star** $\mathrm{st}_\sigma(\Sigma)$ is a fan in $N_{\mathbb{R}}/\operatorname{span}(\sigma)$ whose cones are the images under the projection $N_{\mathbb{R}} \to N_{\mathbb{R}}/\operatorname{span}(\sigma)$ of the cones of Σ containing σ. Geometrically, the toric variety $X_{\mathrm{st}_\sigma(\Sigma)}$ of the star is the closure of the torus-orbit of X_Σ corresponding to the cone σ.

Exercise 4.12.

(1) The star of the Bergman fan Σ_{M} at the ray $\mathbb{R}_{\geq 0}\mathbf{e}_F$ corresponding to a nonempty proper flat F is isomorphic to the product $\Sigma_{\mathrm{M}|F} \times \Sigma_{\mathrm{M}/F}$.
(2) Show that the Chow ring $A^\bullet(\mathrm{st}_\sigma \Sigma)$ of the star is isomorphic to the quotient ring $A^\bullet(\Sigma)/\langle a \in A^\bullet(\Sigma) : a \cdot \prod_{\rho \in \sigma} x_\rho = 0 \rangle$.

A **positive Minkowski weight** on a (pure) d-dimensional fan Σ is a linear map $\deg : A^d(\Sigma) \to \mathbb{R}$ such that $\deg(\prod_{\rho \in \sigma} x_\rho) > 0$ for any maximal cone σ of Σ. It defines a symmetric bilinear pairing $A^\bullet \times A^{d-\bullet} \to \mathbb{R}$ by $(x, y) \mapsto \deg(xy)$, which by abuse of notation we also denote deg.

Geometrically, Minkowski weights in general are fundamental objects in tropical intersection theory, serving the role of *Chow homology classes*. For their definition and properties, see [FS97, KP08] and [MS15, Chapter 6], as well as [AHK18, Section 5]. Fans arising in the context of tropical compactifications (Remark 4.9) provide many examples of positive Minkowski weights. For instance, the following can be deduced from what is known as the cover-partition property of flats.

Proposition 4.13 ([AHK18, Proposition 5.2])**.** *For a matroid* M *of rank* r*, the assignment* $x_{F_1} \cdots x_{F_{r-1}} \mapsto 1$ *for any maximal chain* $\varnothing \subsetneq F_1 \subsetneq \cdots \subsetneq F_{r-1} \subsetneq E$ *of nonempty proper flats of* M *gives a well-defined linear map* $\deg_{\mathrm{M}} : A^{r-1}(\mathrm{M}) \to \mathbb{R}$*.*

Similarly, the conormal fan of a matroid also has a natural positive Minkowski weight $\deg_{\mathrm{M},\mathrm{M}^\perp}$. The key notion for the statement of the two fundamental theorems is the notion of Lefschetz-ness of fans, introduced in [ADH22].

Definition 4.14. A fan Σ of dimension d is said to be **Lefschetz** if the following are satisfied:

(1) $\mathrm{Hom}_{\mathbb{R}}(A^d(\Sigma), \mathbb{R})$ is spanned by a positive Minkowski weight deg.
(2) The triple $(A^\bullet(\Sigma), \deg, \mathcal{K}(\Sigma))$ satisfies the Kähler package (Definition 1.2).
(3) For any positive-dimensional cone σ of Σ, the star $\mathrm{st}_\sigma(\Sigma)$ is Lefschetz.

We can now state the two fundamental theorems.

Theorem 4.15 ([AHK18, Theorem 1.4])**.** *The Bergman fan* Σ_{M} *of a matroid* M *is Lefschetz.*

Theorem 4.16 ([ADH22, Theorem 1.6])**.** *Let* Σ *and* Σ' *be fans in* $N_{\mathbb{R}}$ *with the same support, and suppose* $\mathcal{K}(\Sigma)$ *and* $\mathcal{K}(\Sigma')$ *are nonempty. Then* Σ *is Lefschetz if and only if* Σ' *is Lefschetz.*

The product of two Lefschetz fans is again Lefschetz [AHK18, Section 7.2]. Combining this with the two theorems yields the following.

Corollary 4.17. *The conormal fan of the matroid is Lefschetz.*

Exercise 4.18. Let M be a matroid of rank 3. Let $\alpha = \sum_{G \ni i} x_G$ for any $i \in E$. Note that $\alpha \in A^\bullet(\mathrm{M})$ is independent of the choice of i due to the linear relations J_{M}.

(1) Show that $\deg_{\mathrm{M}}(x_F^2) = -1$ for any flat F of rank 2.
(2) Show that $\deg_{\mathrm{M}}\left(\alpha^2\right) = 1$ for any element $i \in E$.
(3) Show that $\deg_{\mathrm{M}}(\alpha x_F) = 0$ for any element $i \in E$ and a flat F of rank 2.
(4) Use these steps to establish the Kähler package for $A^\bullet(\mathrm{M})$ with $\mathcal{K} = \mathbb{R}_{>0}\alpha$.

Let us give a broad overview of the proofs of the two theorems. Both employ the following strategy for establishing the Kähler package for a graded $\mathbb{R}$-algebra $A^\bullet$ of *dimension* d in the sense that $A^\bullet = \bigoplus_{i=0}^d A^i$. This general strategy and variations thereof appear in several previous works on the Kähler package across varied mathematical fields, such as the works of McMullen [McM93] on simple polytopes, of Elias and Williamson [EW14] on Soergel bimodules, and of de Cataldo and Migliorini [dCM09] on the topology of algebraic maps.

(i) It suffices to show the statements of (HL) and (HR) in Definition 1.2 in the special case where $L_0 = L_1 = \cdots = L_{d-2i}$, because this *nonmixed* version of the Kähler package implies the original *mixed* version of the Kähler package [Cat08] (see also [ADH22, Theorem 5.20]).

(ii) Next, one can set up an induction on the dimension d as follows. In many situations, the quotient algebra $A^\bullet/\operatorname{ann}(x)$ for certain choices of $x \in A^1$ is again in the family of graded $\mathbb{R}$-algebras that one is seeking to establish the Kähler package for. Here, $\operatorname{ann}(x)$ denotes the annihilator $\{a \in A^\bullet : ax = 0\}$. The key observation then is that

$$\begin{pmatrix}\text{(HR) in degree } i \text{ of } A^\bullet/\operatorname{ann}(x)\\ \text{for sufficiently many } x \in A^1\end{pmatrix} \implies \begin{pmatrix}\text{(HL) in degree } i\\ \text{for the original } A^\bullet\end{pmatrix}$$

(see for instance [BES, Proposition 6.1.6]). Because the quotient $A^\bullet/\operatorname{ann}(x)$ has smaller dimension, i.e., its dth graded part is zero, one can now proceed by induction on d.

(iii) By the validity of (HL) from the inductive hypothesis, the validity of (HR) for any single element $L \in \mathcal{K}$ then implies (HR) for all $L \in \mathcal{K}$. Thus, the last step is to finish the induction by establishing (HR) for a well-chosen $L \in \mathcal{K}$. This is often the most intricate step.

Returning to the case of matroids, step (ii) is provided by Exercise 4.12, which showed that $A^\bullet(\mathrm{M})/\operatorname{ann}(x_F) \simeq A^\bullet(\Sigma_{\mathrm{M}|F} \times \Sigma_{\mathrm{M}/F})$ for a nonempty proper flat F of M. Hence, one can induct on the rank of the matroid. In the case of Theorem 4.16, step (ii) is essentially built into the definition that the stars of Lefschetz fans are Lefschetz, so that one can induct on the dimension of the fan.

In the original proof [AHK18] of Theorem 4.15, in order to carry out step (iii), Adiprasito, Huh, and Katz introduced the notion of *flips*, inspired by the proof of the Kähler package for simple polytopes by McMullen [McM93]. This process converts the Bergman fan of Σ_{M} through a sequence of fans until it reaches a fan for which (HR) can be verified easily, and the process is set up such that the validity of (HR) for any one fan in the sequence implies (HR) for all fans in the sequence. Geometrically, one may interpret the process of flips as a combinatorial abstraction of the process of constructing the wonderful compactification W_L as a sequence of blowups (Definition 3.6).

Afterwards, it was recognized that a key property of *semi-small maps* [dCM02] inspires a strategy that can greatly simplify step (iii). For a map $f : X \to Y$ of smooth projective varieties, the pullbacks to X of ample divisors on Y generally fail (HL) and (HR) in the cohomology ring of X, since the pullbacks are generally not ample on X but only nef (i.e., is a limit of ample divisors). However, a characterizing property of a semi-small map is that the pullbacks still satisfy (HL) and (HR). This inspires the following approach: One can look for a map $\widetilde{A}^\bullet \to A^\bullet$ of graded algebras, behaving like a pullback along a semi-small map. If (HR) is known to hold for $\widetilde{A}^\bullet$, say by induction, step (iii) would follow.

This insight allowed Braden, Huh, Matherne, Proudfoot, and Wang to give a considerably simplified proof of Theorem 4.15 in [BHM$^+$22]. Using that the deletion $\mathrm{M}\setminus e$ of a matroid M by a non-coloop element e behaves like a semi-small map, they carry out step (iii) by an induction that reduces to the case of Boolean matroids. This insight on semi-small maps is also essential in the proof of Theorem 4.16 by Ardila, Denham, and Huh. A key step [ADH22, Theorem 5.9], building upon the works [Wło97, AKMW02], states that any two fans with the same support can be related by a sequence of edge stellar subdivisions, which are operations on fans that play the role of semi-small maps in toric geometry.

4.3. **Applications of Hodge–Riemann relations in degree 1.** The Kähler package gives rise to log-concave sequences in the following way.

Proposition 4.19. *Suppose Σ is a Lefschetz fan of dimension d with a positive Minkowski weight* deg*. Then, for any nef divisor classes α and β,*

$$\text{the sequence } (a_0, a_1, \dots, a_d) \text{ defined by } a_i = \deg(\alpha^{d-i}\beta^i)$$

is log-concave with no internal zeros.

Proof. We may assume that α, β are ample, and show that the sequence is strictly positive and log-concave, since a limit of such sequences is necessarily log-concave with no internal zeros. Strict positivity is then implied by Hodge–Riemann relations (HR) in degree 0. For log-concavity, (HR) in degree 1, with $L_1 \cdots L_{d-2} = \alpha^{d-i-1}\beta^{i-1}$, implies that the symmetric bilinear pairing $A^1(\Sigma) \times A^1(\Sigma) \to \mathbb{R}$ given by $(x, y) \mapsto \deg(xy \cdot \alpha^{d-i-1}\beta^{i-1})$ has at most one positive eigenvalue. This implies that the symmetric matrix

$$\begin{bmatrix} \deg(\alpha^2 \cdot \alpha^{d-i-1}\beta^{i-1}) & \deg(\alpha\beta \cdot \alpha^{d-i-1}\beta^{i-1}) \\ \deg(\alpha\beta \cdot \alpha^{d-i-1}\beta^{i-1}) & \deg(\beta^2 \cdot \alpha^{d-i-1}\beta^{i-1}) \end{bmatrix}$$

cannot be positive definite, but it also cannot be negative definite because all of its entries are positive. Hence, the determinant of the matrix is nonpositive or, equivalently, $a_i^2 \geq a_{i-1}a_{i+1}$. □

Returning to showing log-concavity for a matroid M of rank r, one now searches for appropriate divisor classes on the Bergman fan Σ_{M} or the conormal fan $\Sigma_{\mathrm{M},\mathrm{M}^\perp}$. This step benefits heavily from the geometry of realizable matroids explained in Section 3.

The divisor classes for the log-concavity of the sequence (c) come from an involutive symmetry of the permutohedral fan $\Sigma_{A_{n-1}}$ (Example 4.7). As $-\overline{\mathbf{e}}_i = \overline{\mathbf{e}}_{E\setminus i}$, the minus map $x \mapsto -x$ on $\mathbb{R}^E/\mathbb{R}\mathbf{e}_E$ gives an involution of $\Sigma_{A_{n-1}}$. This equips $\Sigma_{A_{n-1}}$ with two distinguished coarsenings to a normal fan of a simplex: Let Σ_Δ (resp., be Σ_∇) be the fan whose rays are $\{\overline{\mathbf{e}}_i : i \in E\}$ (resp., $\{-\overline{\mathbf{e}}_i : i \in E\}$) and whose cones are generated by any subsets of the rays with cardinality $\leq n-1$. The fan $\Sigma_{A_{n-1}}$ is a common refinement of both the fans Σ_Δ and Σ_∇.

Let us fix an element $i \in E$, and let φ_Δ be the piecewise-linear function on Σ_Δ given by $\overline{\mathbf{e}}_i \mapsto 1$ and $\overline{\mathbf{e}}_j \mapsto 0$ for all $j \neq i$, which is clearly a convex (but not strictly convex) function. Restricting this piecewise-linear function to Σ_{M}, it defines a divisor on Σ_{M} whose divisor class is denoted $\alpha \in A^1(\mathrm{M})$. Similarly, starting with the fan Σ_∇, we obtain a divisor class β. Both α and β are nef, and independent of the choice of $i \in E$ we fixed. Algebraically, we may take $\alpha = \sum_{F \ni i} x_F$ and $\beta = \sum_{F \not\ni i} x_F$ for any choice of $i \in E$. We have the following.

Proposition 4.20 ([HK12, Proposition 5.2], [AHK18, Proposition 9.5])**.** *With the notations as above, we have*

$$\frac{1}{1+q}\mathrm{T}_{\mathrm{M}}(1+q, 0) = \sum_{i=0}^{r-1} \deg_{\mathrm{M}}(\alpha^{r-1-i}\beta^i)q^{r-1-i}.$$

Combining the proposition with Theorem 4.15 and Proposition 4.19, we obtain that the coefficients of $\frac{1}{1+q}\mathrm{T}_{\mathrm{M}}(1+q, 0)$ are log-concave with no internal zeros, which implies the same property for the coefficients of $\mathrm{T}_{\mathrm{M}}(1+q, 0)$, i.e., the log-concavity of the sequence (c). One can also consider other kinds of divisor classes

on Σ_{M} and their values under $\deg_{\mathrm{M}}$ to study properties of matroids; see for instance [Eur20, BES, BST, DR22].

Remark 4.21. In geometric terms, the toric variety of $\Sigma_{A_{n-1}}$ is the permutohedral variety defined in Section 3.2.1. That the fan $\Sigma_{A_{n-1}}$ coarsens to Σ_Δ and Σ_∇ gives the two blow-down maps $\pi_1, \pi_2 : X_{A_{n-1}} \to \mathbb{P}^{n-1}$, related by the Cremona transformation, which were described in (†). Thus, the divisor classes α and β here and the computations involving them agree with those described in Section 3.2.1.

For the sequence (d), recall that the conormal fan of a matroid M is a fan in $\mathbb{R}^E/\mathbb{R}\mathbf{e}_E \times \mathbb{R}^E/\mathbb{R}\mathbf{e}_E$ with support equal to the support of $\Sigma_{\mathrm{M}} \times \Sigma_{\mathrm{M}^\perp}$. Let

$p : \mathbb{R}^E/\mathbb{R}\mathbf{e}_E \times \mathbb{R}^E/\mathbb{R}\mathbf{e}_E \to \mathbb{R}^E/\mathbb{R}\mathbf{e}_E$ be the projection to the first factor, and

$s : \mathbb{R}^E/\mathbb{R}\mathbf{e}_E \times \mathbb{R}^E/\mathbb{R}\mathbf{e}_E \to \mathbb{R}^E/\mathbb{R}\mathbf{e}_E$ be the addition map $(x, y) \mapsto x + y$.

By pulling back the piecewise-linear function φ_Δ on $\mathbb{R}^E/\mathbb{R}\mathbf{e}_E$ along these two maps, we obtain two divisor classes γ and δ.

Proposition 4.22 ([ADH22, Theorem 1.2])**.** *With the notations as above, we have*

$$\frac{1}{q}\mathrm{T}_{\mathrm{M}}(q, 0) = \sum_{i=0}^{r-1} \deg_{\mathrm{M},\mathrm{M}^\perp}(\gamma^{r-1-i}\delta^{n-r-1+i})q^{r-1-i}.$$

Combining the proposition with Corollary 4.17 and Proposition 4.19, we conclude the log-concavity of the sequence (d). The proposition was proved via an intricate combinatorics of biflags in [ADH22] and the Chern–Schwartz–MacPherson classes of matroids [LdMRS20]. By developing a new framework of *tautological classes of matroids*, Berget, Spink, Tseng and the author proved a formula [BEST, Theorem A and Theorem 9.7] that contains both Proposition 4.20 and Proposition 4.22 as special cases.

Note that only a part of the Kähler package, the Hodge–Riemann relations in degrees at most 1, was required for concluding log-concavity. Extracting the essence of the analytic properties behind (HR) in degrees at most 1 leads to the fascinating theory of **Lorentzian polynomials** [BH20] and (equivalently) **completely log-concave polynomials** [ALOGV18]. One powerful feature of this theory is that it often allows one to reduce to dimension 2 cases, mirroring the feature in classical algebraic geometry that (HR) in degree 1 can be reduced to the case of surfaces. For instance, by reducing to an analysis of rank 2 matroids, Brändén and Huh [BH20] and independently Anari, Liu, Oveis-Gharan, and Vinzant [ALOGV18] proved that the sequence (a) is in fact *ultra*-log-concave, in the sense that

$$\frac{I_i^2}{\binom{n}{i}^2} \geq \frac{I_{i-1}I_{i+1}}{\binom{n}{i-1}\binom{n}{i+1}} \quad \text{for all } i,$$

which was conjectured by Mason [Mas72]. Moreover, by analyzing rank 3 matroids (cf. Exercise 4.18), one can use Lorentzian polynomials to give a simplified proof of the log-concavity of the sequence (c) [BES, BL]. We point to [Huh22, Section 2] for a survey of Lorentzian polynomials and their applications.

5. Intersection cohomology of a matroid

We describe the intersection cohomology of a matroid, and its role in showing the top-heaviness of the sequence (e). We begin by considering the following graded algebra.

Definition 5.1. For a matroid M of rank r, its **Möbius algebra** is a graded $\mathbb{R}$-algebra $B^\bullet(\mathrm{M}) = \bigoplus_{i=0}^{r} B^i(\mathrm{M})$ where $B^i(\mathrm{M})$ has basis $\{y_F : F \text{ a rank } i \text{ flat of M}\}$ for each $0 \le i \le r$, and multiplication is given by

$$y_F \cdot y_{F'} = \begin{cases} y_{F \vee F'} & \text{if } \mathrm{rk_M}(F) + \mathrm{rk_M}(F') = \mathrm{rk_M}(F \vee F') \\ 0 & \text{otherwise.} \end{cases}$$

A strategy for the top-heaviness is to show that there is an injective linear map $B^i(\mathrm{M}) \to B^{r-i}(\mathrm{M})$ for every $i \le r/2$. The statement of the hard Lefschetz property inspires a candidate for such a map: the multiplication by a power of an element in $B^1(\mathrm{M})$. We then immediately face the difficulty that $B^\bullet(\mathrm{M})$ usually cannot satisfy Poincaré duality (PD) or the hard Lefschetz property (HL), since the sequence $(W_0, \ldots, W_r)$ of the dimensions of graded pieces is usually not symmetric.

The intersection cohomology $IH^\bullet(\mathrm{M})$, introduced in [BHM$^+$], is a graded vector space containing $B^\bullet(\mathrm{M})$ that *most efficiently* amends the failure of (PD) and (HL). We give a broad outline of their construction and their properties. The following remark explains some geometric motivation.

Remark 5.2. Recall from Section 3.3 the matroid Schubert variety Y_L of a realization $L \subseteq \mathbb{C}^E$ of a matroid M. One deduces from Theorem 3.9 that the algebra $B^\bullet(\mathrm{M})$ is the cohomology ring (in even degrees) of the *matroid Schubert variety* Y_L (see [HW17, Theorem 14]). The variety Y_L is usually quite singular, which witnesses the failure of (PD) and (HL) for $B^\bullet(\mathrm{M})$. Motivated by the proof of Theorem 3.8, one seeks to understand the intersection cohomology $IH^\bullet(Y_L)$, which contains $B^\bullet(\mathrm{M})$ as a subalgebra.

To do so, let $f : X \to Y_L$ be a resolution of singularities of Y_L. The decomposition theorem of Beilinson, Bernstein, Deligne, and Gabber [BBD82] implies that $H^\bullet(X)$ can be decomposed into a direct sum of $B^\bullet(\mathrm{M})$-modules, and that $IH^\bullet(Y_L)$ is a direct summand. In general, computing these decompositions to get a handle on $IH^\bullet(Y_L)$ can be intractible, but for Y_L there is a resolution $f : \widetilde{Y}_L \to Y_L$ by the *augmented wonderful variety* $\widetilde{Y}_L$ *of* L [BHM$^+$22] (see also [EHL]), such that its cohomology ring $H^\bullet(\widetilde{Y}_L)$ and the injection $B^\bullet(\mathrm{M}) \hookrightarrow H^\bullet(\widetilde{Y}_L)$ have explicit combinatorial descriptions in terms of the matroid M.

We first find a bigger graded $\mathbb{R}$-algebra containing $B^\bullet(\mathrm{M})$ that satisfies the Kähler package. The **augmented Bergman fan** [BHM$^+$22, Definition 2.4] of a matroid M of rank r is an r-dimensional fan in $\mathbb{R}^E$ closely related to the Bergman fan Σ_{M}. Its Chow ring has the following explicit description.

Definition 5.3. The **augmented Chow ring** (with real coefficients) of a matroid M is the graded $\mathbb{R}$-algebra

$$\mathrm{CH}^\bullet(\mathrm{M}) = \frac{\mathbb{R}[y_i, x_F : i \in E,\ F \text{ a (possibly empty) proper flat of M}]}{\widetilde{I}_{\mathrm{M}} + \widetilde{J}_{\mathrm{M}}},$$

where $\widetilde{I}_{\mathrm{M}}$ and $\widetilde{J}_{\mathrm{M}}$ are the ideals

$$\widetilde{I}_{\mathrm{M}} = \Big\langle x_F x_{F'} : F \not\subseteq F' \text{ and } F \not\supseteq F' \Big\rangle + \Big\langle y_i x_F : i \notin F \Big\rangle \quad \text{and}$$
$$\widetilde{J}_{\mathrm{M}} = \Big\langle y_i - \sum_{F \not\ni i} x_F : i \in E \Big\rangle.$$

The augmented Chow ring has the following useful features:

- The assignment $y_F \mapsto \prod_{i \in F} y_i$ defines an injection $B^\bullet(\mathrm{M}) \hookrightarrow \mathrm{CH}^\bullet(\mathrm{M})$ of graded $\mathbb{R}$-algebras [BHM$^+$22, Proposition 2.28].
- Theorem 4.15 combined with Theorem 4.16 implies that the augmented Bergman fan is Lefschetz, because the support of the augmented Bergman fan of M can be identified with the support of the usual Bergman fan of the *free co-extension matroid* of M (see [EHL, Section 5.3]). Thus, the Chow ring $\mathrm{CH}^\bullet(\mathrm{M})$ satisfies the Kähler package.

Thus, we have found a bigger algebra containing $B^\bullet(\mathrm{M})$ that satisfies the Kähler package. However, we are not done because $\mathrm{CH}^\bullet(\mathrm{M})$ is *too big*: To conclude injectivity properties for $B^\bullet(\mathrm{M})$, we need the graded linear operators $\mathcal{K}$ satisfying (HL) on $\mathrm{CH}^\bullet(\mathrm{M})$ to come from $B^1(\mathrm{M})$, but this is almost never the case—a positive linear combination of the y_i's usually does not satisfy (HL) on $\mathrm{CH}^\bullet(\mathrm{M})$. One instead must consider the following $B^\bullet(\mathrm{M})$-submodule of $\mathrm{CH}^\bullet(\mathrm{M})$ that *most efficiently* repairs the the failure of (HL) on $B^\bullet(\mathrm{M})$.

Theorem 5.4. *Up to isomorphism there is a unique indecomposable $B^\bullet(\mathrm{M})$-module direct summand of $\mathrm{CH}^\bullet(\mathrm{M})$ containing $B^\bullet(\mathrm{M})$. This direct summand is the* ***intersection cohomology*** *$IH^\bullet(\mathrm{M})$ of* M.

In fact, the authors of [BHM$^+$] establish a canonical decomposition of $\mathrm{CH}^\bullet(\mathrm{M})$ as a $B^\bullet(\mathrm{M})$-module, and they identify the direct summand $IH^\bullet(\mathrm{M})$. This decomposition along with the Kähler package for $\mathrm{CH}^\bullet(\mathrm{M})$ is then fed into a highly intricate version of the general strategy for establishing the Kähler package outlined in Section 4.2, resulting in the following main theorem.

Theorem 5.5 ([BHM$^+$, Theorem 1.6])**.** *The intersection cohomology $IH^\bullet(\mathrm{M})$ of a matroid* M *satisfies the Kähler package with $\mathcal{K} = \{\sum_{e \in E} c_e y_e : c_e > 0\}$.*

As a corollary, for any positive linear combination $\ell = \sum_{e \in E} c_e y_e$ and $0 \leq i \leq j \leq r - i$, we have a commuting diagram

$$\begin{array}{ccc} B^i(\mathrm{M}) & \hookrightarrow & IH^i(\mathrm{M}) \\ {\scriptstyle \cdot \ell^{j-i}}\downarrow & & \downarrow{\scriptstyle \cdot \ell^{j-i}} \\ B^j(\mathrm{M}) & \hookrightarrow & IH^j(\mathrm{M}) \end{array}$$

where the right vertical map is injective by the hard Lefschetz property of $IH^\bullet(\mathrm{M})$. The left vertical map is thus injective, and the desired top-heaviness of the sequence (e) follows.

6. Conclusion

Matroids are combinatorial structures that capture the essence of independence. There were several conjectures about the behavior of sequences of invariants of a matroid, involving log-concavity or top-heaviness. June Huh and his collaborators made fundamental contribution to matroid theory [AHK18, ADH22, BHM$^+$], resolving many of these conjectures. They began by answering the conjectures for realizable matroids using algebraic geometry, a significant step on its own. Then, with considerable effort, they were able to extract the combinatorial heart, and

establish Hodge-theoretic properties for arbitrary, not necessarily realizable, matroids. This development of the Hodge theory of matroids forms an integral part of the foundation for studying matroids from an algebro-geometric perspective.

Acknowledgments

The author thanks June Huh for helpful conversations, and Matt Baker, Andrew Berget, Tom Braden, Eric Katz, Matt Larson, and Lauren Williams for helpful suggestions and comments on a preliminary draft of this article.

About the author

Chris Eur received his PhD at University of California, Berkeley, under David Eisenbud in 2020. He is currently a postdoctoral fellow at Harvard University. His research interests concern the interplay between combinatorics and algebraic geometry, particularly in the context of matroids and their generalizations.

References

[AA] Marcelo Aguiar and Federico Ardila, *Hopf monoids and generalized permutahedra*, Memoirs of the American Mathematical Society (to appear).

[AB16] Federico Ardila and Adam Boocher, *The closure of a linear space in a product of lines*, J. Algebraic Combin. **43** (2016), no. 1, 199–235, DOI 10.1007/s10801-015-0634-x. MR3439307

[ADH22] Federico Ardila, Graham Denham, and June Huh, *Lagrangian geometry of matroids*, J. Amer. Math. Soc. **36** (2023), no. 3, 727–794, DOI 10.1090/jams/1009. MR4583774

[AHK18] Karim Adiprasito, June Huh, and Eric Katz, *Hodge theory for combinatorial geometries*, Ann. of Math. (2) **188** (2018), no. 2, 381–452, DOI 10.4007/annals.2018.188.2.1. MR3862944

[AK06] Federico Ardila and Caroline J. Klivans, *The Bergman complex of a matroid and phylogenetic trees*, J. Combin. Theory Ser. B **96** (2006), no. 1, 38–49, DOI 10.1016/j.jctb.2005.06.004. MR2185977

[AKMW02] Dan Abramovich, Kalle Karu, Kenji Matsuki, and Jarosław Włodarczyk, *Torification and factorization of birational maps*, J. Amer. Math. Soc. **15** (2002), no. 3, 531–572, DOI 10.1090/S0894-0347-02-00396-X. MR1896232

[ALOGV18] Nima Anari, Kuikui Liu, Shayan Oveis-Gharan, and Cynthia Vinzant, *Log-concave polynomials II: High-dimensional walks and an FPRAS for counting bases of a matroid*, STOC'19—Proceedings of the 51st Annual ACM SIGACT Symposium on Theory of Computing, ACM, New York, 2019, pp. 1–12, DOI 10.1145/3313276.3316385. MR4003314

[Alu99] Paolo Aluffi, *Differential forms with logarithmic poles and Chern-Schwartz-MacPherson classes of singular varieties* (English, with English and French summaries), C. R. Acad. Sci. Paris Sér. I Math. **329** (1999), no. 7, 619–624, DOI 10.1016/S0764-4442(00)80012-9. MR1717120

[Alu05] Paolo Aluffi, *Characteristic classes of singular varieties*, Topics in cohomological studies of algebraic varieties, Trends Math., Birkhäuser, Basel, 2005, pp. 1–32, DOI 10.1007/3-7643-7342-3_1. MR2143071

[AP] Omid Amini and Matthieu Piquerez, *Hodge theory for tropical varieties*, `arXiv:2007.07826`, 2020.

[Ard22] Federico Ardila, *The geometry of matroids*, Notices Amer. Math. Soc. **65** (2018), no. 8, 902–908. MR3823027

[Bak18] Matthew Baker, *Hodge theory in combinatorics*, Bull. Amer. Math. Soc. (N.S.) **55** (2018), no. 1, 57–80, DOI 10.1090/bull/1599. MR3737210

[BBD82] A. A. Beĭlinson, J. Bernstein, and P. Deligne, *Faisceaux pervers* (French), Analysis and topology on singular spaces, I (Luminy, 1981), Astérisque, vol. 100, Soc. Math. France, Paris, 1982, pp. 5–171. MR751966

[BE09] Anders Björner and Torsten Ekedahl, *On the shape of Bruhat intervals*, Ann. of Math. (2) **170** (2009), no. 2, 799–817, DOI 10.4007/annals.2009.170.799. MR2552108

[BES] Spencer Backman, Christopher Eur, and Connor Simpson, *Simplicial generation of Chow rings of matroids*, Sém. Lothar. Combin. **84B** (2020), Art. 52, 11. MR4138680

[BEST] Andrew Berget, Christopher Eur, Hunter Spink, and Dennis Tseng, *Tautological classes of matroids*, Invent. Math. **233** (2023), no. 2, 951–1039, DOI 10.1007/s00222-023-01194-5. MR4607725

[BH20] Petter Brändén and June Huh, *Lorentzian polynomials*, Ann. of Math. (2) **192** (2020), no. 3, 821–891, DOI 10.4007/annals.2020.192.3.4. MR4172622

[BHM+] Tom Braden, June Huh, Jacob Matherne, Nicholas Proudfoot, and Botong Wang, *Singular Hodge theory for combinatorial geometries*, `arXiv:2010.06088`, 2020

[BHM+22] Tom Braden, June Huh, Jacob P. Matherne, Nicholas Proudfoot, and Botong Wang, *A semi-small decomposition of the Chow ring of a matroid*. Part A, Adv. Math. **409** (2022), Paper No. 108646, 49, DOI 10.1016/j.aim.2022.108646. MR4477425

[BL] Petter Brändén and Jonathan Leake, *Lorentzian polynomials on cones and the Heron-Rota-Welsh conjecture*, `arXiv:2110.00487`, 2021.

[BO92] Thomas Brylawski and James Oxley, *The Tutte polynomial and its applications*, Matroid applications, Encyclopedia Math. Appl., vol. 40, Cambridge Univ. Press, Cambridge, 1992, pp. 123–225, DOI 10.1017/CBO9780511662041.007. MR1165543

[Bre94] Francesco Brenti, *Log-concave and unimodal sequences in algebra, combinatorics, and geometry: an update*, Jerusalem combinatorics '93, Contemp. Math., vol. 178, Amer. Math. Soc., Providence, RI, 1994, pp. 71–89, DOI 10.1090/conm/178/01893. MR1310575

[Bri96] Michel Brion, *Piecewise polynomial functions, convex polytopes and enumerative geometry*, Parameter spaces (Warsaw, 1994), Banach Center Publ., vol. 36, Polish Acad. Sci. Inst. Math., Warsaw, 1996, pp. 25–44. MR1481477

[Bry77] Tom Brylawski, *The broken-circuit complex*, Trans. Amer. Math. Soc. **234** (1977), no. 2, 417–433, DOI 10.2307/1997928. MR468931

[Bry82] Thomas Brylawski, *The Tutte polynomial. I. General theory*, Matroid theory and its applications, Liguori, Naples, 1982, pp. 125–275. MR863010

[BST] Andrew Berget, Hunter Spink, and Dennis Tseng, *Log-concavity of matroid h-vectors and mixed eulerian numbers*, `arXiv:2005.01937`, 2020.

[Cat08] Eduardo Cattani, *Mixed Lefschetz theorems and Hodge-Riemann bilinear relations*, Int. Math. Res. Not. IMRN **10** (2008), Art. ID rnn025, 20, DOI 10.1093/imrn/rnn025. MR2429243

[CDFV11] D. Cohen, G. Denham, M. Falk, and A. Varchenko, *Critical points and resonance of hyperplane arrangements*, Canad. J. Math. **63** (2011), no. 5, 1038–1057, DOI 10.4153/CJM-2011-028-8. MR2866070

[CLS11] David A. Cox, John B. Little, and Henry K. Schenck, *Toric varieties*, Graduate Studies in Mathematics, vol. 124, American Mathematical Society, Providence, RI, 2011, DOI 10.1090/gsm/124. MR2810322

[Cra69] Henry H. Crapo, *The Tutte polynomial*, Aequationes Math. **3** (1969), 211–229, DOI 10.1007/BF01817442. MR262095

[Dan78] V. I. Danilov, *The geometry of toric varieties* (Russian), Uspekhi Mat. Nauk **33** (1978), no. 2(200), 85–134, 247. MR495499

[Daw84] Jeremy E. Dawson, *A collection of sets related to the Tutte polynomial of a matroid*, Graph theory, Singapore 1983, Lecture Notes in Math., vol. 1073, Springer, Berlin, 1984, pp. 193–204, DOI 10.1007/BFb0073117. MR761018

[dBE48] N. G. de Bruijn and P. Erdős, *On a combinatorial problem*, Nederl. Akad. Wetensch., Proc. **51** (1948), 1277–1279 = Indagationes Math. 10, 421–423 (1948). MR28289

[dCM02] Mark Andrea A. de Cataldo and Luca Migliorini, *The hard Lefschetz theorem and the topology of semismall maps* (English, with English and French summaries), Ann. Sci. École Norm. Sup. (4) **35** (2002), no. 5, 759–772, DOI 10.1016/S0012-9593(02)01108-4. MR1951443

[dCM09] Mark Andrea A. de Cataldo and Luca Migliorini, *The decomposition theorem, perverse sheaves and the topology of algebraic maps*, Bull. Amer. Math. Soc. (N.S.) **46** (2009), no. 4, 535–633, DOI 10.1090/S0273-0979-09-01260-9. MR2525735

[DCP95] C. De Concini and C. Procesi, *Wonderful models of subspace arrangements*, Selecta Math. (N.S.) **1** (1995), no. 3, 459–494, DOI 10.1007/BF01589496. MR1366622

[Del71] Pierre Deligne, *Théorie de Hodge. II* (French), Inst. Hautes Études Sci. Publ. Math. **40** (1971), 5–57. MR498551

[DGS12] Graham Denham, Mehdi Garrousian, and Mathias Schulze, *A geometric deletion-restriction formula*, Adv. Math. **230** (2012), no. 4-6, 1979–1994, DOI 10.1016/j.aim.2012.04.003. MR2927361

[Dim17] Alexandru Dimca, *Hyperplane arrangements*: *An introduction*, Universitext, Springer, Cham, 2017, DOI 10.1007/978-3-319-56221-6. MR3618796

[DP03] Alexandru Dimca and Stefan Papadima, *Hypersurface complements, Milnor fibers and higher homotopy groups of arrangments*, Ann. of Math. (2) **158** (2003), no. 2, 473–507, DOI 10.4007/annals.2003.158.473. MR2018927

[DR22] Jeshu Dastidar and Dustin Ross, *Matroid psi classes*, Selecta Math. (N.S.) **28** (2022), no. 3, Paper No. 55, 38, DOI 10.1007/s00029-022-00771-5. MR4405747

[DW74] Thomas A. Dowling and Richard M. Wilson, *The slimmest geometric lattices*, Trans. Amer. Math. Soc. **196** (1974), 203–215, DOI 10.2307/1997023. MR345849

[DW75] Thomas A. Dowling and Richard M. Wilson, *Whitney number inequalities for geometric lattices*, Proc. Amer. Math. Soc. **47** (1975), 504–512, DOI 10.2307/2039773. MR354422

[EH16] David Eisenbud and Joe Harris, *3264 and all that—a second course in algebraic geometry*, Cambridge University Press, Cambridge, 2016, DOI 10.1017/CBO9781139062046. MR3617981

[EHL] Christopher Eur, June Huh, and Matt Larson, *Stellahedral geometry of matroids*, `arXiv:2207.10605`, 2022.

[EPW16] Ben Elias, Nicholas Proudfoot, and Max Wakefield, *The Kazhdan-Lusztig polynomial of a matroid*, Adv. Math. **299** (2016), 36–70, DOI 10.1016/j.aim.2016.05.005. MR3519463

[Eur20] Christopher Eur, *Divisors on matroids and their volumes*, J. Combin. Theory Ser. A **169** (2020), 105135, 31, DOI 10.1016/j.jcta.2019.105135. MR4011081

[EW14] Ben Elias and Geordie Williamson, *The Hodge theory of Soergel bimodules*, Ann. of Math. (2) **180** (2014), no. 3, 1089–1136, DOI 10.4007/annals.2014.180.3.6. MR3245013

[FS97] William Fulton and Bernd Sturmfels, *Intersection theory on toric varieties*, Topology **36** (1997), no. 2, 335–353, DOI 10.1016/0040-9383(96)00016-X. MR1415592

[Ful93] William Fulton, *Introduction to toric varieties*, Annals of Mathematics Studies, vol. 131, Princeton University Press, Princeton, NJ, 1993. The William H. Roever Lectures in Geometry, DOI 10.1515/9781400882526. MR1234037

[Ful98] William Fulton, *Intersection theory*, 2nd ed., Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 2, Springer-Verlag, Berlin, 1998, DOI 10.1007/978-1-4612-1700-8. MR1644323

[FY04] Eva Maria Feichtner and Sergey Yuzvinsky, *Chow rings of toric varieties defined by atomic lattices*, Invent. Math. **155** (2004), no. 3, 515–536, DOI 10.1007/s00222-003-0327-2. MR2038195

[GGMS87] I. M. Gel′fand, R. M. Goresky, R. D. MacPherson, and V. V. Serganova, *Combinatorial geometries, convex polyhedra, and Schubert cells*, Adv. in Math. **63** (1987), no. 3, 301–316, DOI 10.1016/0001-8708(87)90059-4. MR877789

[GM83] Mark Goresky and Robert MacPherson, *Intersection homology. II*, Invent. Math. **72** (1983), no. 1, 77–129, DOI 10.1007/BF01389130. MR696691

[Har77] Robin Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, No. 52, Springer-Verlag, New York-Heidelberg, 1977. MR0463157

[Her72] A. P. Heron, *Matroid polynomials*, Combinatorics (Proc. Conf. Combinatorial Math., Math. Inst., Oxford, 1972), Inst. Math. Appl., Southend-on-Sea, 1972, pp. 164–202. MR0340058

[Hib89] Takayuki Hibi, *Flawless O-sequences and Hilbert functions of Cohen-Macaulay integral domains*, J. Pure Appl. Algebra **60** (1989), no. 3, 245–251, DOI 10.1016/0022-4049(89)90085-6. MR1021849

[Hib92] Takayuki Hibi, *Face number inequalities for matroid complexes and Cohen-Macaulay types of Stanley-Reisner rings of distributive lattices*, Pacific J. Math. **154** (1992), no. 2, 253–264. MR1159510

[HK12] June Huh and Eric Katz, *Log-concavity of characteristic polynomials and the Bergman fan of matroids*, Math. Ann. **354** (2012), no. 3, 1103–1116, DOI 10.1007/s00208-011-0777-6. MR2983081

[Hog74] S. G. Hoggar, *Chromatic polynomials and logarithmic concavity*, J. Combinatorial Theory Ser. B **16** (1974), 248–254, DOI 10.1016/0095-8956(74)90071-9. MR342424

[HS14] June Huh and Bernd Sturmfels, *Likelihood geometry*, Combinatorial algebraic geometry, Lecture Notes in Math., vol. 2108, Springer, Cham, 2014, pp. 63–117, DOI 10.1007/978-3-319-04870-3_3. MR3329087

[Huh12] June Huh, *Milnor numbers of projective hypersurfaces and the chromatic polynomial of graphs*, J. Amer. Math. Soc. **25** (2012), no. 3, 907–927, DOI 10.1090/S0894-0347-2012-00731-0. MR2904577

[Huh13] June Huh, *The maximum likelihood degree of a very affine variety*, Compos. Math. **149** (2013), no. 8, 1245–1266, DOI 10.1112/S0010437X13007057. MR3103064

[Huh15] June Huh, *h-vectors of matroids and logarithmic concavity*, Adv. Math. **270** (2015), 49–59, DOI 10.1016/j.aim.2014.11.002. MR3286530

[Huh18] June Huh, *Tropical geometry of matroids*, Current developments in mathematics 2016, Int. Press, Somerville, MA, 2018, pp. 1–46. MR3837872

[Huh22] June Huh, *Combinatorics and Hodge theory*, 2022.

[Huy05] Daniel Huybrechts, *Complex geometry: An introduction*, Universitext, Springer-Verlag, Berlin, 2005. MR2093043

[HW17] June Huh and Botong Wang, *Enumeration of points, lines, planes, etc*, Acta Math. **218** (2017), no. 2, 297–317, DOI 10.4310/ACTA.2017.v218.n2.a2. MR3733101

[IKMZ19] Ilia Itenberg, Ludmil Katzarkov, Grigory Mikhalkin, and Ilia Zharkov, *Tropical homology*, Math. Ann. **374** (2019), no. 1-2, 963–1006, DOI 10.1007/s00208-018-1685-9. MR3961331

[JKL18] Martina Juhnke-Kubitzke and Dinh Van Le, *Flawlessness of h-vectors of broken circuit complexes*, Int. Math. Res. Not. IMRN **5** (2018), 1347–1367, DOI 10.1093/imrn/rnw284. MR3801465

[Kat16] Eric Katz, *Matroid theory for algebraic geometers*, Nonarchimedean and tropical geometry, Simons Symp., Springer, [Cham], 2016, pp. 435–517. MR3702317

[KP08] Eric Katz and Sam Payne, *Piecewise polynomials, Minkowski weights, and localization on toric varieties*, Algebra Number Theory **2** (2008), no. 2, 135–155, DOI 10.2140/ant.2008.2.135. MR2377366

[Laf03] L. Lafforgue, *Chirurgie des grassmanniennes* (French), CRM Monograph Series, vol. 19, American Mathematical Society, Providence, RI, 2003, DOI 10.1090/crmm/019. MR1976905

[Laz04a] Robert Lazarsfeld, *Positivity in algebraic geometry. I: Classical setting: line bundles and linear series*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 48, Springer-Verlag, Berlin, 2004, DOI 10.1007/978-3-642-18808-4. MR2095471

[Laz04b] Robert Lazarsfeld, *Positivity in algebraic geometry. II: Positivity for vector bundles, and multiplier ideals*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 49, Springer-Verlag, Berlin, 2004, DOI 10.1007/978-3-642-18808-4. MR2095472

[LdMRS20] Lucía López de Medrano, Felipe Rincón, and Kristin Shaw, *Chern-Schwartz-MacPherson cycles of matroids*, Proc. Lond. Math. Soc. (3) **120** (2020), no. 1, 1–27, DOI 10.1112/plms.12278. MR3999674

[Len13] Matthias Lenz, *The f-vector of a representable-matroid complex is log-concave*, Adv. in Appl. Math. **51** (2013), no. 5, 543–545, DOI 10.1016/j.aam.2013.07.001. MR3118543

[Mac74] R. D. MacPherson, *Chern classes for singular algebraic varieties*, Ann. of Math. (2) **100** (1974), 423–432, DOI 10.2307/1971080. MR361141

[Mas72] J. H. Mason, *Matroids: unimodal conjectures and Motzkin's theorem*, Combinatorics (Proc. Conf. Combinatorial Math., Math. Inst., Oxford, 1972), Inst. Math. Appl., Southend-on-Sea, 1972, pp. 207–220. MR0349445

[McM93] Peter McMullen, *On simple polytopes*, Invent. Math. **113** (1993), no. 2, 419–444, DOI 10.1007/BF01244313. MR1228132

[MS15] Diane Maclagan and Bernd Sturmfels, *Introduction to tropical geometry*, Graduate Studies in Mathematics, vol. 161, American Mathematical Society, Providence, RI, 2015, DOI 10.1090/gsm/161. MR3287221

[Nel18] Peter Nelson, *Almost all matroids are nonrepresentable*, Bull. Lond. Math. Soc. **50** (2018), no. 2, 245–248, DOI 10.1112/blms.12141. MR3830117

[OS80] Peter Orlik and Louis Solomon, *Combinatorics and topology of complements of hyperplanes*, Invent. Math. **56** (1980), no. 2, 167–189, DOI 10.1007/BF01392549. MR558866

[OT92] Peter Orlik and Hiroaki Terao, *Arrangements of hyperplanes*, Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 300, Springer-Verlag, Berlin, 1992, DOI 10.1007/978-3-662-02772-1. MR1217488

[Oxl11] James Oxley, *Matroid theory*, 2nd ed., Oxford Graduate Texts in Mathematics, vol. 21, Oxford University Press, Oxford, 2011, DOI 10.1093/acprof:oso/9780198566946.001.0001. MR2849819

[Pro18] Nicholas Proudfoot, *The algebraic geometry of Kazhdan-Lusztig-Stanley polynomials*, EMS Surv. Math. Sci. **5** (2018), no. 1-2, 99–127, DOI 10.4171/EMSS/28. MR3880222

[PS06] Nicholas Proudfoot and David Speyer, *A broken circuit ring*, Beiträge Algebra Geom. **47** (2006), no. 1, 161–166. MR2246531

[PXY18] Nicholas Proudfoot, Yuan Xu, and Ben Young, *The Z-polynomial of a matroid*, Electron. J. Combin. **25** (2018), no. 1, Paper No. 1.26, 21, DOI 10.37236/7105. MR3785005

[Rea68] Ronald C. Read, *An introduction to chromatic polynomials*, J. Combinatorial Theory **4** (1968), 52–71. MR224505

[Rot71] Gian-Carlo Rota, *Combinatorial theory, old and new*, Actes du Congrès International des Mathématiciens (Nice, 1970), Gauthier-Villars, Paris, 1971, pp. 229–233. MR0505646

[Sch93] Werner Schwärzler, *The coefficients of the Tutte polynomial are not unimodal*, J. Combin. Theory Ser. B **58** (1993), no. 2, 240–242, DOI 10.1006/jctb.1993.1040. MR1223696

[Spe08] David E. Speyer, *Tropical linear spaces*, SIAM J. Discrete Math. **22** (2008), no. 4, 1527–1558, DOI 10.1137/080716219. MR2448909

[Sta80] Richard P. Stanley, *The number of faces of a simplicial convex polytope*, Adv. in Math. **35** (1980), no. 3, 236–238, DOI 10.1016/0001-8708(80)90050-X. MR563925

[Sta89] Richard P. Stanley, *Log-concave and unimodal sequences in algebra, combinatorics, and geometry*, Graph theory and its applications: East and West (Jinan, 1986), Ann. New York Acad. Sci., vol. 576, New York Acad. Sci., New York, 1989, pp. 500–535, DOI 10.1111/j.1749-6632.1989.tb16434.x. MR1110850

[Stu02] Bernd Sturmfels, *Solving systems of polynomial equations*, CBMS Regional Conference Series in Mathematics, vol. 97, Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2002, DOI 10.1090/cbms/097. MR1925796

[Swa03] Ed Swartz, *g-elements of matroid complexes*, J. Combin. Theory Ser. B **88** (2003), no. 2, 369–375, DOI 10.1016/S0095-8956(03)00038-8. MR1983365

[Ter02] Hiroaki Terao, *Algebras generated by reciprocals of linear forms*, J. Algebra **250** (2002), no. 2, 549–558, DOI 10.1006/jabr.2001.9121. MR1899865

[Tev07] Jenia Tevelev, *Compactifications of subvarieties of tori*, Amer. J. Math. **129** (2007), no. 4, 1087–1104, DOI 10.1353/ajm.2007.0029. MR2343384

[Tut67] W. T. Tutte, *On dichromatic polynominals*, J. Combinatorial Theory **2** (1967), 301–320. MR223272

[Var95] A. Varchenko, *Critical points of the product of powers of linear functions and families of bases of singular vectors*, Compositio Math. **97** (1995), no. 3, 385–401. MR1353281

[Voi02] Claire Voisin, *Hodge theory and complex algebraic geometry. I*, Cambridge Studies in Advanced Mathematics, vol. 76, Cambridge University Press, Cambridge, 2002. Translated from the French original by Leila Schneps, DOI 10.1017/CBO9780511615344. MR1967689

[Web04] Andrzej Weber, *Pure homology of algebraic varieties*, Topology **43** (2004), no. 3, 635–644, DOI 10.1016/j.top.2003.09.001. MR2041634

[Wel71] D. J. A. Welsh, *Combinatorial problems in matroid theory*, Combinatorial Mathematics and its Applications (Proc. Conf., Oxford, 1969), Academic Press, London, 1971, pp. 291–306. MR0278975

[Wel76] D. J. A. Welsh, *Matroid theory*, L. M. S. Monographs, No. 8, Academic Press [Harcourt Brace Jovanovich, Publishers], London-New York, 1976. MR0427112

[Whi32] Hassler Whitney, *The coloring of graphs*, Ann. of Math. (2) **33** (1932), no. 4, 688–718, DOI 10.2307/1968214. MR1503085

[Wło97] Jarosław Włodarczyk, *Decomposition of birational toric maps in blow-ups & blow-downs*, Trans. Amer. Math. Soc. **349** (1997), no. 1, 373–411, DOI 10.1090/S0002-9947-97-01701-7. MR1370654

[Zas87] Thomas Zaslavsky, *The Möbius function and the characteristic polynomial*, Combinatorial geometries, Encyclopedia Math. Appl., vol. 29, Cambridge Univ. Press, Cambridge, 1987, pp. 114–138. MR921071

[Zie95] Günter M. Ziegler, *Lectures on polytopes*, Graduate Texts in Mathematics, vol. 152, Springer-Verlag, New York, 1995, DOI 10.1007/978-1-4613-8431-1. MR1311028

Harvard University
Email address: ceur@math.harvard.edu