BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 61, Number 1, January 2024, Pages 173–185
https://doi.org/10.1090/bull/1818
Article electronically published on October 12, 2023

SELECTED MATHEMATICAL REVIEWS

related to the paper in the previous section by
HENRY COHN

MR0920369 (89a:11067) 11H31; 05B40, 11H06, 20E32, 52A43, 52A45, 94C30

Conway, J. H.; Sloane, N. J. A.

Sphere packings, lattices and groups. (English)

Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 290.

Springer-Verlag, *New York*, 1988, xxviii+663 pp., \$87.00, ISBN 0-387-96617-X

The book is a landmark in the literature on sphere packings. It is mainly concerned with the problem of packing spheres in d-dimensional Euclidean space E^d. The authors also study closely related problems as, e.g., the kissing number problem (how many spheres touch a given central sphere?); the problem of covering E^d in the least dense way by equal spheres; and the classification of lattices and quadratic forms. The book also deals with applications of these geometric problems to other areas of mathematics (mainly number theory) and to areas outside mathematics, mainly the channel coding problem, but also crystals and quasicrystals.

Two mathematical objects play a central role in this book, namely the famous sphere packings in the E_8-lattice and in the Leech lattice Λ_{24}. As the authors formulate, one could say "that the book is devoted to studying these two lattices and their properties". Also remarkable are the informative tables and graphs which are helpful for the reader.

The book contains a lot of new material, mainly from papers by the two authors, but also remarkable contributions by others (e.g. by Bannai, Leech, Norton, Odlyzko, Parker, Queen, and Venkov). In fact the book was originally planned as a collection of some important papers by these authors which can be seen from its structure and the different chapters. However, in its final form it has become much more than such a collection—it is a successful synthesis of these papers.

J. M. Wills

From MathSciNet, October 2023

MR1681101 (2000b:57039) 57R17; 53D35, 57R57

Biran, Paul

A stability property of symplectic packing.

Inventiones Mathematicae **136** (1999), *no.* 1, 123–155.

Optimally filling the entire volume of a compact manifold with N balls of the same radius r by volume preserving maps for given N is clearly unobstructed. On the other hand there is an obvious obstruction to full packings when filling by isometric embeddings, like for instance Euclidean embeddings in the famous Kepler conjecture about densest sphere packings. In this perspective packings by symplectic embeddings represent an intermediate case. Given a compact symplectic manifold (M,ω), let $v_N(M,\omega)$ denote the supremum of the ratio of volumes $N \cdot \mathrm{vol}(B^{2n}(r))/\mathrm{vol}(M,\omega)$ such that there exist disjoint symplectic embeddings of N

copies of the ball $B^{2n}(r)$ of any radius r into M. The situation $v_N(M,\omega) < 1$ represents a packing obstruction, $v_N = 1$ signifies full packings.

This symplectic packing problem was first addressed by M. Gromov in his seminal paper [Invent. Math. **82** (1985), no. 2, 307–347; MR0809718] as an extension of the non-squeezing phenomenon. Using the method of pseudoholomorphic curves for almost complex structures tamed by ω he showed for example that $v_N(B^{2n}(1)) \leq N/2^n$ for $1 < N < 2^n$. Packing $B^{2n}(1)$ is equivalent to packing $(\mathbf{C}\mathrm{P}^n, \omega_0)$ with $\omega_0([\mathbf{C}\mathrm{P}^1]) = \pi$. Since one can find a line through the centers of any two of the embedded balls, the lower bound πr^2 for the area of a holomorphic curve through the center of a ball $B^{2n}(r)$ implies that $2r^2 < 1$. Thus, packing obstructions are obtained by finding holomorphic curves through a prescribed number of points in general position.

Further essential progress was achieved by D. McDuff and L. Polterovich [Invent. Math. **115** (1994), no. 3, 405–434; MR1262938] based on the unique correspondence between symplectically embedded balls and symplectic blowing-up. Given an embedded ball of radius r, one can find a symplectic form on the complex blow-up $\tilde{M}$ of the center such that its cohomology class is given by $[\Theta^*\omega] - \pi r^2 e$, where e is the Poincaré dual of the exceptional divisor E resulting from the blowing-up. In fact, symplectic blowing-up can be understood as removing the interior of the symplectic ball of radius r and collapsing the remaining bounding sphere to the exceptional divisor. In dimension 4 symplectic blowing-up for N embedded balls thus encounters obstructions in terms of evaluating the symplectic form in the class $[\Theta^*\omega] - \pi r^2 \sum_{q=1}^N e_q$ on $\tilde{M}$ on holomorphic curves in the classes $d \cdot \mathrm{PD}(\Theta^*\omega) - \sum_{q=1}^N m_q E_q$, viz. necessarily $d \int_M \omega \wedge \omega \geq \pi r^2 \sum m_q$. The crucial problem is therefore to determine tuples $(d, m_1, \cdots, m_N) \in \mathbf{Z}_+^{N+1}$ which can be represented by holomorphic curves. On the other hand, symplectic blowing-down provides a construction of symplectic packings. The problem here is to find symplectic forms representing the class $[\Theta^*\omega] - \pi r^2 \sum_{q=1}^N e_q$ which are Kähler near the exceptional divisors.

Whereas the results by Gromov and McDuff-Polterovich mainly focus on packing obstructions, the article under review deals with the existence of full packings. The main result states that for any four-dimensional (M,ω) with rational class $[\omega] \in H^2(M, \mathbf{Q})$ there exists N_0 such that $v_N(M,\omega) = 1$ for $N \geq N_0$. That is, for any such symplectic four-manifold there are at most finitely many packing obstructions. Moreover, if the Poincaré dual of $k_0[\omega]$ can be represented by a symplectic manifold of genus at least 1, then one has $N_0 = k_0^2 \int_M \omega \wedge \omega$. The latter statement relates the search for the threshold N_0 of stable full packings to the minimal k_0 in S. K. Donaldson's seminal theorem [J. Differential Geom. **44** (1996), no. 4, 666–705; MR1438190] providing symplectic submanifolds Poincaré dual to $k_0[\omega]$ for k_0 sufficiently large. In examples, Biran finds the optimal N_0 by this approach, e.g. $N_0 = 9$ for $\mathbf{C}\mathrm{P}^2$.

The essential idea behind the proof of the main theorem is to focus on a symplectic tubular neighborhood of the symplectic submanifold Σ Poincaré dual to $k_0[\omega]$. Namely, symplectically compactifying the normal bundle $N \to \Sigma$ by adding a section Z_∞, at infinity one obtains a so-called ruled surface $S \to \Sigma$. In [Geom. Funct. Anal. **7** (1997), no. 3, 420–437; MR1466333] Biran already developed techniques for achieving full packings of such ruled surfaces. Using a crucial inflating technique due to Lalonde and McDuff one can enlarge the symplectic neighborhood of Σ until it essentially fills out the entire volume of M. Thus it remains to fill $S \backslash Z_\infty$. Biran shows that $v_N(S\backslash Z_\infty, \omega_S) = 1$ if $N \geq (\int_S \omega_S \wedge \omega_S)/(\int_F \omega_S)^2$, where F is the class of the fiber of $S \to \Sigma$.

In order to obtain the full packing of the ruled surface $S\backslash Z_\infty$, one proceeds by starting with N sufficiently small disjointly embedded balls not intersecting the section at infinity. Using again a suitably generalized inflation technique the balls are enlarged without intersecting Z_∞ until the volume is filled. The inflation lemma works by deforming the symplectic structure of the ruled surface symplectically blown-up at the centers of the embedded small balls. The deformation adds to $\tilde{\omega}$ a form Poincaré dual to a suitably chosen homology-2-class represented by a symplectic 2-dimensional submanifold $C \subset \tilde{S}$. In order to apply this inflation technique, however, the crucial condition of positivity of intersections of C with Z_∞ and all exceptional divisors E_q, $q = 1, \cdots, N$, has to be guaranteed. This essential condition constitutes the hardest part of the construction of the packing. It is here, where one observes a lower bound for N, where the genus of Σ has to be at least 1 and where the structure of S as a ruled surface plays a role, namely in terms of a four-manifold of Seiberg-Witten non-simple type. The idea is to represent all necessary 2-dimensional symplectic submanifolds by pseudoholomorphic curves, so that positivity of intersection in dimension 4 follows. The essential technique to achieve this is due to McDuff [in *Topics in symplectic 4-manifolds (Irvine, CA, 1996)*, 85–99, Internat. Press, Cambridge, MA, 1998; MR1635697]. One of the crucial conditions is that the class of C has to intersect all exceptional spheres nonnegatively. This restricts N from below.

The article under review fills a large white area on the map of symplectic topology, answering the important question of how far one can expect symplectic packing obstructions. The link to Donaldson's symplectic submanifold theorem is fascinating. Biran makes elegant use of the rich and powerful theory of holomorphic curves at his disposal, based on essential contributions due to Gromov, Taubes, McDuff and Donaldson. The article provides valuable insight into a highly modern field and is very clearly written. In particular, the introduction and the outline of the proof are perfectly accessible even to the non-expert in this area. It is fascinating to note how strongly the holomorphic curve method relies on dimension 4, largely due to positivity of intersections. It is an intriguing problem to find similar results on symplectic packing obstructions for higher dimensions.

Matthias Schwarz

From MathSciNet, October 2023

MR1797293 (2002a:52020) 52C20; 51M16

Hales, T. C.

The honeycomb conjecture.

Discrete & Computational Geometry. An International Journal of Mathematics and Computer Science **25** (2001), *no.* 1, 1–22.

If the Euclidean plane is to be partitioned into cells of equal area, how should the shapes of the cells be designed to minimize the average perimeter of the cells? Both the question and the "obvious" answer (each cell should be a regular hexagon, as in a honeycomb) date back to antiquity, but the precise statement of the problem had to wait for the development of the notions and language of modern mathematics (What exactly is a cell? How is a cell's perimeter measured and what is meant by the average?). Then it turned out that the answer is not as obvious as it seemed, and the question became known as the honeycomb conjecture. Simple to state but surprisingly difficult to solve, the conjecture remained open for a long time. A special case of the question, requiring that all cells be congruent, is attributed to H. Steinhaus by H. T. Croft, K. J. Falconer, and R. K. Guy in Problem C15 of [*Unsolved problems in geometry*, Springer, New York, 1991; MR1107516].

L. Fejes Tóth made the first step towards proving the honeycomb conjecture. He confirmed it in [Math. Naturwiss. Anz. Ungar. Akad. Wiss. **62** (1943), 349–354; MR0024155] assuming convexity of the cells. While the convexity assumption seems natural (at first glance it seems that nothing can be gained by designing nonconvex cells), it forces the cells to be polygonal, excluding from consideration the potential counterexamples with cells whose boundary might contain circular arcs, as suggested by the isoperimetric inequality. Fejes Tóth expected that the honeycomb conjecture should hold true even without the convexity assumption, but he remarked that "its proof seems to involve considerable difficulties" [*Regular figures*, Macmillan, New York, 1964; MR0165423]. Nevertheless, his result was a significant breakthrough.

More recently, M. N. Bleicher [Studia Sci. Math. Hungar. **22** (1987), no. 1-4, 123–137; MR0913901] obtained some closely related results, concerning the structure of the optimal configurations of finitely many cells of prescribed areas, convexity not assumed. F. Morgan [Pacific J. Math. **165** (1994), no. 2, 347–361; MR1300837] made further progress in the direction of affirming the honeycomb conjecture. For a survey of the topic see Morgan's article [Trans. Amer. Math. Soc. **351** (1999), no. 5, 1753–1763; MR1615934].

In the paper under review, Thomas C. Hales presents a complete proof of the honeycomb conjecture. Several versions of the conjecture are proved, corresponding to various interpretations of the terms involved, and of varied degrees of generality. Theorem 1-A corresponds precisely to the classical, elementary statement of the conjecture: Let Γ be a locally finite graph in $\mathbf{R}^2$ consisting of smooth curves and such that $\mathbf{R}^2\backslash\Gamma$ has infinitely many bounded connected components, each of unit area. Let C be the union of these bounded components. Then

$$\limsup_{r\to\infty} \frac{\operatorname{perim}(C\cap B(0,r))}{\operatorname{area}(C\cap B(0,r))} \geq \sqrt[4]{12}.$$

Here $\operatorname{perim}(C\cap B(0,r))$ denotes the sum of the lengths of the parts of the edges of Γ that are contained in the circular disk $B(0,r)$ of radius r, centered at the origin. The inequality is sharp, since equality is attained by the honeycomb tiling of the plane with regular hexagons.

A very general form of the conjecture is stated as the Honeycomb Conjecture for Disconnected Regions (Theorem 1-B). Here the cells are just disjoint measurable sets with rectifiable boundaries, whose union is contained in a compact set. The cells need not be of equal areas, they need not be connected, and need not even form a tiling. A special case of this general form is the Finite Version (Theorem 2), followed by the Honeycomb Conjecture on a Torus (Theorem 3). In the latter formulation, the cells on a (flat) torus are rather simple, each being connected, simply connected, and bounded by a finite number of analytic arcs (edges). Yet this is the version that implies the general result. Theorem 3 includes uniqueness of the optimal solution: the regular hexagonal honeycomb on the torus. In the process of reduction to the torus version, Hales follows the approach outlined by Morgan and he uses Morgan's partial result for the case in which the number of cells is not too large. The solution of the torus version is based on the so-called Hexagonal Isoperimetric Inequality for Closed Plane Curves, a technical yet very crucial detail, displaying the author's impressive analytical skill and ingenuity.

W. Kuperberg

From MathSciNet, October 2023

MR1973059 (2004b:11096) 11H31; 52C17

Cohn, Henry; Elkies, Noam

New upper bounds on sphere packings. I.

Annals of Mathematics. Second Series **157** (2003), *no.* 2, 689–714.

The sphere packing problem asks for the largest fraction of $\mathbf{R}^n$ that can be covered by congruent balls, which may only intersect along their boundaries. This problem is solved currently only for dimension ≤ 3. The authors develop linear programming bounds and apply these to derive the currently best bounds for sphere packings in dimensions 4 to 36. The authors obtain particularly sharp bounds for dimension 8 and 24 and conjecture that their techniques can be extended to prove sharp bounds in these dimensions. The main tool of the article is the following theorem: Let $\widehat{f}(t) := \int_{\mathbf{R}^n} f(x)e^{2\pi i\langle x,t\rangle}dx$ denote the Fourier transform of $f\colon \mathbf{R}^n \to \mathbf{R}$; we assume that there is a constant $\delta > 0$ such that $|f(x)|$ and $|\widehat{f}(t)|$ are bounded from above by a constant factor of $(1+|x|)^{-n-\delta}$. Suppose, in addition, that f satisfies (1) $f(0) = \widehat{f}(0) > 0$, (2) $f(x) \leq 0$ for $|x| \geq r$, and (3) $\widehat{f}(t) \geq 0$ for all t. Then the center density of sphere packings in $\mathbf{R}^n$ is bounded from above by $(r/2)^n$.

Matthias Beck

From MathSciNet, October 2023

MR2409678 (2010b:11047) 11F11; 11-02, 11E45, 11F20, 11F25, 11F27, 11F67

Zagier, Don

Elliptic modular forms and their applications.

The 1-2-3 of modular forms, 1–103, *Universitext*, *Springer, Berlin*, 2008.

This article is part of a book of lecture notes from a summer school on modular forms and their applications which covers the classical theory of elliptic modular forms. At the same time, it gives a first glimpse into a book in preparation on this topic by the author, in which he plans to treat in much more detail some of the topics that are covered (or rather introduced) here. The present article is hence not an introduction from which the reader is supposed to learn the subject from scratch but rather a guided tour through a gallery of masterpieces from the art of modular forms, stopping briefly at each of the items on exhibit, explaining its genesis and calling the viewer's attention to some of its most striking features. The connoisseur, being led by an extraordinarily competent guide who is in fact himself one of the main artists, encounters enlightening new views of objects which had seemed quite familiar to him before this tour; the novice, who may have come across this tour accidentally, will probably leave the gallery amazed by the beauty of what he has seen but perhaps also somewhat bewildered, in any case hopefully with the firm intention to come back and learn more about what he has seen.

The article starts in the first two sections with a complete introduction to the basic function theory of modular forms for the full modular group $SL_2(\mathbb{Z})$, treating as applications the finiteness of the class number of positive-definite integral binary quadratic forms, divisor sum identities and congruences for the Ramanujan τ-function. We notice here that throughout the article the word "application" does not refer to something related to improvements in the industrial production of goods (sometimes also called "real world applications") but to inner mathematical applications, in fact mostly (but not exclusively) applications to other problems of number theory.

Section 3 deals with theta series and their applications to the theory of quadratic forms, the Kac-Wakimoto conjecture and isospectral manifolds ("drums"). Section 4 sketches Hecke's theory of Dirichlet series attached to modular forms and the striking connections of these series to arithmetic geometry. In Section 5, the action of differential operators on the ring of modular and quasimodular forms and the differential equations satisfied by these are discussed. Finally, Section 6, entitled "Singular moduli and complex multiplication", not only deals with these themes, but takes us to topics like Borcherds products, Taylor expansions of modular forms and (as an application) a discussion of the problem of which primes can be represented as the sum of two cubes.

Rainer Schulze-Pillot

From MathSciNet, October 2023

MR2601036 (2011d:52037) 52C17

Hales, Thomas C.; McLaughlin, Sean

The dodecahedral conjecture.

Journal of the American Mathematical Society **23** (2010), *no.* 2, 299–344.

A packing of balls of unit radius in $\mathbb{E}^3$ can be identified with the set Λ of the centers of the balls. For each center $v \in \Lambda$, let $\Omega(\Lambda, v)$ be its Voronoi cell. It consists of all points $x \in \mathbb{E}^3$ which are at least as close to v as to any other point $w \in \Lambda$. $\Omega(\Lambda, v)$ is a convex polytope. The *dodecahedral conjecture* asserts that in any packing of unit balls the volume of each Voronoi cell has volume at least that of a regular dodecahedron of inradius 1. Equality holds precisely in the case when the Voronoi cell is a regular dodecahedron of inradius 1.

The dodecahedron conjecture goes back to an article of L. Fejes Tóth [Math. Z. **48** (1943), 676–684; MR0009129]. It has resisted a series of attacks by geometers such as Fejes Tóth, C. A. Rogers and D. J. Muder and finally was solved in 1998. The present article contains the revised and many times rewritten proof. It is essentially the same as the original proof. For certain details the reader is referred to the expanded version of 2002.

The *Kepler conjecture* asserts that the maximum density of a packing of balls of unit radius in $\mathbb{E}^3$ is attained by lattice packings with face-centered cubic lattices. Its solution is due to T. C. Hales with the help of S. P. Ferguson [T. C. Hales, Discrete Comput. Geom. **36** (2006), no. 1, 5–20; MR2229657; T. C. Hales and S. P. Ferguson, Discrete Comput. Geom. **36** (2006), no. 1, 21–69; MR2229658; T. C. Hales, Discrete Comput. Geom. **36** (2006), no. 1, 71–110; MR2229659; Discrete Comput. Geom. **36** (2006), no. 1, 111–166; MR2229660; S. P. Ferguson, Discrete Comput. Geom. **36** (2006), no. 1, 167–204; MR2229661; T. C. Hales, Discrete Comput. Geom. **36**

(2006), no. 1, 205–265; MR2229662]. Neither conjecture follows from the other one, but the proofs share a series of ideas and methods.

The authors work on a project with the aim of providing a complete formalization of the proofs of both conjectures. This means that every logical inference of the proof can and has been checked by a computer.

The present work is a major breakthrough in discrete geometry.

Peter M. Gruber

From MathSciNet, October 2023

MR3012355 52C17; 03B35, 68T15

Hales, Thomas C.

Dense sphere packings. (English)

A blueprint for formal proofs.

London Mathematical Society Lecture Note Series, 400.

Cambridge University Press, *Cambridge*, 2012, xiv+271 pp., $60.00,

ISBN 978-0-521-61770-3

The Kepler conjecture states that the densest sphere packing of 3-dimensional space by equal spheres is attained by the FCC (face-centered cubic) lattice packing, which has density $\frac{\pi}{\sqrt{18}}$. In 2005 and 2006 T. C. Hales, together with S. P. Ferguson, published a proof of the Kepler Conjecture [T. C. Hales, Ann. of Math. (2) **162** (2005), no. 3, 1065–1185; MR2179728; Discrete Comput. Geom. **36** (2006), no. 1, 5–20; MR2229657; T. C. Hales and S. P. Ferguson, Discrete Comput. Geom. **36** (2006), no. 1, 21–69; MR2229658; T. C. Hales, Discrete Comput. Geom. **36** (2006), no. 1, 71–110; MR2229659; Discrete Comput. Geom. **36** (2006), no. 1, 111–166; MR2229660; S. P. Ferguson, Discrete Comput. Geom. **36** (2006), no. 1, 167–204; MR2229661; T. C. Hales, Discrete Comput. Geom. **36** (2006), no. 1, 205–265; MR2229662]. A revision to this proof made in 2010 appears in [T. C. Hales et al., Discrete Comput. Geom. **44** (2010), no. 1, 1–34; MR2639816]. All the papers on this proof, including the revision and supporting work, are collected in the volume [T. C. Hales and S. P. Ferguson, *The Kepler conjecture*, Springer, New York, 2011; MR3075372].

The Kepler conjecture is an asymptotic statement about packing large volumes of space with spheres. A general approach to the Kepler Conjecture was suggested in the 1950's by L. Fejes Tóth [*Lagerungen in der Ebene, auf der Kugel und im Raum*, Die Grundlehren der Mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete, Band LXV, Springer, Berlin, 1953; MR0057566] via proving some type of local density inequality around each sphere center; see [J. C. Lagarias, Discrete Comput. Geom. **27** (2002), no. 2, 165–193; MR1880936] for general background. A local density inequality is a logically stronger statement than the Kepler Conjecture because it also proves that certain local densities within a finite region of a fixed size cannot exceed the Kepler Conjecture density. The proof of Hales and Ferguson proceeds by formulating a particular local density inequality near each individual sphere in a given packing, which has the property of implying Kepler's conjecture, and which could in principle be verified by a finite computation. The inequality was very complicated, having features which would simplify the subsequent computer calculations to a point where they became feasible, and in particular giving a way to list all the possible configurations to be checked. The actual proof had many thousands of cases and intensive computer calculations.

Since 2006 Hales has engaged in a project to produce a formal proof of the Kepler Conjecture, terming the project Flyspeck. A formal proof is a proof written in a formal logical system, which can itself be verified by computer by a proof assistant that checks the proof logically line by line. Formal proofs are more reliable than proofs written in mathematics journals; the latter are like computer program specifications, while a formal proof is analogous to a computer program itself. The planned formal proof of Kepler's Conjecture is intended to be checked by computer using the proof assistant HOL light [J. R. Harrison, in *Theorem proving in higher order logics*, 60–66, Lecture Notes in Comput. Sci., 5674, Springer, Berlin, 2009; MR2550931]. Some statements in the formal proof will be inequalities checked by computer. Other parts require first formalizing some parts of Euclidean geometry going beyond D. Hilbert's work on foundations of geometry [*Grundlagen der Geometrie*, fourteenth edition, Teubner-Arch. Math. Suppl., 6, Teubner, Stuttgart, 1999; MR1732507]. Such a formalization, which includes notions of point set topology, is described in work of Harrison [J. Automat. Reason. **50** (2013), no. 2, 173–190; MR3016800]. When completed such a formal proof will be among the largest formal proofs, comparable with the formal proof of the Feit-Thompson theorem that all odd-order finite groups are solvable, undertaken by G. Gonthier [in *Interactive theorem proving*, 2, Lecture Notes in Comput. Sci., 6898, Springer, Heidelberg, 2011; MR2877865].

The present book reports on part of this project. Its object is to present a blueprint version of the formal proof: a structure which will be a guide to constructing the detailed formal proof organized in a suitable way for conversion to a formal proof. The book is self-contained and does not require any knowledge of the previous proofs.

This formal proof aims to establish an entirely new set of local inequalities that imply Kepler's conjecture. It will constitute a new second-generation proof of the Kepler Conjecture logically independent of the previous proof, which proves entirely new inequalities. The new local inequalities are motivated by work of C. Marchal [Math. Z. **267** (2011), no. 3-4, 737–765; MR2776056] which we now review. Marchal associated to a given saturated packing of spheres a partition of space into cells of four types, plus possibly some unused extra volume. (The cells are closed and overlap on volume zero sets.) The cells of type 1 are tetrahedra with vertices at 4 different sphere centers. The cells of type 2 are tetrahedra with vertices at 3 sphere centers plus one extra vertex which is strictly outside any sphere. The cells of type 3 are (roughly) unions of two truncated half-cones having 2 vertices at sphere centers. The cells of type 4 are truncated half-cones with a vertex at a sphere center (and edge length $\sqrt{2}$). The cells of all types having a given sphere center as vertex contain within them the entire sphere.

The Marchal approach has as a main ingredient a specific choice of weight function $f(h)$ defined for $h \geq 0$ which is compactly supported, which weights nearby volume. Associated to a sphere center $\mathbf{v}$ is the set of all Marchal cells having $\mathbf{v}$ as a vertex. These cells fill out the whole solid angle at $\mathbf{v}$ and each one of them will be assessed for a volume proportional to their solid angle subtended at $\mathbf{v}$, using the weight function, corrected by their shape. Then to prove Kepler's conjecture two types of inequalities are required to be satisfied. The first local inequality says that the assessed volume at each sphere center will total up to at least the amount required by Kepler's conjecture. It is actually expressed in a negative form, that a certain amount of "given up" volume is not too large, taking the form that $\mathcal{L}_f(V, \mathbf{v}) \leq 12$, where $\mathcal{L}_f(V, \mathbf{v}) := \sum_{\mathbf{w} \in V \setminus \mathbf{v}} f((\|\mathbf{w} - \mathbf{v}\|)/2)$. The second type of local inequality says that for each Marchal cell X the total volume assessed to it by all the sphere centers at its vertices is at most the total volume of X. For type 1 and type 2 cells this inequality is expressed as $\gamma(X, f) \geq 0$, where $\gamma(X, f) := \mathrm{vol}(X) - \frac{2m_1}{\pi}\mathrm{tsol}(X) + \frac{8m_2}{\pi}\sum_e \mathrm{dih}(X, e) f(\|e\|)$, where e is an edge and $\|e\|$ its length. Here $\mathrm{tsol}(X)$ is the total solid angle of the sphere center vertices (up to 4π), while $\mathrm{dih}(X, e)$ is the dihedral angle (up to 2π), and $f(\cdot)$ is the weight function. Marchal proposed a function $f(h) = M(h)$ which is positive up to $h_+ = 1.3254$ and is zero above $h = \sqrt{2}$, and presented evidence that both types of inequalities above should hold for this function. This is Marchal's approach to the Kepler conjecture, but the details he supplies are not complete.

The new local inequalities of Hales use the Marchal partition of space into cells, together with a simpler function $f(h) = L(h)$ which is piecewise linear, having $L(h) = 1$ for $h \leq 1$, $L(h) = 0$ for $h \geq 1.26$ and linearly interpolates between these values. The important point is its cutoff value 1.26 which is smaller than the 1.3254 used by Marchal and so reduces the complexity of the later analysis of counterexample configurations. A price paid for this is that the individual cell inequalities $\gamma(X, L) \geq 0$ for the function $L(h)$ do not always hold. The bad cases, treated in section 6.4, concern cells having an edge of length in interval $[h-, h_+] \approx [1.23175, 1.3254]$, which are termed *critical edges*. Cells with such edges are weighted and grouped into *clusters* sharing a common critical edge; the weights for cells with several critical edges are arranged so that their contributions to different clusters add up to 1. The replacement for the second inequality $\gamma(X, L) \geq 0$ in this case is Theorem 6.93, which asserts nonnegativity for a sum of weighted $\gamma(X, L)$ over a cluster, after an additional correction term is included. It requires an enormous computer calculation.

The heart of the new blueprint proof is then a proof of the first local inequality $\mathcal{L}_L(V, \mathbf{v}) = \sum_{\mathbf{w} \in V \setminus \mathbf{v}} L((\|\mathbf{w} - \mathbf{v}\|)/2) \leq 12$. This part of the proof is now a finite-dimensional nonlinear optimization problem, and Hales now develops and formalizes ideas used in the earlier proof, to systematize finding and discarding a large number of cases. It shows that a counterexample local configuration can be taken to have several extra desirable properties, which he terms being a *contravening configuration.* By locating $\mathbf{v} = \mathbf{0}$ and letting W now be the finite set of sphere centers with $\|\mathbf{w}\| \leq 2.52$, these properties are: one may reduce to the case where W has cardinality 13, 14 or 15, it is a local maximum of the function $\mathcal{L}_L(W, \mathbf{v})$, and the projections $\mathbf{w}/\|\mathbf{w}\|$ on the unit sphere, when connected with geodesic arcs for any two $\mathbf{w}_1, \mathbf{w}_2$ within distance 2.52 forms a planar graph on the surface of the sphere, with all faces being geodesic polygons having all angles less than π. The argument then classifies all such planar graphs having some extra structure, which are termed tame hypermaps. Part of the proof shows that there is a finite list of such tame hypermaps, and determines them all. To each hypermap is associated a family of linear programs to obtain upper bounds on $\mathcal{L}_L(W, \mathbf{v})$. The linear programs now show for each such configuration that $\mathcal{L}_L(W, \mathbf{v}) < 12$, i.e. the linear program is infeasible. This can be certified by a certificate of infeasibility. and such bound may be established by suitable linear programs, concocted using many auxiliary geometric properties of such sphere center configurations W.

The book is divided into three parts. Part I of the book has one chapter, giving a historical overview of work on the problem in Sections 1.1 to 1.5, finishing in Section 1.6 with a sketch of the planned blueprint proof.

Part II, Chapters 2 to 5, gives geometric foundations for the proof. Chapter 2 treats trigonometry equalities and inequalities. Chapter 3 treats volumes. Chapter 4 treats the concept of a hypermap, which is a combinatorial structure which replaces the notion of planar map used in the original proof of the Kepler conjecture. Chapter 5 introduces a notion of "fan" to describe hypermaps further; this notion is different from that of "fan" in toric varieties.

Part III, Chapters 6 to 8, then presents a detailed outline of the blueprint of a new proof of the Kepler conjecture which is planned to be converted to a formal proof. Chapter 6 details the Marchal cell partition and formulates local inequalities. Chapter 7 presents main technical estimates for looseness of a packing in terms of fan parameters, given in Theorem 7.43. Chapter 8 addresses the first local inequality above. Section 8.6 uses similar ideas to present a new proof of the Dodecahedral Conjecture, done originally by Hales and S. McLaughlin [J. Amer. Math. Soc. **23** (2010), no. 2, 299–344; MR2601036], proving a strong form that characterizes the case of equality.

J. C. Lagarias

From MathSciNet, October 2023

MR3044452 52C17

Venkatesh, Akshay

A note on sphere packings in high dimension.

International Mathematics Research Notices. IMRN (2013), *no.* 7, 1628–1642.

A theorem of Minkowski and Hlawka says that there is an origin-centered ellipsoid $E \subset \mathbf{R}^n$ of volume 1, containing no nonzero integer vector (i.e., to say: $E \cap \mathbf{Z}^n = \{0\}$). Let $c_n = \sup\{\text{volume}(E)\}$ where E is an origin-centered ellipsoid and $E \cap \mathbf{Z}^n = \{0\}$. The first substantial improvement on Minkowski's work was given by Rogers (1947) by showing that $c_n > 0.73n$ for large enough n. It is known by work of Ball (1992) that $c_n \geq 2(n-1)$ always, and due to Vance (2011) that $c_n \geq 2.2n$ when n is divisible by 4. These results yield the best known lower bounds on the sphere-packing problem. Minkowski's result furnishes a periodic sphere packing of density 2^{-n}. Similarly, Ball's result yields a sphere packing of density at least $2(n-1)2^{-n}$ in every dimension. The goal here is to improve the linear bound by a large constant, and also to show that in many dimensions the asymptotic growth can be improved. The main result states that there exist infinitely many dimensions n for which $c_n > \frac{1}{2} n \log\log n$. Also, in every sufficiently large dimension, $c_n > 65,963n$. The constant 65,963 here could be replaced by any number less than $2\frac{\sinh^2(\pi e)}{\pi^2 e^3}$. In the first case, one considers random lattices with automorphism group containing $\mathbf{Z}/k\mathbf{Z}$. The improvement comes from the fact that the degree of the field extension $[\mathbf{Q}(\mu_k) : \mathbf{Q}]$ can be as small as $\frac{k}{\log\log k}$; here $\mathbf{Q}(\mu_k)$ denotes the field obtained by adjoining all kth roots of unity to $\mathbf{Q}$. In the second case, one considers "random orthogonal lattices". The improvement comes from the fact that integrality sometimes forces vectors to be longer than they otherwise would be. The theorem uses two special properties of the sphere: it is preserved by a large subgroup of linear automorphisms and its boundary is defined by a polynomial equation. One needs to tweak the free parameter (the discriminant) to get the most out of it.

Ranjeet Kaur Sehmi
From MathSciNet, October 2023

MR3229046 05B40; 11H31; 52C17

Cohn, Henry; Zhao, Yufei

Sphere packing bounds via spherical codes.

Duke Mathematical Journal **163** (2014), *no.* 10, 1965–2002.

The sphere packing problem is one of the most important problems in discrete geometry. Probably the most significant part of this problem is the problem of asymptotic behaviour of the maximal packing density $\Delta_{\mathbb{R}^n}$ for spheres in n-dimensional Euclidean space.

The current best upper bound in all sufficiently high dimensions is due to G. A. Kabatyanskiĭ and V. I. Levenshtein [Problemy Peredači Informacii **14** (1978), no. 1, 3–25; MR0514023]. The proof of this bound consists of two main steps.

(1) Firstly, Kabatyanskiĭ and Levenshtein proved the inequality

$$\Delta_{\mathbb{R}^n} \leq \sin^n(\theta/2)A(n+1,\theta),$$

where $A(n,\theta)$ is the greatest size of the spherical code in dimension n with minimum angle θ.

(2) Further, they used a linear programming method to obtain upper bounds for $A(n,\theta)$.

In the reviewed paper it is proved that in the first step of the described scheme we have the inequality

$$\Delta_{\mathbb{R}^n} \leq \sin^n(\theta/2)A(n,\theta)$$

with supplementary restriction $\pi/3 \leq \theta \leq \pi$. Note that $A(n,\theta) \leq A(n+1,\theta)$ for any $n \geq 1$. Combining this estimate with known linear programming estimates for $A(n,\theta)$, the authors show that the Kabatyanskiĭ-Levenshtein bound for $\Delta_{\mathbb{R}^n}$ can be improved by a constant factor.

An alternative approach to upper bounds for $\Delta_{\mathbb{R}^n}$ was introduced by H. Cohn and N. D. Elkies [Ann. of Math. (2) **157** (2003), no. 2, 689–714; MR1973059]. Let $f\,\mathbb{R}^n \to \mathbb{R}$ be a continuous, positive definite, and integrable function such that $f(x) \leq 0$ for all $|x| \geq 2$. Let $\widehat{f}$ be the Fourier transform of f and suppose that $\widehat{f}(0) > 0$. Then

$$\Delta_{\mathbb{R}^n} \leq \operatorname{vol}(B_1^n)\frac{f(0)}{\widehat{f}(0)}.$$

Unfortunately, asymptotic behaviour of this estimate is unknown. In the reviewed paper it is proved that there exists a function f such that the Cohn-Elkies bound applied to this function gives us exactly the Kabatyanskiĭ-Levenshtein bound.

Further, in the reviewed paper the authors consider the hyperbolic version of the sphere packing problem. Let $\Delta_{\mathbb{H}^n}(r)$ be the optimal packing density for balls of radius r in n-dimensional hyperbolic space. Then it is proved that for all $n \geq 2$, $\pi/3 \leq \theta \leq \pi$, and $r > 0$ we have

$$\Delta_{\mathbb{H}^n}(r) \leq \sin^{n-1}(\theta/2)A(n,\theta).$$

With linear programming estimates for $A(n,\theta)$ that bound allows them to prove that

$$\sup_{r>0} \Delta_{\mathbb{H}^n}(r) \leq 2^{-(0.5990\cdots+o(1))n}.$$

This result is a new best upper bound for $\Delta_{\mathbb{H}^n}(r)$.

In the last part of the paper, some problems on the conjectural hyperbolic analogue of the Cohn-Elkies bound are discussed.

Also one of the appendices of the reviewed paper is devoted to numerical computation of known upper bounds for $\Delta_{\mathbb{R}^n}$. It is important to note that the results obtained disagree with the well-known book [J. H. Conway and N. J. A. Sloane, *Sphere packings, lattices and groups*, third edition, Grundlehren Math. Wiss., 290, Springer, New York, 1999; MR1662447].

Anton Vladimirovich Shutov

From MathSciNet, October 2023

MR3643686 52C17; 01A70, 05B40, 11H31

de Laat, David; Vallentin, Frank

A breakthrough in sphere packing: the search for magic functions.
Includes an interview with Henry Cohn, Abhinav Kumar, Stephen D. Miller and Maryna Viazovska.

Nieuw Archief voor Wiskunde. Vijfde Serie **17** (2016), *no.* 3, 184–192.

This paper is not so much about a research result, which has interest in its own right, but an interview with the researchers and an exposition of certain features of the research result. Consequently, this paper would be of interest to graduate and post-doctorate students who are interested in how a new method can be used to generate further results as well as an understanding of the collaboration process.

The paper starts by reviewing the definition and history of the sphere-packing problem and the successes in dimensions 1, 2 and 3. The upper bound on sphere packings of H. L. Cohn and N. D. Elkies [Ann. of Math. (2) **157** (2003), no. 2, 689–714; MR1973059] is also mentioned.

On March 14, 2016, Maryna Viazovska solved the sphere-packing problem in 8 dimensions [Ann. of Math. (2) **185** (2017), no. 3, 991–1015; MR3664816]. One week later (March 21, 2016), Cohn, Abhinav Kumar, Stephen D. Miller, Danylo Radchenko and Viazovska generalized the new methods used and solved the sphere-packing problem in dimension 24 [Ann. of Math. (2) **185** (2017), no. 3, 1017–1033; MR3664817].

A good mathematical result typically involves methods from several disparate mathematical subfields which converge to produce something new. In this case, the sphere-packing problem is solved using the fields of error correcting codes, theta functions and modular forms. Use of computers for number crunching was also crucial for the result.

The interview, as mentioned above, sheds light on the collaboration process. Little things, like the use of Dropbox or showing tablet drawings on Skype, are mentioned and should prove useful to those starting their careers and interested in how things happen.

Russell Jay Hendel

From MathSciNet, October 2023

MR3664816 52C17; 11F03, 11F06, 11H31

Viazovska, Maryna S.

The sphere packing problem in dimension 8.

Annals of Mathematics. Second Series **185** (2017), *no.* 3, 991–1015.

The author proves in this article that the E_8 lattice gives the densest sphere packing in dimension 8. This long-standing problem was reduced by H. L. Cohn and N. D. Elkies [Ann. of Math. (2) **157** (2003), no. 2, 689–714; MR1973059] to the problem of finding a function f on $\mathbb{R}^8$ for which both f and its Fourier transform $\widehat{f}$ satisfy certain conditions, and for which $\frac{f(0)}{\widehat{f}(0)}$ attains under these conditions the smallest possible value 16. The author constructs such a function explicitly, using integral transforms of some carefully chosen quotients of modular forms.

Rainer Schulze-Pillot

From MathSciNet, October 2023

MR3664817 52C17; 05B40, 11H31

Cohn, Henry; Kumar, Abhinav; Miller, Stephen D.; Radchenko, Danylo; Viazovska, Maryna

The sphere packing problem in dimension 24.

Annals of Mathematics. Second Series **185** (2017), *no.* 3, 1017–1033.

This is another breakthrough result in sphere packing. After the recent spectacular solution of the sphere packing problem in dimension 8 by M. S. Viazovska [Ann. of Math. (2) **185** (2017), no. 3, 991–1015; MR3664816], the paper under review solves the problem in dimension 24. The authors show that the maximum sphere packing density in dimension 24 is achieved by the Leech lattice packing and, up to scaling and isometries, it is the only periodic packing of this density.

The proof follows the eight-dimensional approach of Viazovska. First, based on the linear programming bound of H. L. Cohn and N. D. Elkies [Ann. of Math. (2) **157** (2003), no. 2, 689–714; MR1973059], the optimal density of a packing of spheres in $\mathbb{R}^n$ is upper bounded by $\mathrm{vol}((1/2)B_n)f(0)/\widehat{f}(0)$, where $f\,\mathbb{R}^n \to \mathbb{R}$, $f \not\equiv 0$, is a Schwartz function, which is assumed to be non-positive outside the unit ball B_n, $\widehat{f}$ is its Fourier transform which has to be non-negative everywhere and vol( ) denotes the n-dimensional volume (Lebesgue measure).

Hence, the sphere packing problem can be solved by determining a magic function f, such that the linear programming bound matches the density of a conjectured optimal packing. It is not clear at all, however, in which dimensions such an optimal magic function exists.

In [op. cit.], Viazovska constructed such a magic function in dimension 8 via a new connection with quasimodular forms. The construction of the optimal function f in dimension 24 is also based on this connection. More precisely, the magic function is a linear combination of two Fourier eigenfunctions with eigenvalues ± 1, and both are constructed via weakly holomorphic quasimodular forms.

Martin Henk

From MathSciNet, October 2023