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TRANSVERSELY PROJECTIVE STRUCTURES ON A TRANSVERSELY HOLOMORPHIC FOLIATION

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Abstract. The space of transversely projective structures on a transversely holomorphic foliation is described. Some applications are given.

1. Introduction

Let $\mathcal{F}$ be a codimension two foliation on a C^∞ manifold M. A transversely holomorphic foliation on $\mathcal{F}$ is defined by data of the following type:

Let $\{U_i\}_{i\in I}$ be an open covering of M, and let $\phi_i : U_i \longrightarrow \mathbb{C}$ be submersions onto the image such that $\mathcal{F}$ is the kernel of the differential $d\phi_i$, or in other words, the fibers of ϕ_i are leaves for $\mathcal{F}$. If there is a biholomorphic map

$$f_{i,j} : \phi_i(U_i \cap U_j) \longrightarrow \phi_j(U_i \cap U_j)$$

for each pair $i, j \in I$ such that the following diagram is commutative

$$\begin{array}{ccc} U_i \cap U_j & = & U_i \cap U_j \\ \Big\downarrow \phi_i & & \Big\downarrow \phi_j \\ \phi_i(U_i \cap U_j) & \xrightarrow{f_{i,j}} & \phi_j(U_i \cap U_j), \end{array}$$

then $\{U_i, \phi_i, f_{i,j}\}$ defines a *transversely holomorphic* structure on $\mathcal{F}$.

A projective structure on a Riemann surface X is defined by giving a covering of X by holomorphic coordinate charts such that all the transition functions are restrictions of Möbius transformations.

The notion of a projective structure can be extended to the situation of foliations. We recall that a transversely holomorphic foliation $\mathcal{F}$ as above is a *transversely projective foliation* if each $f_{i,j}$ is the restriction of a Möbius transformation to $\phi_i(U_i \cap U_j)$.

Let $\mathcal{F}$ be a transversely holomorphic foliation. The normal bundle $N := TM/\mathcal{F}$ has a Bott connection, as explained in Section 2. Moreover, it has a holomorphic structure in the transverse direction in a sense which is briefly described below (the details are given in Section 2).

The pullback, using the above maps ϕ_i, of anti-holomorphic differentials on $\mathbb{C}$ define a C^∞ line bundle $\overline{N^*}$ over M equipped with a Bott connection. The holomorphic structure on N in the transverse direction is a homomorphism from (locally defined) flat sections (with respect to the Bott partial connection) of N to the flat sections of $\overline{N^*} \otimes N$ with respect to the Bott partial connection satisfying a Leibniz

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identity. The holomorphic structure on N in the transverse direction induces a holomorphic structure in the transverse direction on any tensor power of N.

We prove that the space of transversely projective structures on $\mathcal{F}$ (with the transversely holomorphic structure fixed) is an affine space for the space of global sections of $N^* \otimes N^*$ that are both flat with respect to the Bott partial connection and holomorphic with respect to the transversely holomorphic structure of $N^* \otimes N^*$ (Theorem 2.6).

Let M be a compact Kähler manifold and $\mathcal{F}$ be a holomorphic foliation of codimension one. As a corollary of Theorem 2.6 we prove that if the holomorphic tangent bundle T_M is semistable of positive degree, then there can be at most one transversely projective structure on $\mathcal{F}$ (Lemma 2.7).

2. Transversely holomorphic foliations

Let M be a connected smooth real manifold of dimension $d+2$. Let $\mathcal{F}$ be a C^∞ subbundle of rank d of the tangent bundle TM.

We will recall the definition of a transversely holomorphic structure.

Definition 2.1. A transversely holomorphic structure on $\mathcal{F}$ is defined by giving the following data [4], [2] :

(1) A covering of M by open subsets U_i, where i runs over an index set I. Hence we have $\bigcup_{i\in I} U_i = M$.
(2) For each $i \in I$, a submersion ϕ_i of U_i to an open subset D_i of $\mathbb{C}$. The restriction $\mathcal{F}|_{U_i}$ is the kernel of the differential map $d\phi_i : TU_i \longrightarrow \phi_i^* TD_i$.
(3) For every pair $i, j \in I$, there is a commutative diagram of maps

$$\begin{array}{ccc} U_i \cap U_j & \xrightarrow{\text{Id}} & U_i \cap U_j \\ \Big\downarrow \phi_i & & \Big\downarrow \phi_j \\ \phi_i(U_i \cap U_j) & \xrightarrow{f_{i,j}} & \phi_j(U_i \cap U_j) \end{array}$$

where $f_{i,j}$ is a holomorphic map.

Note that the second condition implies that $\mathcal{F}$ is closed under Lie bracket, or in other words, $\mathcal{F}$ is a foliation.

Two such data $\{U_i, \phi_i\}_{i\in I}$ and $\{U_i, \phi_i\}_{i\in J}$ are called *equivalent* if their union, namely

$$\{U_i\,, \phi_i\}_{i\in I\cup J}\,,$$

also satisfies the above conditions. A *transversely holomorphic* structure on $\mathcal{F}$ will mean an equivalence class of data of the above type satisfying the above three conditions.

Next we will recall the definition of a transversely projective foliation. Let $\mathcal{F}$, as before, be a C^∞ subbundle of rank d.

Definition 2.2. A transversely projective structure on $\mathcal{F}$ is defined by giving a data $\{U_i, \phi_i\}_{i\in I}$ exactly as in Definition 2.1, but satisfying the extra condition (apart from the three conditions in Definition 2.1) that the holomorphic maps $f_{i,j}$ in condition (3) are of the form $z \longmapsto (az+b)/(cz+d)$, where $a, b, c, d \in \mathbb{C}$ are constant scalars and $ad - bc = 1$, or in other words, each $f_{i,j}$ is the restriction of some Möbius transformation. The scalars a, b, c, d may depend on the index i. As before, two such data $\{U_i, \phi_i\}_{i\in I}$ and $\{U_i, \phi_i\}_{i\in J}$ are called *equivalent* if their union

$\{U_i, \phi_i\}_{i\in I\cup J}$ is also a data for a transversely projective structure. A *transversely projective* structure on $\mathcal{F}$ will mean an equivalence class of such data.

Clearly, a transversely projective structure on $\mathcal{F}$ defines a transversely holomorphic structure on $\mathcal{F}$. If $\overline{\mathcal{F}}$ is a transversely holomorphic structure on $\mathcal{F}$, then a transversely projective structure on $\overline{\mathcal{F}}$ is a transversely projective structure on $\mathcal{F}$ such that the transversely holomorphic structure defined by it coincides with $\overline{\mathcal{F}}$.

If $d = 0$, then a transversely holomorphic structure on $\mathcal{F}$ is simply a complex structure on the surface M, and a transversely projective structure on $\mathcal{F}$ is a projective structure on M.

Any two projective structures on a Riemann surface C differ by a quadratic differential on C, that is, a holomorphic section of $K_C^{\otimes 2}$, where $K_C := T^*C$ is the holomorphic cotangent bundle. More precisely, the space of projective structures on C is an affine space for the space of quadratic differentials [3]. Let P be a projective structure on C and $\omega \in H^0(C,\, K_C^{\otimes 2})$ a quadratic differential. Let (U, z) be a coordinate chart compatible with P. So the restriction of ω to U is of the form $hdz\otimes dz$, where h is a holomorphic function. Let w be a function on U such that $\mathcal{S}_z(w) = h$, where $\mathcal{S}_z$ is the Schwarzian differential operator with respect to z defined by

$$\mathcal{S}_z(f) := \frac{2f'(z)f'''(z) - 3(f''(z))^2}{2(f'(z))^2}\,.$$

Then (U, w) is a coordinate chart compatible with the projective structure $P + \omega$ (see [3] for the details).

Let $\overline{\mathcal{F}}$ be a transversely holomorphic structure on the foliation $\mathcal{F}$. The normal bundle

$$N := TM/\mathcal{F} \tag{2.3}$$

clearly has a structure of a complex line bundle. Indeed, this is an immediate consequence of the condition (3) in Definition 2.1. Therefore, for every integer $k \in \mathbb{Z}$, we have a complex line bundle $N^{\otimes k}$ obtained by taking the kth tensor power of the complex line bundle N. By $N^{\otimes -1}$ we mean the dual line bundle N^*.

The line bundle N, and hence any $N^{\otimes k}$, has the Bott partial connection [6]. Recall that the Lie bracket operation on the sheaf of sections of the tangent bundle TM defines the Bott partial connection

$$N \longrightarrow \mathcal{F}^* \otimes N \tag{2.4}$$

on N along the foliation $\mathcal{F}$. Since $\mathcal{F}$ is closed under the Lie bracket operation, the composition $[TM\,, TM] \longrightarrow TM \longrightarrow N$ gives the operator in (2.4). The Jacobi identity for Lie bracket ensures that this partial connection is flat. The partial connection of $N^{\otimes k}$ is induced by the partial connection on N.

The conjugate of N^* will be denoted by $\overline{N^*}$. Over each U_i in Definition 2.1, consider $\phi_i^*\overline{K}_{\mathbb{C}}$, where $\overline{K}_{\mathbb{C}}$ is the space of anti-holomorphic differentials. Denote this line bundle $\phi_i^*\overline{K}_{\mathbb{C}}$ over U_i by $\overline{L}_i$. For any pair U_i and U_j, glue $\overline{L}_i$ with $\overline{L}_j$ over $U_i \cap U_j$ using the differential $\overline{\partial}(\overline{f_{i,j}})$ for the map $f_{i,j}$ in Definition 2.1(3). It is easy to see that these line bundles $\overline{L}_i$ patch compatibly to give a line bundle over M. The line bundle over M obtained this way clearly coincides with $\overline{N^*}$. The Bott partial connection on N induces a partial connection on $\overline{N^*}$.

Any such line bundle $N^{\otimes k}$ has a natural transversely holomorphic structure. This means that there is a first order differential operator

$$\overline{\partial}_{N^{\otimes k}} : N^{\otimes k} \longrightarrow \overline{N^*} \otimes N^{\otimes k} \tag{2.5}$$

from the local section of $N^{\otimes k}$ flat under the Bott partial connection to flat sections of $\overline{N^*} \otimes N^{\otimes k}$ satisfying the Leibniz identity. For any flat local section s of $N^{\otimes k}$, defined over some open subset of some U_i in Definition 2.1, and for any C^∞ function f defined on some open subset of $\phi_i(U_i)$, the Leibniz identity says

$$\overline{\partial}_{N^{\otimes k}}((f \circ \phi_i)s) = (f \circ \phi_i)\overline{\partial}_{N^{\otimes k}}(s) + \phi_i^* \overline{\partial} f \otimes s\,.$$

The operator $\overline{\partial}_{N^{\otimes k}}$ is simply the Dolbeault operator on the holomorphic tangent bundle $T_{\mathbb{C}}^{\otimes k}$ of the complex line $\mathbb{C}$ transported to M using the projections ϕ_i. It may be noted that the condition in Definition 2.1(3), that every $f_{i,j}$ is holomorphic, ensures that these locally defined operators patch compatibly to define the global differential operator $\overline{\partial}_{N^{\otimes k}}$.

A (local) section s of $N^{\otimes k}$ will be called *holomorphic* if $\overline{\partial}_{N^{\otimes k}}(s) = 0$.

It is easy to see that both the complex structure of N and the transversely holomorphic structure of N are compatible with respect to the Bott partial connection. In other words, both the complex vector space structure of the fibers of N and the Dolbeault operator $\overline{\partial}_N$ defined in (2.5) commute with the differential operator in (2.4) defining the Bott connection. Equivalently, parallel translation (for the Bott connection) along the leaves of the foliation $\overline{\mathcal{F}}$ of holomorphic sections of N remain holomorphic. Also, parallel translations for the Bott connection commute with multiplication by $\sqrt{-1}$ of the fibers of N.

As we noted, the Bott partial connection on N induces a flat partial connection on any $N^{\otimes k}$. The above compatibility properties of the Bott connection with the complex structure and the holomorphic structure evidently remain valid for any $N^{\otimes k}$.

Consider the space of all globally defined smooth sections over M of the complex line bundle N^{-2} that are flat with respect to the Bott partial connection. Let $\mathcal{V}_{\overline{\mathcal{F}}}$ denote its subspace consisting of all sections that are transversely holomorphic for the transversely holomorphic foliation $\overline{\mathcal{F}}$. In other words, $s \in \mathcal{V}_{\overline{\mathcal{F}}}$ if

$$\overline{\partial}_{N^{\otimes -2}}(s) = 0$$

for the operator $\overline{\partial}_{N^{\otimes -2}}$ in (2.5). The complex vector space $\mathcal{V}_{\overline{\mathcal{F}}}$ need not be of finite dimension.

Let $\mathcal{P}(\overline{\mathcal{F}})$ denote the space of transversely projective structures on M such that the underlying transversely holomorphic structure coincides with $\overline{\mathcal{F}}$.

Theorem 2.6. *The space $\mathcal{P}(\overline{\mathcal{F}})$ is an affine space for the vector space $\mathcal{V}_{\overline{\mathcal{F}}}$.*

Proof. Let $\mathcal{P} \in \mathcal{P}(\overline{\mathcal{F}})$ be a transversely projective structure defined by

$$\{U_i\,, \phi_i\,, f_{i,j}\}_{i,j \in I}$$

as in Definition 2.2. Let $\mathcal{P}_1 \in \mathcal{P}(\overline{\mathcal{F}})$ be another transversely projective structure. Choosing a finer open covering for the data defining $\mathcal{P}$ and $\mathcal{P}_1$, we may assume that $\mathcal{P}_1$ is defined by the data $\{U_i\,, \psi_i\,, g_{i,j}\}_{i \in I}$. In other words, the open coverings are the same for both $\mathcal{P}$ and $\mathcal{P}_1$.

For every $i \in I$, we have the commutative diagram of maps

$$\begin{array}{ccc} U_i & = & U_i \\ \downarrow \phi_i & & \downarrow \psi_i \\ \phi_i(U_i) & \xrightarrow{F_i} & \psi_i(U_i) \end{array}$$

where F_i is a biholomorphism. Now, $\psi_i(U_i)$ being an open subset of $\mathbb{C}$, it has a canonical projective structure. The pullback of this projective structure, using the isomorphism F_i, defines a projective structure on $\phi_i(U_i)$. This projective structure on $\phi_i(U_i)$ will be denoted by $\mathcal{P}_{i,1}$. On the other hand, since $\phi_i(U_i)$ is an open subset of $\mathbb{C}$, it has a natural projective structure. This projective structure on $\phi_i(U_i)$ will be denoted by $\mathcal{P}_{i,0}$.

There is a quadratic differential ω_i on $\phi_i(U_i)$, i.e., a holomorphic section of $K^2_{\mathbb{C}}$ over $\phi_i(U_i)$, such that

$$\mathcal{P}_{i,1} = \mathcal{P}_{i,0} + \omega_i \,.$$

Indeed, as we have already noted, any two projective structures on a Riemann surface differ by a quadratic differential.

The pullback $\phi_i^*\omega_i$ defines a transversely holomorphic section of $N^{\otimes -2}$ over U_i. This section is evidently flat with respect to the Bott partial connection.

For any $j \in I$, define ω_j on $\phi_j(U_j)$ as above. On $U_i \cap U_j$ the two sections of $N^{\otimes -2}$, defined by $\phi_i^*\omega_i$ and $\phi_j^*\omega_j$ respectively, clearly coincide. In other words, these locally defined sections $\phi_i^*\omega_i$ patch together to give a globally defined section ω of N^{-2}. This section is clearly in $\mathcal{V}_{\overline{\mathcal{F}}}$.

Conversely, if we have a transversely projective structure $\mathcal{P}$ and a holomorphic section $\omega \in \mathcal{V}_{\overline{\mathcal{F}}}$, then using the fact that the space of projective structures on a Riemann surface C is an affine space for the space of quadratic differentials on C, it is a straight-forward exercise to construct a new transversely projective structure $\mathcal{P}_1$ on $\overline{\mathcal{F}}$. We earlier described the affine space structure of the space of projective structures on a Riemann surface. That construction can be reproduced in the context of transversely holomorphic foliations. In fact, we can just retrace the earlier construction of a section in $\mathcal{V}_{\overline{\mathcal{F}}}$ from a pair of transversely projective structure on $\overline{\mathcal{F}}$.

Finally, it is straight-forward to check that the map

$$\mathcal{P}(\overline{\mathcal{F}}) \times \mathcal{V}_{\overline{\mathcal{F}}} \longrightarrow \mathcal{P}(\overline{\mathcal{F}})$$

defined by $(\mathcal{P}\,,\omega) \longmapsto \mathcal{P}_1$ makes $\mathcal{P}(\overline{\mathcal{F}})$ an affine space for the vector space $\mathcal{V}_{\overline{\mathcal{F}}}$. This completes the proof of the theorem. □

We will now specialize to the situation where the manifold M is a compact Kähler manifold. Furthermore, $\mathcal{F}$ is assumed to be a holomorphic subbundle of the holomorphic tangent bundle T_M of M. In other words, $\mathcal{F}$ is a holomorphic foliation of codimension one on the complex manifold M. In particular, $\mathcal{F}$ has an induced structure of a transversely holomorphic foliation. So, the maps ϕ_i in Definition 2.1 are taken to be holomorphic.

Let $P(\mathcal{F})$ denote the space of all transversely projective structures on $\mathcal{F}$ satisfying the following two conditions:

(1) for any $P \in P(\mathcal{F})$, its underlying transversely holomorphic foliation coincides with $\mathcal{F}$;

(2) P is holomorphic, or in other words, the maps ϕ_i in Definition 2.2 are holomorphic submersions.

From Theorem 2.6 it follows that $P(\mathcal{F})$ is an affine space for the space of all holomorphic sections of $N^{\otimes -2}$ that are flat with respect to the Bott partial connection.

We recall the definition of semistability of a vector bundle over a compact Kähler manifold [5]. Fix a Kähler form β on M. A vector bundle E over M is called *semistable* if for any coherent analytic subsheaf $F \subset E$ of positive rank, the inequality

$$\frac{\text{degree}(F)}{\text{rank}(F)} \leq \frac{\text{degree}(E)}{\text{rank}(E)}$$

is valid. The degree of a coherent sheaf F is defined as

$$\text{degree}(F) = \int_M c_1(F)\beta^{\dim M-1}$$

where $\dim M$ is the complex dimension of M. Note that $c_1(F)\beta^{\dim M-1}$ is a top degree cohomology class of M.

Lemma 2.7. *Let the holomorphic tangent bundle T_M be semistable of positive degree. Then given any holomorphic foliation $\mathcal{F}$ of codimension one, there is at most one transversely projective structure on $\mathcal{F}$. In other words, the cardinality of the set $P(\mathcal{F})$ is not more than one.*

Proof. From Theorem 2.6 it follows that if $P, P' \in P(\mathcal{F})$, then the difference $P'-P$ is a holomorphic section of the holomorphic line bundle $N^{\otimes -2}$ over M. Therefore, it suffices to show that $N^{\otimes -2}$ does not admit any nonzero holomorphic section.

Since N is a quotient bundle of T_X, the semistability condition ensures that the degree of N is positive. Hence $\text{degree}(N^{\otimes -2}) = -2\text{degree}(N)$ is negative. Therefore, we have

$$H^0(M, N^{\otimes -2}) = 0$$

since any semistable vector bundle of negative degree does not admit any nonzero section. This completes the proof of the lemma. □

Now assume that M is a complex projective manifold with $\dim M \geq 2$. We further assume that

$$\text{NS}(M) := H^{1,1}(M) \cap H^2(M, \mathbb{Z}) = \mathbb{Z},$$

where $H^{1,1}$ corresponds to the Hodge decomposition of $H^2(M, \mathbb{C})$. The group $\text{NS}(M)$ is known as the *Neron–Severi group*.

An immediate consequence of the assumption on $\text{NS}(M)$ is that if L is a holomorphic line bundle over M with $c_1(L) \neq 0$, then for any other line bundle L' over M, the first Chern class $c_1(L')$ is a scalar multiple of $c_1(L)$.

Lemma 2.8. $\dim P(\mathcal{F}) \leq 1$.

Proof. Fix an ample class $\omega_0 \in \text{NS}(M)$. So ω_0 is represented by a positive $(1,1)$-form (also called a Kähler form). Let $c \in \mathbb{Q}$ be such that

$$c_1(N) = c\omega_0 .$$

The existence of such a scalar is ensured by the assumption that $\text{NS}(M) = \mathbb{Z}$.

A theorem of Bott says that $c_1(N)^2 = 0$ [6, Theorem 3.4], [1]. So we have

$$c^2(\omega_0)^2 = 0 .$$

Now, since the cohomology class ω_0 is ample and $\dim M \geq 2$, we have $(\omega_0)^2 \neq 0$. Therefore, $c = 0$. In other words, we have $c_1(N) = 0$. Consequently, N is topologically trivial. This implies that

$$\dim H^0(M, N) \leq 1 . \tag{2.9}$$

Indeed, if $\dim H^0(M, N) \geq 2$, then given any two linearly independent sections of N, say s_1 and s_2, some linear combination of them, say $s = \lambda_1 s_1 + \lambda_2 s_2$, vanishes somewhere but it is not identically zero. Let $\theta \in H^2(M, \mathbb{Z})$ be the cohomology class represented (using Poincaré duality) by the divisor of M defined by s. Since $\theta \neq 0$ and $\theta = c_1(N)$, we have a contradiction. This establishes (2.9).

In view of Theorem 2.6, the inequality (2.9) implies that $\dim P(\mathcal{F}) \leq 1$. This completes the proof of the lemma. □

A topologically trivial holomorphic line bundle admits a nonzero section if and only if the line bundle is holomorphically trivial. Therefore, in Lemma 2.8 we have $\dim P(\mathcal{F}) = 1$ if and only if N is holomorphically trivial and the trivialization of N is flat with respect to the Bott partial connection on N.

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