DYNAMICS OF SHIFT-LIKE POLYNOMIAL DIFFEOMORPHISMS OF $\mathbb{C}^N$

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Abstract. We identify a family of polynomial diffeomorphisms of $\mathbb{C}^N$ and show that these mappings may be studied using certain methods (filtration and potential-theoretic) which were developed for the study of polynomial diffeomorphisms of $\mathbb{C}^2$.

0. Introduction

The dynamics of polynomial diffeomorphisms of $\mathbb{C}^2$ have been studied intensively in recent years, starting with the work of Hubbard [H] and Friedland-Milnor [FM]. The methods of pluri-potential theory, i.e., the methods related to pluri-subharmonic functions and positive, closed currents, have proven effective in the further study of the dynamics of these maps. This new direction started with the works of [BS] and [FS] and has progressed further in a series of papers by John Smillie and one of the authors. The reader is referred to the survey paper [BuS] for a good overview of this area. The purpose of the present paper is to introduce a family of polynomial diffeomorphisms of $\mathbb{C}^N$, $N \geq 2$, and to show that similar potential-theoretic tools may be developed for them. Our hope is that many of the methods and results from the case $N = 2$ will extend naturally to this more general case.

Let us review some of the features of the dynamics of polynomial diffeomorphisms of $\mathbb{C}^2$. We consider the sets

$$K^\pm = \{ x \in \mathbb{C}^2 : \{ f^{\pm n}(x) : n \geq 0 \} \text{ is bounded} \},$$

$$U^\pm = \mathbb{C}^N - K^\pm, \quad K = K^+ \cap K^-,$$

$$J^\pm = \partial K^\pm, \quad \text{and} \quad J = J^+ \cap J^-.$$

Friedland and Milnor [FM] showed that a polynomial diffeomorphism $f$ which is dynamically nontrivial has several interesting properties. One property is that such an $f$ is conjugate to a finite composition of mappings of the form $f : (x, y) \mapsto (y, p(y) - ax)$. For these mappings there are sets $V^-, V$, and $V^+$ such that $(V^-, V, V^+)$ forms a filtration for $f$ in the following sense:

1. A point not already in $V^-$ cannot enter $V^-$, and an $f$-orbit can remain in $V^-$ for finite positive time,
2. $V$ is compact, and a forward orbit $\{ f^n(x) : n \geq 0 \}$ is bounded if and only if it is eventually contained in $V$,
3. Every point of $V^+$ remains in $V^+$ and tends to infinity in forward time.

Another property of $f$ as given above is that it has minimal degree within its conjugacy class. In fact, if we set $\text{deg}(f) = d$, then $\text{deg}(f^n) = d^n$, where $f^n = f \circ \cdots \circ f$ denotes the $n$-fold composition. We may use $d$ to measure the (super-exponential) rate of escape to infinity in forward/backward time by defining

$$G^\pm(x) := \lim_{n \to \infty} \frac{1}{d^n} \log^+ ||f^n(x)||.$$ 

$G^\pm$ transforms under composition as: $G^\pm \circ f = d^{\pm 1} \cdot G^\pm$. Thus the stable/unstable currents, which are defined by $\mu^\pm := \frac{1}{2\pi} \text{dd}^c G^\pm$ may be wedged together to give an invariant measure $\mu := \mu^+ \wedge \mu^-$. The currents $\mu^\pm$ and the measure $\mu$ have been important in gaining a deeper understanding of $f$.

For $x = (x_1, \ldots, x_N) \in \mathbb{C}^N$, we set $||x|| = \max_{1 \leq j \leq N} |x_j|$. For $R < \infty$ large, $1 \leq j \leq N$, and $0 \leq \nu \leq N - 1$, we define

$$V_j = \{x \in \mathbb{C}^N : |x_j| \geq R, |x| = ||x||\},$$

$$V = \{x \in \mathbb{C}^N : ||x|| \leq R\},$$

$$V^- = \bigcup_{j=1}^{N-\nu} V_j, \quad \text{and} \quad V^+ = \bigcup_{j=N-\nu+1}^{N} V_j.$$ 

In this paper we introduce a family of polynomial diffeomorphisms of $\mathbb{C}^N$, which we call shift-like of type $\nu$. In Lemmas 1, 2 and 3 we show that the sets $V^+, V, V^-$ give a filtration (in the sense of 1, 2, and 3 above) for the dynamical system generated by these mappings.

In Theorem 9 we show that the limits defining the rate of escape functions $G^\pm$ converge uniformly on compact subsets of $\mathbb{C}^N$. Thus we may define the corresponding stable/unstable currents $\mu^\pm := \frac{1}{2\pi} \text{dd}^c G^\pm$ and $\mu^- := \frac{1}{2\pi} \text{dd}^c G^-\nu$, and we define a measure $\mu := \mu^+ \wedge \mu^-$. From Theorem 11 it follows that $\mu$ coincides with the harmonic measure of $K$ in the sense of pluri-potential theory.

1. **Shift-Like Mappings**

We will say that a (holomorphic) polynomial diffeomorphism $f : \mathbb{C}^N \to \mathbb{C}^N$, $N \geq 2$, is shift-like if the orbit of a point $x \in \mathbb{C}^N$ under $f$ determines a bi-infinite sequence $(\zeta_j)_{j \in \mathbb{Z}}$ such that

$$f^k(x) = (\zeta_{k+1}, \ldots, \zeta_{k+N}) \in \mathbb{C}^N.$$ 

Thus the forward iteration of $f$ corresponds to shifting the sequence to the left. In this case it has the form $f(x_1, \ldots, x_N) = (x_2, \ldots, x_N, g(x_2, \ldots, x_N) - ax_1)$ for some polynomial $g$ and some nonzero $a \in \mathbb{C}$, and the sequence $\zeta_n$ is generated by the recurrence relation: $\zeta_j = x_j$, for $1 \leq j \leq N$, and

$$\zeta_{n+N} = g(\zeta_{n+1}, \ldots, \zeta_{n+N-1}) - a\zeta_n \quad \text{for} \quad n \in \mathbb{Z}.$$ 

Note that this may be used as a recurrence relation for both increasing and decreasing $n$. We will also refer to a finite composition $f = f_m \circ \cdots \circ f_1$ of such mappings as shift-like. In this case we let $g_s$ and $a_s$ denote the polynomial and constant defining $f_s$. We use the notation $[s]$ for the integer satisfying $1 \leq [s] \leq m$. 


and \([s] \equiv s \mod m\). Thus \(f^n = f_{[m,n]} \circ \cdots \circ f_{[1]}\). It follows that \(f\) generates a sequence \((\zeta_n)_{n \in \mathbb{Z}}\) such that \(\zeta_j = x_j\) for \(1 \leq j \leq N\), and

\[
\zeta_{N+n} = g_{[n]}(\zeta_{n+1}, \ldots, \zeta_{n+N-1}) - a_{[n]}\zeta_n.
\]

The action of \(f\) on \((\zeta_n)\) corresponds to a shift by \(m\) units:

\[
f^k(x) = (\zeta_{mk+1}, \ldots, \zeta_{mk+N}).
\]

Let \(f\) be a shift-like map. We will say that \(f\) has type \(\nu\), for some \(1 \leq \nu \leq N-1\), if \(f\) has the form \(f = f_m \circ \cdots \circ f_1\), where each \(f_s\) is as follows: there is a polynomial \(p_s(z) = \sum_{j=0}^{d_s} c_{s,j}z^j\), \(d_s \geq 2\), \(c_{s,d_s} \neq 0\), and a nonzero constant \(a_s \in \mathbb{C}\) such that

\[
f_s(x_1, \ldots, x_N) = (x_2, \ldots, x_N, p_s(x_{N-\nu+1}) - a_s x_1).
\]

By [FM], the polynomial automorphisms of \(\mathbb{C}^2\) that are dynamically interesting are all conjugate to shift-like mappings of type 1.

**Example.** The mapping \(h(x, y, z) = (y, z, yz + \beta x)\) is shift-like but is not of type \(\nu\) for any \(\nu\). If \(|\beta| = 1\), then the coordinate axes are contained both in \(K\) and in the nonwandering set. In this case, neither \(K\) nor the nonwandering set is compact.

If \(f\) is shift-like of type \(\nu\), it is natural to iterate \(f^\nu\) rather than \(f\); for \(1 \leq k \leq m\) we write

\[
f^\nu = g_m \circ \cdots \circ g_1, \quad \text{with} \quad g_k(x) = f_{[k\nu]} \circ \cdots \circ f_{[(k-1)\nu+1]}.
\]

We will use the notation \(\pi_q(y_1, \ldots, y_N) = y_q\). The expression for \(g_k(x)\) in (4) is given by

\[
g_k(x) = (x_{\nu+1}, \ldots, x_N, \pi_{N-\nu+1}g_k(x), \ldots, \pi_N g_k(x)),
\]

where by (1)

\[
\pi_q g_k(x) = p_{[q-N+k\nu]}(x_q) - a_{[q-N+k\nu]}x_q - (N-\nu)
\]

for \(N-\nu+1 \leq q \leq N\).

It follows that the degree of the \(q\)th coordinate of \(f^\nu\) is

\[
\hat{d}_q = \prod_{k=1}^{m} d_{[q-N+k\nu]}
\]

for \(N-\nu+1 \leq q \leq N\). In §3 we will assume that the numbers \(\hat{d}_j\) satisfy

\[
d = d_1 \cdots d_m = \hat{d}_{N-\nu+1} = \cdots = \hat{d}_N.
\]

This occurs if \(d_1 = \cdots = d_m\). Also, if \(m\) and \(\nu\) are relatively prime, then \(\{[q-N+k\nu] : 1 \leq k \leq m\} = \{1, \ldots, m\}\); and so (6) holds.

If (6) holds, then for each \(N-\nu+1 \leq q \leq N\) there exists a constant \(\alpha_q\) such that

\[
de(\pi_q f^\nu(x) - \alpha_q x_q^d) < \hat{d}.
\]

The inverse of \(f\) is given by \(f_1^{-1} \circ \cdots \circ f_m^{-1}\), where the inverse of each \(f_s\) is given by

\[
f_s^{-1}(x_1, \ldots, x_N) = (a_s^{-1}(p_s(x_{N-\nu}) - x_N), x_1, \ldots, x_{N-1}).
\]
In order to work in negative time, we find it convenient to iterate \( f^{-(N-\nu)} \), rather than \( f^{-1} = f_{-1}^{-1} \circ \cdots \circ f_{-N}^{-1} \). We write
\[
f^{-(N-\nu)} = h_m \circ \cdots \circ h_1, \quad \text{with} \quad h_k = f_{[m-k(N-\nu)+1]}^{-1} \circ \cdots \circ f_{[m-(k-1)(N-\nu)]}^{-1},
\]
Thus we have
\[
(9) \quad h_k(x_1, \ldots, x_N) = (\pi_1 h_k(x), \ldots, \pi_{N-\nu} h_k(x), x_1, \ldots, x_{N-\nu}),
\]
where by (8) we have
\[
\pi_q h_k(x) = a_{m-k(N-\nu)+q}^{-1} [p_{m-k(N-\nu)+q}(x_q) - x_{q+\nu}]
\]
for \( 1 \leq q \leq N - \nu \). In §3 we will assume that for \( 1 \leq q \leq N - \nu \) the numbers
\[
\hat{d}_q = \prod_{k=1}^{m} d_{[m-k(N-\nu)+q]}
\]
satisfy
\[
(10) \quad d = \hat{d}_1 = \cdots = \hat{d}_{N-\nu}.
\]
This holds if \( d_1 = \cdots = d_m \) or if \( (m, N - \nu) = 1 \). In this case there are constants \( \alpha_q, 1 \leq q \leq N - \nu \) such that
\[
\pi_q f^{-(N-\nu)} x = \alpha_q x_q^d + \cdots.
\]

Remark 1. The involution \( I(x_1, \ldots, x_N) = (x_N, \ldots, x_1) \) conjugates \( f^{-1} \) to
\[
(11) \quad (x_1, \ldots, x_N) \mapsto (x_2, \ldots, x_N, a_s^{-1}(p_s(x_{\nu+1}) - x_1)),
\]
which is shift-like of type \((N - \nu)\). This observation allows us to deduce that the results proved for a general map \( f = f_m \circ \cdots \circ f_1 \) of any type \( \nu \) will also apply to \( f^{-1} = f_1^{-1} \circ \cdots \circ f_m^{-1} \), since this is the general map of type \((N - \nu)\).

Remark 2. If \( \delta > 0 \) is an integer which divides \( \nu \) and \( N \), then for \( 0 \leq c < \delta \), the subsequence \( \{\zeta_n : n \equiv c \pmod{\delta}\} \) is invariant under each \( f_s \). If we write \( \nu' = \nu/\delta \) and \( N' = N/\delta \), it follows that the mapping
\[
f'_s(y_1, \ldots, y_{N'}) = (y_2, \ldots, y_{N'}, p_s(y_{N'-\nu'+1}) - a_s y_1)
\]
is shift-like of type \( \nu' \) on \( \mathbb{C}^{N'} \), and thus each \( f_s \), and the composition \( f = f_m \circ \cdots \circ f_1 \), are biholomorphically conjugate to a \( \delta \)-fold product of the mappings \( f'_s : \mathbb{C}^{N'} \to \mathbb{C}^{N'} \). Thus there is no loss of generality if we assume that \( \nu \) and \( N \) are relatively prime.

2. Filtration Properties

Since \( f \) acts as a shift, it follows that
\[
(12) \quad f(V) \subset V \cup V_N \subset V \cup V^+ \quad \text{and} \quad f(V_j) \subset V_{j-1} \cup V_N \quad \text{for} \quad 2 \leq j \leq N.
\]
Similarly, since \( f^{-1} \) is a shift in the opposite direction, it follows that
\[
(13) \quad f^{-1}(V) \subset V \cup V_1 \subset V \cup V^- \quad \text{and} \quad f^{-1}(V_j) \subset V_{j+1} \cup V_1 \quad \text{for} \quad 1 \leq j \leq N - 1.
\]
We assume that \( f = f_m \circ \cdots \circ f_1 \), with \( f_s \) as in (1) and with \( d_s \geq 2 \). We let \( \rho < 1 \) be given and choose \( R \geq 1 \) sufficiently large that
\[
(14) \quad \rho^{-1} |c_s d_s \zeta^{d_s}| > \left| \frac{1}{p_s(\zeta)} \pm (1 + |a_s|)|\zeta| \right| > \rho |c_s d_s \zeta^{d_s}|
\]
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holds for all $|\zeta| \geq R$ and all $1 \leq s \leq m$. We also assume that

$$R > \frac{2(1 + |a_s|)}{|c_d|}$$

holds for $1 \leq s \leq m$. We noted in Remark 1 that $f_s^{-1}$ is also shift-like, although the type is $(N - \nu)$. When we work with $f^{-1}$, we will assume that the corresponding inequalities (14) and (15) hold for the inverses $f_s^{-1}$.

Remark 3. In this section on filtrations, we will often treat the iteration $f^n = (f_m \circ \cdots \circ f_1) \circ \cdots \circ (f_m \circ \cdots \circ f_1)$ as part of the composition $f_{s_1} \circ f_{s_2} \circ \cdots \circ f_{s_k}$ of an infinite sequence of mappings $f_{s_1}, f_{s_2}, f_{s_3}, \ldots$, such that the degrees $d_s$ are uniformly bounded, and the conditions (14) and (15) hold for the inverses $f_{s_1}^{-1}$.

Lemma 1. If (14) holds, then $f_s V_{N-\nu+1} \subset V_N$ holds for each $1 \leq s \leq m$, and thus $fV^+ \subset V^+$.

Proof. If $x \in V_{N-\nu+1}$, then $\|x\| = |x|_{N-\nu+1} > R$. Thus, by (14), it follows that $|x|_{N+1} > \|x\|$. Thus $fx \in V_N \subset V^+$. \hfill $\square$

Under the involution $I(x_1, \ldots, x_N) = (x_N, \ldots, x_1)$, the sets $V^\pm$ of type $(N - \nu)$ are taken to $V^\mp$ of type $\nu$. If we apply the argument of Lemma 1 to $f^{-1}$ we obtain:

Lemma 2. If (14) holds for $f^{-1}$, then $f_s^{-1} V_{N-\nu} \subset V_1$, holds for $1 \leq s \leq m$, and thus $f^{-1} V^- \subset V^-$. We begin by giving a weak estimate which shows that points of $V^+$ escape to infinity in forward time.

Lemma 3. There exists $c' > 0$, depending only on $f$, such that if $R$ is large,

$$c'||x||^2 \leq \|f'(x)||$$

for all $x \in V^+$. In particular, if we take $R$ such that $R c' > 1$, then there exists $\kappa > 1$ such that for every $x \in V^+$, we have $\|f^n x\| \geq \kappa^{2n}/c'$.

Proof. We write $f^n = g_m \circ \cdots \circ g_1$ as above. It suffices to prove the Lemma for each of these mappings $g_k$. If $x \in V^+$, then there exists $N - \nu + 1 \leq j \leq N$ such that $|x_j| = ||x||$. By (5), we have that the size of the $j$-th component of $g_k x$ is

$$|p_s(x_j) - a_s x_j| \geq c_s x_j^{d_s} - |a_s x_j| \geq c(|x_j|^{d_s} - ||x||) \geq c_k ||x||^2,$$

since $|x_j| = ||x|| \geq |x_j|$. If $R c' > 1$, we may write $R = c'^{-1} \kappa$ with $\kappa > 1$. The final inequality follows by repeatedly substituting $||x|| \geq c'^{-1} \kappa$ into the first inequality. \hfill $\square$

Lemma 4. There exists $c$, depending only on $f$ such that if $R < ||x|| \leq M$, and if $f_{s_j} \circ \cdots \circ f_{s_1} x \in V^-$ for $0 \leq j \leq T + N + \nu$, then

$$\|f_{s_T} \circ \cdots \circ f_{s_1} x\|^2 \leq \max_s \left\{ \frac{2(1 + |a_s|)}{|c_d|} M, R^2 \right\}.$$
Proof. Let \((\zeta_n)_{n \in \mathbb{Z}}\) denote the sequence associated with the orbit of \(x\) under the family of mappings \(f_s, f_{s+1}, \ldots\) as in (16). From the condition \(f_s \circ \cdots \circ f_{s+r}x = (\zeta_1, \ldots, \zeta_{s+r}) \in V^\nu\), we have \(\max_{1 \leq j \leq N - \nu} |\zeta_j| \geq |\zeta_{s+j}|\) for \(1 \leq k \leq N\). Applying this inductively, starting with \(j = 0\), and extending to \(0 \leq j \leq T + N\), we have

\[
\max_{1 \leq j \leq N - \nu} |\zeta_j| \geq |\zeta_k| \quad \text{for} \quad k \leq T + N + \nu.
\]

Now we use the fact that \(M \geq \|x\|\) and (16) to obtain

\[
M \geq |\zeta_k| = |p_s(\zeta_{k - \nu}) - a_s\zeta_{k - N}|.
\]

Now either \(|\zeta_{k - \nu}| \leq R\), or we may apply (14) to obtain

\[
M \geq \frac{|c_{d_s}|}{2}|\zeta_{k - \nu}|^d_s - |a_s\zeta_{k - N}| \geq \frac{|c_{d_s}|}{2}|\zeta_{k - \nu}|^2 - |a_s| M.
\]

This gives

\[
\frac{2(1 + |a_s|)}{|c_{d_s}|} M \geq |\zeta_{k - \nu}|^2.
\]

Finally, since \(f_{s} \circ \cdots \circ f_{s+r}x = (\zeta_{T+1}, \ldots, \zeta_{T+N})\), and since the previous inequality holds for \(k \leq T + N + \nu\), we have the desired estimate. \(\square\)

**Lemma 5.** Any orbit \(f^n x\) can remain in \(V^-\) for only finitely many values of \(n \geq 0\).

Proof. Let us suppose that the forward orbit of \(f^n x\) remains in \(V^-\) for all \(n \geq 0\). It follows from Lemma 1 that \(f_{[T]} \circ \cdots \circ f_{[1]} x \in V^-\) for all \(T \geq 0\). Let \(c = \max \frac{2(1 + |a_s|)}{|c_{d_s}|}\), and define \(M_j\) by \(M_0 := \|x\|\) and \(M_{j+1} := (cM_j)^{1/2}\). Since \(f_{j(N - \nu)} x \in V^-\), we have \(\|f_{j(N - \nu)} x\| > R\). We apply Lemma 4 inductively in \(j\) to the map \(f_{[T]} \circ \cdots \circ f_{[1]}\) with \(T = j(N - \nu)\) and obtain that

\[
\|f^{(j+1)(N - \nu)} x\| \leq c\|f_{j(N - \nu)} x\| \leq cM_j = M_{j+1}.
\]

On the other hand, it is easily seen that the sequence \(\{M_j\}\) decreases to \(c\), and \(c < R\) by (15), which is a contradiction. \(\square\)

We may summarize our work so far with the following:

**Theorem 6.** If \(f\) is a shift-like mapping of type \(\nu\), and if \(R\) is chosen sufficiently large, then the sets \(V^-\), \(V\), and \(V^+\) have the filtration properties 1, 2, and 3 for \(f\) as given above. Further, \(V^+\), \(V\), and \(V^-\) have the same filtration properties for the mapping \(f^{-1}\).

**Proposition 7.** \(U^\pm = \bigcup_{n=0}^{\infty} f^{-n}V^\pm\), and this union is increasing.

Proof. By Lemma 1, \(fV^+ \subset V^+\), so \(f^{-n}V^+ \subset f^{-n-1}V^+\) for \(n \geq 0\), so the union is increasing. By Lemma 3, if \(x \in V^+\), then \(\lim_{n \to \infty} \|f^{-n}x\| = \infty\). Thus \(U^+ \supset V^+\). By the invariance of \(U^\pm\) under \(f\), we obtain \(U^+ \supset f^{-n}V^+\). On the other hand, if \(x \in U^+\), the forward orbit is unbounded. Since a forward orbit cannot remain in \(V^-\) for all positive time, we must have \(f^n x \in V^+\), which is to say that \(x \in f^{-n}V^+\). The arguments for \(V^-\) are analogous. \(\square\)

**Corollary 8.** \(K \subset V\) and \(K^\pm \cap V^\pm = \emptyset\).
Homology of $U^\pm$. For $N-\nu+1 \leq j \leq N$, the circle $\sigma_j : \theta \mapsto (R, \ldots, e^{2\pi i \theta} R, \ldots, R)$ (with the exponential in the $j$-th coordinate) generates $H_1(V_j; \mathbb{Z})$. The action of one of the component mappings $f_s$ on homology is: $f_s \cdot \sigma_j = \sigma_{j-1}$ for $N-\nu+1 < j \leq N$, and $f_s \sigma_{N-\nu+1} = d_s \sigma_N$. The inverse gives a homeomorphism $f^{-1} : V^+ \to f^{-1}V^+$. The action of $f_s^{-\nu}$ on the homology of $V^+$ is to divide all $\nu$ of the generators $\sigma_{N-\nu+1}, \ldots, \sigma_N$ by $d = d_1 \cdots d_m$. By Proposition 7, $\bigcup_{n \geq 0} f^{-n}V^+ = U^+$, so the homology of the limit $U^+$ is given as the $\nu$-fold product:

$$H_1(U^+; \mathbb{Z}) = \mathbb{Z}[\frac{1}{d}] \times \cdots \times \mathbb{Z}[\frac{1}{d}].$$

A similar argument gives $H_1(U^-; \mathbb{Z})$ as the $(N-\nu)$-fold product of $\mathbb{Z}[\frac{1}{d}]$.

3. Green Functions

In this Section we study the rate of escape functions for the forward iterates of $f^\nu$ and the backward iterates of $f^{N-\nu}$. For the rest of this paper we assume that (6) and (10) hold.

**Example.** If $f = f_2 \circ f_1$ with $f_1(x, y, z) = (y, z, y^2 + x)$ and $f_2(x, y, z) = (y, z, y^2 + x)$, then $f(x, y, z) = (z, y^3 + x, z^2 + y)$. The condition (6) does not hold, and the arguments below do not apply to this function.

For $n \geq 0$, we define:

$$G^+_n(x) := \frac{1}{dn} \log^+ ||f^{n\nu}(x)||,$$

$$G^-_n(x) := \frac{1}{dn} \log^+ ||f^{-n(N-\nu)}(x)||.$$

**Theorem 9.** The limits $G^\pm : = \lim_{n \to \infty} G_n^\pm$ are uniform on compact subsets of $C^N$, and $G^\pm$ are continuous and pluri-subharmonic on $C^N$. We have

\begin{equation}
G^+ \circ f^\nu = d \cdot G^+ \quad \text{and} \quad G^- \circ f^{N-\nu} = d^{-1} \cdot G^-.
\end{equation}

Further, $K^\pm = \{G^\pm = 0\}$, and

$$G^\pm(x) = \log ||x|| + O(1)$$

holds uniformly on $V^\pm$ as $x \to \infty$.

**Proof.** Without loss of generality we consider only $G^+$. We will show that the limit defining $G^+$ converges uniformly on compact sets. Thus $G^+$ is continuous and pluri-subharmonic. By Lemma 5, any compact subset of $V^-$ will be mapped to $V \cup V^+$ in finite positive time. Thus it suffices to show that the series $\sum(G^+_{n+1} - G^+_n)$ converges uniformly on compact subsets of $V \cup V^+$.

We may assume that $V$ is contained in the polydisk of radius $R$. For the points $x$ such that $f^{n\nu}x \notin V$ for all $n \geq 0$, we have

$$G^+_{n+1}(x) - G^+_n(x) \leq \frac{1}{dn} \log R.$$

If $f^{n\nu}x \notin V$ for some $n \geq 0$, then $f^{n\nu}x \in V^+$ for $n \geq n_0$. Thus it will suffice to show that the series converges uniformly on $V^+$. In order to estimate $G^+_{n+1} - G^+_n$ on $V^+$, let us write $y = f^{nu}(x)$ and $z = f^\nu(y)$. Let $N-\nu+1 \leq m, k \leq N$ be indices such that $|z_k| = ||z||$ and $|y_m| = ||y||$. Thus

$$G^+_{n+1}(x) - G^+_n(x) = \frac{1}{dn+1} \log \frac{|z_k|}{|y_m|^d}.$$
By (7) we have \( z_i = \alpha_i y_i^d + \mathcal{O}(\|y\|^{d-1}) \) for \( N - \nu + 1 \leq i \leq N \). Thus since \( |y_k| \leq |y_m| = \|y\| \), we have

\[
\frac{|z_k|}{|y_m|^d} \leq \frac{|\alpha_k y_k^d| + \mathcal{O}(\|y\|^{d-1})}{|y_m|^d} \leq |\alpha_k| + \mathcal{O}(\|y\|^{-1}).
\]

Now we only need to bound \( |z_k|/|y_m|^d \) from below. Equation (7) and \( \|z\| = |z_k| \geq |z_m| \) give

\[
|\alpha_k y_k^d| + \mathcal{O}(\|y\|^{d-1}) = |z_k| \geq |z_m| \geq |\alpha_m y_m^d| + \mathcal{O}(\|y\|^{d-1}) = |\alpha_m| \|y\|^d + \mathcal{O}(\|y\|^{d-1}),
\]

from which we conclude that

\[
\frac{|z_k|}{|y_m|^d} \geq |\alpha_m| + \mathcal{O}(\|y\|^{-1}).
\]

Thus

\[
G_{n+1}^+(x) - G_n^+(x) = \mathcal{O}(d^{-(n+1)})
\]
on \( V^+ \), and the series converges uniformly.

The asymptotic behavior of \( G^+ = \log \|x\| + \sum_{n=0}^{\infty} (G_{n+1}^+ - G_n^+) \) is given by the fact that the series converges uniformly on \( V \cup V^+ \). Further, it follows from the definition that \( d \cdot G_{n+1}^+ = G_n^+ \circ f^\nu \). Thus (17) follows from the uniform convergence of \( \{G_n^+\} \).

Finally, let us note that \( G^+ > 0 \) on \( V^+ \). For each \( x \in U^+ \), it follows by Proposition 7 that \( f^n x \in V^+ \) for some \( n \geq 0 \). Thus \( G^+(f^n(x)) > 0 \), so it follows from (17) that \( G^+(x) > 0 \). This shows that \( G^+ > 0 \) on \( U^+ = C^N - K^+ \). Conversely, it is evident that \( G^+ = 0 \) on \( K^+ \).

\[\text{Remark 4. If } m = 1, \text{ i.e. if } f = f_1, \text{ then we actually have}
\]

\[
G^+(x) = \log \|x\| + (d - 1)^{-1} \log |c_d| + o(1),
\]

\[
G^-(x) = \log \|x\| + (d - 1)^{-1} \log |a^{-1} c_d| + o(1),
\]

on \( V^\pm \) as \( x \to \infty \), where \( a = a_1 \) and \( c_d = c_1, d_1 \) in the notation of (3).

4. INVARIANT CURRENTS

Let us begin with some general computations involving currents. We recall (see [BT]) that if \( U \) is a continuous, psh function, and if \( T \) is a positive, closed \((p, p)\)-current, then \( dd^c U \wedge T \) is a \((p - 1, p - 1)\)-current, whose action on a test form \( \varphi \) of degree \((p - 1, p - 1)\) is defined by \( dd^c U \wedge T(\varphi) = T(Udd^c \varphi) \). If \( \tau \) is a Borel measure, and if \( t \mapsto S_t \) is a Borel measurable family of \((p, p)\)-currents, we will define a new \((p, p)\)-current, which acts on a test form \( \varphi \) of degree \((p, p)\) as

\[
(\int \tau(t) S_t)(\varphi) := \int (S_t \varphi) \tau(t).
\]

We define

\[
L^-(\zeta_1, \ldots, \zeta_N) = \max_{j=1,\ldots,N-\nu} \log^+ |\zeta_j|
\]

and

\[
L^+(\zeta_1, \ldots, \zeta_N) = \max_{j=N-\nu+1,\ldots,N} \log^+ |\zeta_j|.
\]
Lemma 10. We have

\[ (dd^c L^-)^{N-\nu} = dd^c L^- \wedge \cdots \wedge dd^c L^- = \int \tau_{N-\nu}(\zeta') \left[ \{\zeta'\} \times \mathbb{C}^\nu \right] \]

and

\[ (dd^c L^+)^\nu = dd^c L^+ \wedge \cdots \wedge dd^c L^+ = \int \tau_\nu(\zeta'') \left[ \mathbb{C}^{N-\nu} \times \{\zeta''\} \right], \]

where \( \tau_j \) denotes the \( j \)-dimensional Hausdorff measure on the \( j \)-torus \( \{|\zeta_1| = \cdots = |\zeta_j| = 1\} \) in \( \mathbb{C}^j \), and \([X]\) denotes the current of integration over the complex manifold \( X \). In particular,

\[ (dd^c \max(L^+, L^-))^N = (dd^c L^-)^{N-\nu} \wedge (dd^c L^+)^\nu. \]

Proof. First we consider \( L^- \) and show that (18) holds; the proof of (19) is similar. Let us introduce the variables \( z_j = x_j + iy_j = \log \zeta_j \) for \( 1 \leq j \leq N-\nu \) and \( z_j = \zeta_j \) for \( N-\nu + 1 \leq j \leq N \). Then \( \log^+ |\zeta_j| = \max(x_j, 0) \) for \( 1 \leq j \leq N-\nu \). Since the operator \( dd^c \) is invariant under holomorphic coordinates, we may compute in the \( z \) coordinates to obtain

\[ (dd^c \max_{1 \leq j \leq N-\nu} (x_j, 0))^{N-\nu} = \int_{\mathbb{R}^{N-\nu}} dy [\{y\} \times \mathbb{C}^\nu], \]

where \( dy \) denotes Lebesgue measure. This identity is remarked in [BT], and the computation is carried out in [HP, Lemma 3.5]. We obtain the formula (18) by transforming this identity (locally) under the exponential map \( y \mapsto e^{iy} \); we observe that under the exponential map, the current of integration over \([\{y\} \times \mathbb{C}^\nu]\) is taken to the current of integration \([\{\zeta'\} \times \mathbb{C}^\nu]\) and that \((N-\nu)\)-dimensional Lebesgue measure on \( \mathbb{R}^{N-\nu} \) is taken (locally) to the measure \( \tau_{N-\nu} \).

For (20) we recall that the wedge product of currents of integration corresponds to the current of integration over the intersection. Thus \( \delta_{(\zeta', \zeta'')} = \{\zeta'\} \times \mathbb{C}^{N-\nu} \wedge \{\zeta'\} \times \mathbb{C}^{N-\nu} = \{\zeta''\} \) is the unit point mass at \((\zeta', \zeta'')\). Integrating this observation with respect to \( \tau_{N-\nu} \) in the variable \( \zeta' \) and \( \tau_\nu \) in \( \zeta'' \), and applying (18) and (19), we have that

\[ \tau_\nu \wedge \tau_{N-\nu} = (dd^c L^+)^\nu \wedge (dd^c L^-)^{N-\nu}. \]

Since \( L = \log^+ |\zeta| = \max(L^-, L^+) \) is equal to \( L^- \) in the case \( \nu = 0 \), we see by (18) that \((dd^c L)^N\) is also equal to the measure \( \tau_N = \tau_\nu \wedge \tau_{N-\nu} \). Thus \((dd^c L)^N\) is equal to the left hand side of this identity, which gives (20).

Since \( G^+ \) and \( G^- \) are continuous, we may define \( \mu^+ := \frac{1}{2\pi} dd^c G^+ \) and \( \mu^- := \frac{1}{2\pi} dd^c G^- \). It follows from Theorem 9 that

\[ (f^\nu)^* \mu^+ = d^\nu \mu^+ \quad \text{and} \quad (f^{N-\nu})^* \mu^- = d^{N-\nu} \mu^- \]

We take the wedge product \( \mu := \mu^+ \wedge \mu^- \) and obtain a Borel measure, which then satisfies

\[ f^{\nu(N-\nu)}(\mu) = \mu. \]

We define \( G := \max(G^+, G^-) \).

Theorem 11. \( \mu^+ = 0 \) on \( U^+ \); \( \mu^- = 0 \) on \( U^- \); \((dd^c G)^N = 0 \) on \( \mathbb{C}^N - K \); and

\[ \frac{1}{2\pi} dd^c G)^N = \mu. \]
Proof. It follows from (18) that the support of $(dd^c L^-)^{N-\nu}$ is disjoint from $\{L^- > 0\}$, and so $(dd^c L^-)^{N-\nu} = 0$ there. Similarly, $(dd^c L^+)^{N-\nu} = 0$ on $\{L^+ > 0\}$. We restrict ourselves now to the case of $G^+$; the case of $G^-$ is similar. By Theorem 9, we have that $d^{-n} L^+(f^{(N-\nu)n})$ converges uniformly on compact sets to $G^+$ as $n \to \infty$. It then follows that $\mu^+ = (dd^c G^+)^{\nu} = 0$ on $\{G^+ > 0\}$.

To work with $G$, we note that the sequence

$$G_n := d^{-n} \max(L^- (f^{(N-\nu)n}), L^+ (f^{\nu n}))$$

converges uniformly on compact sets to $G$. Arguing as from (18) with $\nu = 0$, we have that

$$(dd^c \max(L^+, L^-))^N = 0$$

on $\{\zeta \in \mathbb{C}^N : ||\zeta|| > 1\}$, and so $(dd^c G_n)^N = 0$ on $\{G_n > 0\}$. Taking the limit as $n \to \infty$, we have that $(dd^c G)^N = 0$ on $\{G > 0\} = \mathbb{C}^N - K$.

Finally, we note that by equation (20), we have that

$$(dd^c G_n)^N = \frac{1}{d^n N} (dd^c L^- (f^{(N-\nu)n}))^\nu \wedge (dd^c L^+ (f^{\nu n}))^{N-\nu}.$$ 

The last assertion follows upon taking the limit as $n \to \infty$. \hfill $\square$

Remark 5. Let us recall (see Klimek [K]) that the pluri-complex Green function $G_K$ is characterized as the psh function on $\mathbb{C}^N$ such that $G_K = \log ||x|| + O(1)$ at infinity, $G_K = 0$ on $K$, and $(dd^c G_K)^N = 0$ on $\mathbb{C}^N - K$. It follows from Theorems 9 and 11 that $G := (G^+, G^-)$ coincides with $G_K$. Since $G$ is continuous, it follows by definition that $K$ is pluri-regular. By a Theorem of Siciak, it follows that $G$ is equal to the supremum of $\deg(q)^{-1} \log ||q||$, taken over all polynomials $q$ with $|q|_{K} \leq 1$, and degree equal to $\deg(q)$. By Theorem 11, it follows that $\mu$ is the pluri-complex equilibrium measure of $K$, normalized to have total mass one.

References


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