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ON DISTORTION OF HAUSDORFF MEASURES UNDER QUASICONFORMAL MAPPINGS

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Abstract. Astala (*Acta Math.* **173** (1994), no. 1, 37–60) gave optimal bounds for the distortion of Hausdorff dimension under planar quasiconformal maps. The corresponding estimates on the level of Hausdorff measures remain open. We show that these techniques allow for establishing absolute continuity for some weaker Hausdorff measures.

1. Introduction

A homeomorphism $\phi\colon \Omega \to \Omega'$ between planar domains is called *K-quasiconformal* if it lies in the Sobolev class $W^{1,2}_{\mathrm{loc}}(\Omega)$ and satisfies the distortion inequality

$$\max_{\alpha} |\partial_\alpha \phi| \le K \min_{\alpha} |\partial_\alpha \phi| \quad \text{a.e. in } \Omega.$$

Astala [1], in his seminal work, gave a complete description of dimension distortion of sets under planar quasiconformal mappings.

Theorem 1.1 ([1]). *Let $\phi\colon \Omega \to \Omega'$ be K-quasiconformal and suppose $E \subset \Omega$ is compact. Then*

$$\frac{1}{K}\left(\frac{1}{\dim(E)} - \frac{1}{2}\right) \le \frac{1}{\dim(\phi(E))} - \frac{1}{2} \le K\left(\frac{1}{\dim(E)} - \frac{1}{2}\right). \tag{1.2}$$

This inequality is best possible.

It is then natural to ask (see [1, 3]) whether these estimates hold on the level of Hausdorff measures $\mathcal{H}^t$. That is, if ϕ is a planar K-quasiconformal mapping, $0 < t < 2$ and $d = \frac{2Kt}{2+(K-1)t}$, is it true that

$$\mathcal{H}^t(E) = 0 \;\Rightarrow\; \mathcal{H}^d(\phi(E)) = 0? \tag{1.3}$$

In other words, do we have absolute continuity $\phi^*\mathcal{H}^d \ll \mathcal{H}^t$? It is classical that quasiconformal mappings are absolutely continuous with respect to the Lebesgue measure, and Astala [1] proves this in a quantitatively optimal form,

$$|\phi(E)| \le C|E|^{\frac{1}{K}}, \tag{1.4}$$

with a constant that depends only on K and the normalization.

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A convenient normalization we shall work with is the following. We call a quasiconformal map $\phi\colon \mathbb{C} \to \mathbb{C}$ *principal* if it is conformal outside a compact set and normalized by $\phi(z) = z + O(1/|z|)$ as $z \to \infty$.

The question of absolute continuity in (1.3) has recently been studied in [3]. Let us discuss some of their results in this direction. They reduce the absolute continuity question to the following (for the complementary part, see Theorem 2.2). We use $\mathcal{H}^d_\infty$ to denote the d-dimensional Hausdorff content.

Conjecture 1.5 ([3]). *For any compact set $E \subset \mathbb{C}$ and for any principal K-quasiconformal mapping h which is conformal on $\mathbb{C} \setminus E$, we have for any $d \in (0, 2]$,*

$$\mathcal{H}^d_\infty(h(E)) \simeq \mathcal{H}^d_\infty(E),$$

with constants that depend only on K and d.

The authors were able to prove this for $d = 1$, thus confirming absolute continuity for the special case $t = 2/(K + 1)$ and $d = 1$. In the case $d > 1$, they obtained the following partial result; see Corollary 2.12 of [3].

Given $1 < d < 2$ consider a measure function of the form $\delta(r) = r^d \varepsilon(r)$, where

$$\int_0 \varepsilon(r)^{\frac{1}{d-1}} \frac{dr}{r} < \infty. \tag{1.6}$$

Theorem 1.7 ([3]). *Let $E \subset \mathbb{D}$ be a compact set and $\phi\colon \mathbb{C} \to \mathbb{C}$ be a principal K-quasiconformal mapping conformal outside $\mathbb{D}$. Let $t \in (\frac{2}{K+1}, 2)$ and $d = \frac{2Kt}{2+(K-1)t}$. Then we have*

$$\mathcal{H}^\delta_\infty(\phi(E)) \le C \left(\mathcal{H}^t_\infty(E)\right)^{\frac{d}{Kt}},$$

where the measure function δ satisfies (1.6). *The constant C depends only on δ and K.*

Our objective is to present a similar complementary result with different Hausdorff measures. We consider measure functions $\delta(r) = r^d \varepsilon(r)$ satisfying

$$\int_0 \varepsilon(r)^{\frac{Kt}{Kt-d}} \frac{dr}{r} < \infty. \tag{1.8}$$

We also make the technical assumption that the integrand is decreasing and $\varepsilon(r)$ is increasing in $(0, r_0)$ for some $r_0 > 0$.

Theorem 1.9. *Let $E \subset \mathbb{D}$ be a compact set and let $\phi\colon \mathbb{C} \to \mathbb{C}$ be a principal K-quasiconformal mapping conformal outside $\mathbb{D}$. Let $t \in (0, 2)$ and $d = \frac{2Kt}{2+(K-1)t}$. Then we have*

$$\mathcal{H}^\delta(\phi(E)) \le C \left(\mathcal{H}^t(E)\right)^{\frac{d}{Kt}},$$

where the measure function δ satisfies (1.8). *The constant C depends only on δ and K.*

Remark 1.10. Note that Theorem 1.7 gets sharper as $d \to 1$, while Theorem 1.9 improves as $K \to 1$. Also, Theorem 1.9 is valid for every dimension $0 < t < 2$. In fact, it can be viewed as a result which provides absolute continuity with respect to *some* Hausdorff measure of the right dimension. For instance, we can take $\varepsilon(r) = |\log r|^{-s}$, with $s > 1 - \frac{d}{Kt}$, and then $\mathcal{H}^\delta$ has dimension d.

Let us note that a discrete (stronger) variant of Conjecture 1.5 formulated as Question 2.4 in [3] has recently been disproved by Bishop; see [4]. We shall use the usual convention, that the constant C may change from line to line, but indicate its dependence on the parameters.

2. Distortion of Hausdorff measures

We will need two complementary results from [1]. The first one is a counterpart of Conjecture 1.5 for the area; see Lemma 3.3 and the remark afterwards in [1].

Theorem 2.1 ([1]). *Let $h\colon \mathbb{C} \to \mathbb{C}$ be a principal K-quasiconformal mapping which is conformal outside a compact set E. Then we have*

$$|h(E)| \le K|E|.$$

Another result we need is an optimal discrete version of Theorem 1.1 under *conformality assumption*.

Theorem 2.2. *Let $\phi\colon \mathbb{C} \to \mathbb{C}$ be a principal K-quasiconformal mapping which is conformal outside $\mathbb{D}$. Let $\mathcal{B}$ be a finite family of disjoint disks in $\mathbb{D}$ and assume that ϕ is conformal in $\bigcup \mathcal{B}$. Then, for any $t \in (0,2]$ and $d = \frac{2Kt}{2+(K-1)t}$, we have*

$$\left(\sum_{B\in\mathcal{B}} \operatorname{diam}(\phi B)^d\right)^{\frac{1}{d}} \le C(K)\left(\sum_{B\in\mathcal{B}} \operatorname{diam}(B)^t\right)^{\frac{1}{Kt}}.$$

This result is implicit in [1]; see Corollary 2.3 and the variational principle on p. 48. It can also be deduced from the improved borderline integrability of the Jacobian under conformality assumption [2]. For this latter approach, see [3, (2.6)].

Proof of Theorem 1.9. In [1] the area distortion (cf. (1.4)) is proved via a decomposition reducing it to the two complementary cases above (with $t = d = 2$). The difficulty of establishing absolute continuity for distortion of Hausdorff measures when $0 < t < 2$ is that Conjecture 1.5 is unavailable in general. However, we can still use Theorem 2.1 as a substitute provided that we are concerned with coverings with disks of the same size.

As in the statement of Theorem 1.9, let ϕ be a principal K-quasiconformal mapping, conformal outside $\mathbb{D}$. Let $\mathcal{B}$ be an arbitrary finite family of disjoint disks in $\mathbb{D}$. We assume that the disks have diameter less than $a(K)$, a constant specified later. Define a subfamily $\mathcal{B}_k$ for $k \in \mathbb{N}$ by

$$\mathcal{B}_k = \{B \in \mathcal{B}\colon a2^{-(k+1)K} < \operatorname{diam} B \le a2^{-kK}\}.$$

The constant $a = a(K) > 0$ is specified as follows. Hölder continuity of quasiconformal mappings, i.e. the area distortion estimate (1.4) for disks, assures that

$$\operatorname{diam} \phi(5B) \le 2^{-k},$$

provided that $\operatorname{diam} B \le a(K)2^{-kK}$ with an appropriate constant $a(K)$.

Decompose the map $\phi = \phi_1 \circ h$, where both ϕ_1 and h are principal K-quasiconformal mappings, and h is conformal off $\bigcup \mathcal{B}_k$ while ϕ_1 is conformal in $h(\bigcup \mathcal{B}_k)$. This decomposition can be done due to the measurable Riemann mapping theorem. We shall frequently use the fact that the quasiconformal maps distort disks in a uniform manner, and thus, up to bounded eccentricity, we may consider quasidisks

as disks in our arguments. We apply Theorem 2.1 to find

$$\begin{aligned}\left(\frac{1}{|\mathcal{B}_k|}\sum_{B\in\mathcal{B}_k}\operatorname{diam}(hB)^t\right)^{\frac{1}{t}} &\le \left(\frac{1}{|\mathcal{B}_k|}\sum_{B\in\mathcal{B}_k}\operatorname{diam}(hB)^2\right)^{\frac{1}{2}}\\ \le \left(C(K)\frac{1}{|\mathcal{B}_k|}\sum_{B\in\mathcal{B}_k}\operatorname{diam}(B)^2\right)^{\frac{1}{2}} &\le \left(C(K)\frac{1}{|\mathcal{B}_k|}\sum_{B\in\mathcal{B}_k}\operatorname{diam}(B)^t\right)^{\frac{1}{t}}.\end{aligned} \tag{2.3}$$

The first inequality holds in view of the convexity of the function $r^{2/t}$, while in the last step we used the fact that family $\mathcal{B}_k$ contains disks of essentially the same size (up to a multiplicative constant depending on K).

For the map ϕ_1 we may apply the sharp estimate of Theorem 2.2, as ϕ_1 is conformal in $h(B)$ for every $B \in \mathcal{B}_k$ and is outside $h(\mathbb{D})$. Since we have to take care of normalization, note that according to Koebe's 1/4-theorem we have $h(\mathbb{D}) \subset B(h(0), 4)$. We obtain

$$\sum_{B\in\mathcal{B}_k}\operatorname{diam}(\phi B)^d \le C(K)\left(\sum_{B\in\mathcal{B}_k}\operatorname{diam}(hB)^t\right)^{\frac{1}{K}\frac{d}{t}}.$$

Combining this with (2.3), we have

$$\sum_{B\in\mathcal{B}_k}\operatorname{diam}(\phi B)^d \le C(K,t)\left(\sum_{B\in\mathcal{B}_k}\operatorname{diam}(B)^t\right)^{\frac{1}{K}\frac{d}{t}}. \tag{2.4}$$

We have the same estimate for all $\mathcal{B}_k$'s with constant independent of the decomposition, and thus we may combine them to find that

$$\begin{aligned}&\sum_{B\in\mathcal{B}}\operatorname{diam}(\phi(5B))^d\varepsilon(\operatorname{diam}\phi(5B)) \le C(K)\sum_k \varepsilon(2^{-k})\sum_{B\in\mathcal{B}_k}\operatorname{diam}(\phi B)^d\\ &\le C(K,t)\sum_k\varepsilon(2^{-k})\left(\sum_{B\in\mathcal{B}_k}\operatorname{diam}(B)^t\right)^{\frac{1}{K}\frac{d}{t}}\\ &\le C(K,t)\left(\sum_k\varepsilon(2^{-k})^{\frac{Kt}{Kt-d}}\right)^{1-\frac{d}{Kt}}\left(\sum_{B\in\mathcal{B}}\operatorname{diam}(B)^t\right)^{\frac{1}{K}\frac{d}{t}}.\end{aligned} \tag{2.5}$$

The first sum is finite by the assumption (1.8) on $\varepsilon(r)$. Since we have uniform bounds for all families $\mathcal{B}$ of disjoint disks, a standard $5r$-covering lemma leads to

$$\mathcal{H}^\delta(\phi(E)) \le C(K,\delta)\mathcal{H}^t(E)^{\frac{1}{K}\frac{d}{t}}.$$

□

Remark 2.6. The proof shows that a weaker form of (1.3) holds true. Namely, if we replace the assumption $\mathcal{H}^t(E) = 0$ by the stronger assumption that E has zero t-dimensional lower Minkowski content, then the conclusion of (1.3) is valid; that is, $\mathcal{H}^d(\phi(E)) = 0$.

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