COMPACT NON-ORIENTABLE SURFACES
OF GENUS 4 WITH EXTREMAL METRIC DISCS

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Dedicated to Professor Yoshihiro Mizuta on the occasion of his 60th birthday

ABSTRACT. A compact hyperbolic surface of genus g is said to be extremal if it admits an extremal disc, a disc of the largest radius determined by g. We know how many extremal discs are embedded in a non-orientable extremal surface of genus $g = 3$ or $g > 6$. We show in the present paper that there exist 144 non-orientable extremal surfaces of genus 4, and find the locations of all extremal discs in those surfaces. As a result, each surface contains at most two extremal discs. Our methods used here are similar to those in the case of $g = 3$.

1. INTRODUCTION

Let S be a compact hyperbolic surface of genus g, where g is the number of handles (≥ 2) or that of cross caps (≥ 3) according to whether or not S is orientable. The surface is naturally endowed with a metric induced by the hyperbolic metric of the unit disc $\mathbb{D}$. C. Bavard showed in [1] that the radius of a disc isometrically embedded in S is bounded by the constant $R = R_g$ depending only on g:

$$\cosh R_g = \frac{1}{2\sin\frac{\pi}{6-6\chi_g}},$$

where χ_g denotes the Euler characteristic; that is, $\chi_g = 2 - 2g$ or $2 - g$ if S is orientable or not, respectively. We call S an extremal surface if it contains an extremal disc, a disc of radius R_g. Our problem is to show how many extremal discs are embedded in an extremal surface. In the case that S is orientable, it was solved for every genus $g > 3$ ([2]), $g = 2$ ([3] and [6]), and $g = 3$ ([7]). In the case that S is non-orientable, we know the following for $g > 6$ and $g = 3$ ([4]): an extremal surface of genus $g > 6$ contains a unique extremal disc; there exist 11 extremal surfaces of genus 3, and each of them contains at most two extremal discs. Whereas our methods used here are similar to those in the case of $g = 3$, we have to deal with a large number of surfaces for $g = 4$.

Our main result is as follows:

Theorem 1.1. *There exist* 144 *non-orientable extremal surfaces* S_j ($j = 1, \ldots, 144$) *of genus* 4, *which are derived from the side-pairing patterns* P_j *of the hyperbolic regular* 18-*gon. They admit at most two extremal discs, and* 22 *of them admit exactly two extremal discs.* (P_j *is listed in the last part of this paper (Figures* 3–8), *where lines and dotted lines in the figures indicate pairs of sides pasted by the different and the same direction respectively; bullets correspond to the centers of extremal discs.) Tables* 1 *and* 2 *show the centers of extremal discs and the full group of automorphisms for each* S_j, *where* $\pi : \mathbb{D} \to S_j$ *denotes the natural projection,* $\beta = \pi/18$, *and* $R = R_4$.

TABLE 1. The list of surfaces of genus 4 admitting two extremal discs.

Surface	Centers of extremal discs	Aut$^{\pm}$
S_1, S_{89}	$\pi(0), \pi(\tanh \frac{R}{2})$	$D_2 \times \mathbb{Z}_2$
S_5, S_{22}, S_{65}	$\pi(0), \pi(\frac{1}{\sinh R})$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
$S_9, S_{14}, S_{15}, S_{133}, S_{141}$	$\pi(0), \pi(\tanh \frac{R}{2})$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
S_{17}, S_{21}, S_{94}	$\pi(0), \pi(\frac{i}{2\tanh R \cos \beta})$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
S_{53}	$\pi(0), \pi(\tanh \frac{R}{2})$	$\mathbb{Z}_2$
S_{69}, S_{78}, S_{135}	$\pi(0), \pi(\frac{1}{2\tanh R \cos 2\beta})$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
S_{76}, S_{142}	$\pi(0), \pi(\frac{2\sin 2\beta}{\tanh R}i)$	$\mathbb{Z}_2$
S_{77}, S_{81}, S_{139}	$\pi(0), \pi(\frac{2\sin 2\beta}{\tanh R}i)$	$\mathbb{Z}_2 \times \mathbb{Z}_2$

TABLE 2. The list of surfaces of genus 4 admitting a unique extremal disc.

Surface	Aut$^{\pm}$
S_{138}, S_{144}	D_6
S_4, S_{68}	D_3
$S_{73}, S_{87}, S_{121}, S_{127}, S_{137}, S_{143}$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
$S_3, S_6, S_8, S_{11}, S_{12}, S_{13}, S_{16}, S_{18}, S_{28}, S_{29}, S_{40}, S_{41}, S_{42},$ $S_{43}, S_{55}, S_{56}, S_{63}, S_{64}, S_{66}, S_{67}, S_{70}, S_{74}, S_{82}, S_{83}, S_{84},$ $S_{85}, S_{86}, S_{88}, S_{90}, S_{91}, S_{92}, S_{93}, S_{100}, S_{101}, S_{114}, S_{115},$ $S_{119}, S_{123}, S_{128}, S_{131}, S_{136}, S_{140}$	$\mathbb{Z}_2$
The others	$\{1\}$

In order to obtain this theorem we follow the same line of the proof for the case of genus 3 (cf. [4]).

2. Side-pairing patterns of the 18-gon

Since a fundamental region for a non-orientable extremal surface of genus $g \geq 3$ is a hyperbolic regular $(6g - 6)$-gon, we shall find every side-pairing pattern for

the regular 18-gon which yields a non-orientable surface of genus 4. Identifying the equivalent sides, we have a trivalent graph with 9 edges and 6 vertices on the surface. It is easily shown that there are 17 such trivalent graphs (Figure 1). Note that M, N, O, P, and Q appear in [5].
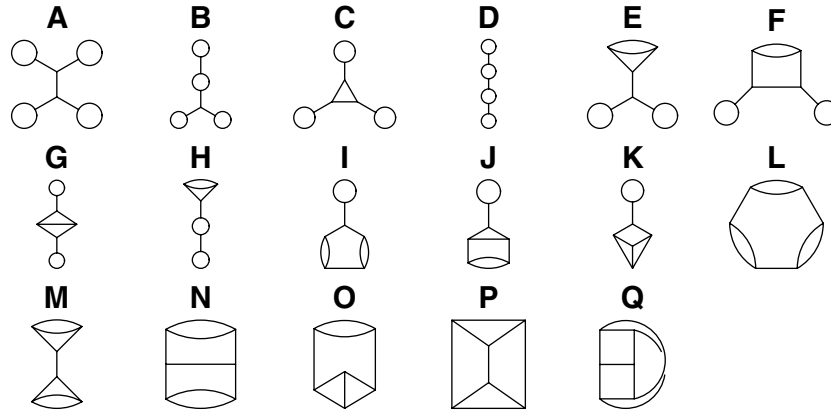

FIGURE 1. Trivalent graphs with 6 vertices

Considering the closed walks on these graphs by computer, we obtain all side-pairing patterns (cf. [4, §5.1]).

Theorem 2.1. *A regular* 18*-gon admits* 144 *side-pairing patterns* $P_1, \ldots, P_{144}$ *which yield non-orientable surfaces of genus* 4 *(Figures* 3–8*). Table* 3 *shows the correspondence between the trivalent graphs and the side-pairing patterns, e.g.,* P_1 *is obtained from* A.

TABLE 3. The trivalent graphs and the side-pairing patterns

A	P_1	G	$P_{17}, \ldots, P_{22}$	M	$P_{69}, \ldots, P_{74}$
B	P_2	H	$P_{23}, \ldots, P_{26}$	N	$P_{75}, \ldots, P_{90}$
C	P_3, P_4	I	$P_{27}, \ldots, P_{33}$	O	$P_{91}, \ldots, P_{115}$
D	P_5, P_6	J	$P_{34}, \ldots, P_{47}$	P	$P_{116}, \ldots, P_{138}$
E	P_7, P_8, P_9	K	$P_{48}, \ldots, P_{62}$	Q	$P_{139}, \ldots, P_{144}$
F	$P_{10}, \ldots, P_{16}$	L	$P_{63}, \ldots, P_{68}$		

3. EXTREMAL DISCS FOR THE 144 SURFACES

In this section we shall sketch the proof of Theorem 1.1. Without loss of generality we may assume that the center of a hyperbolic regular 18-gon is the origin and that the vertices v_n satisfy $\arg v_n = (2n-1)\pi/18$ ($n = 1, \ldots, 18$) in the unit disc $\mathbb{D}$. The origin therefore projects to the center of an extremal disc. We denote by C_n the side between v_n and v_{n+1}, and by w_n the middle point of C_n, where subscripts are regarded as modulo 18.

The following lemma is shown by using the same process as [4, Lemma 5.3].

Lemma 3.1. *Let S be a non-orientable extremal surface of genus 4 and $\pi : \mathbb{D} \to S$ the natural projection. If p is the center of an extremal disc of S, then the hyperbolic distances between two points in the set $\pi^{-1}(p)$ are listed such as $d_1 = 2R \approx 3.438$, $d_2 \approx 4.746$, $d_3 \approx 5.496$, and so on.*

Note that $d_2 = 2l + s$, where l denotes the hyperbolic distance between the origin and a vertex v_n, and s the hyperbolic length of a side C_n (Figure 2).
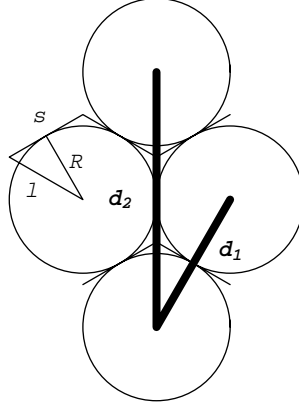

FIGURE 2. d_1 and d_2

Lemma 3.2. *Let K_n $(n = 1, \ldots, 18)$ be the pentagon of which vertices are w_{n-1}, v_n, v_{n+1}, w_{n+1}, and the origin. Let $t_n = t_{n,m}$ be the side-pairing mapping from C_n onto the other side C_m. If z is a point of K_n and is projected to the center of an extremal disc, then $d(z, t_n(z))$, the hyperbolic distance between z and $t_n(z)$, is equal to $2R$.*

Proof. Whereas our proof is similar to that of [4, Lemma 5.4], we shall describe it without numerical values. We know that $d(z, t_n(z))$ is less than or equal to the maximum of distances between a vertex of K_n and its image by t_n, and that $d(0, t_n(0)) = 2R$, $d(v, t_n(v)) \leq 2l$ $(v = v_{n-1}, v_n)$, and $d(w, t_n(w)) \leq R + l + s/2$ $(w = w_{n-1}, w_n)$. It is then obvious from the triangle inequality that $2R < 2l < R + l + s/2$. Therefore we have $d(z, t_n(z)) \leq R + l + s/2$. Since $R + l + s/2 < 2l + s = d_2$ holds by the triangle inequality again, we have $d(z, t_n(z)) = d_1 = 2R$. $\square$

Suppose that $z \in K_n$ is projected to the center of an extremal disc. By the equation $d(z, t_{n,m}(z)) = 2R$, z must be on the curves $L_{n,m}$ or $M_{n,m}$ (or curves $L'_{n,m}$ or $M'_{n,m}$) if $t_{n,m}$ is orientation preserving (or reversing) (see [4, Lemma 5.5]):

$$L_n = L_{n,m} : \left| z - \frac{\tanh R \, e^{i(n+m)\beta}}{2\cos(n-m)\beta} \right| = \frac{\tanh R}{2|\cos(n-m)\beta|} \quad (n - m \not\equiv 9 \pmod{18}),$$

$$M_n = M_{n,m} : z = \coth R \, e^{2in\beta} - t e^{i(n+m+9)\beta} \quad (t \in \mathbb{R}),$$

$$L'_n = L'_{n,m} : \left| z - \frac{\coth R \, e^{i(n+m)\beta}}{2\cos(n-m)\beta} \right| = \frac{\coth R}{2|\cos(n-m)\beta|} \quad (n - m \not\equiv 9 \pmod{18}),$$

$$M'_n = M'_{n,m} : z = \tanh R \, e^{2in\beta} - t e^{i(n+m+9)\beta} \quad (t \in \mathbb{R}).$$
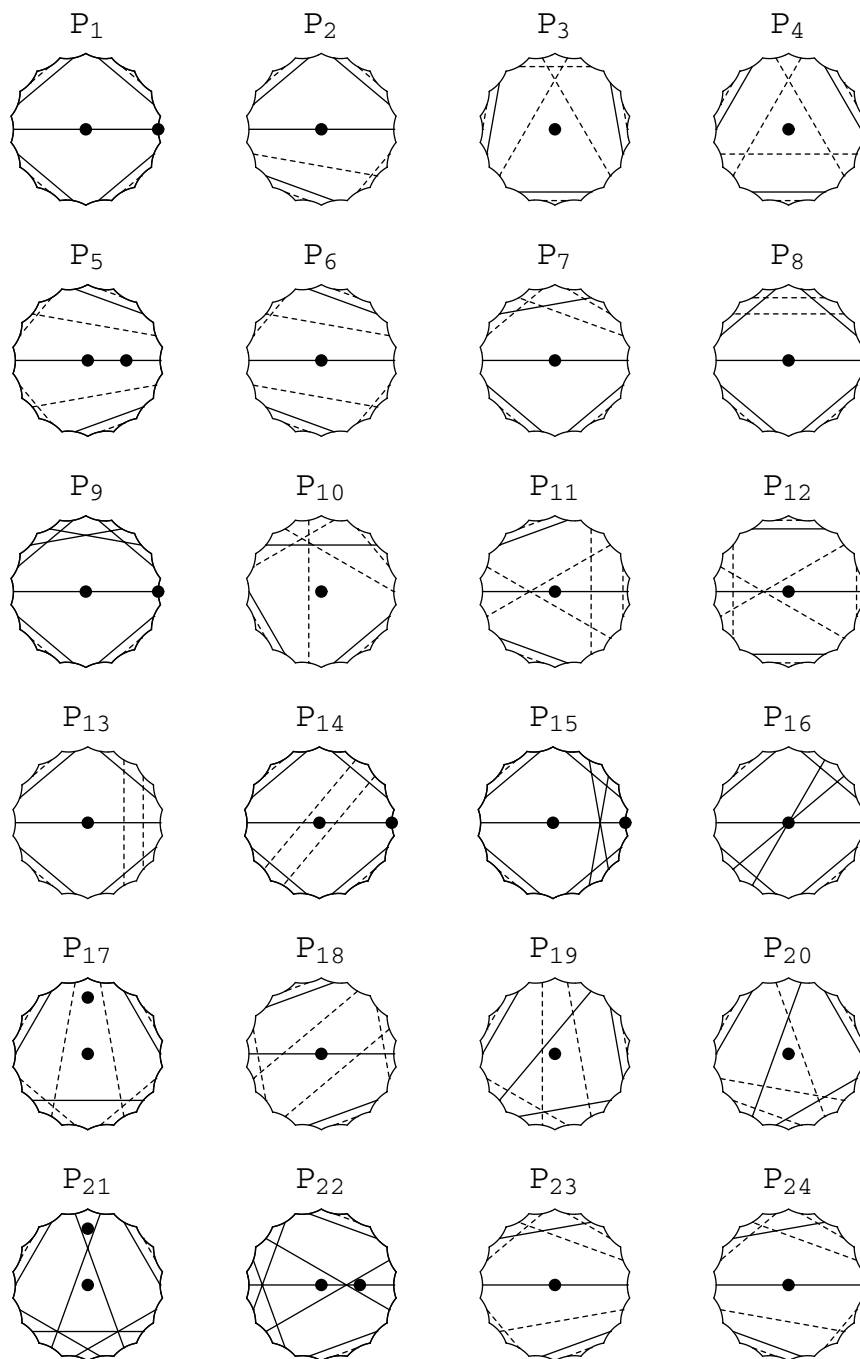
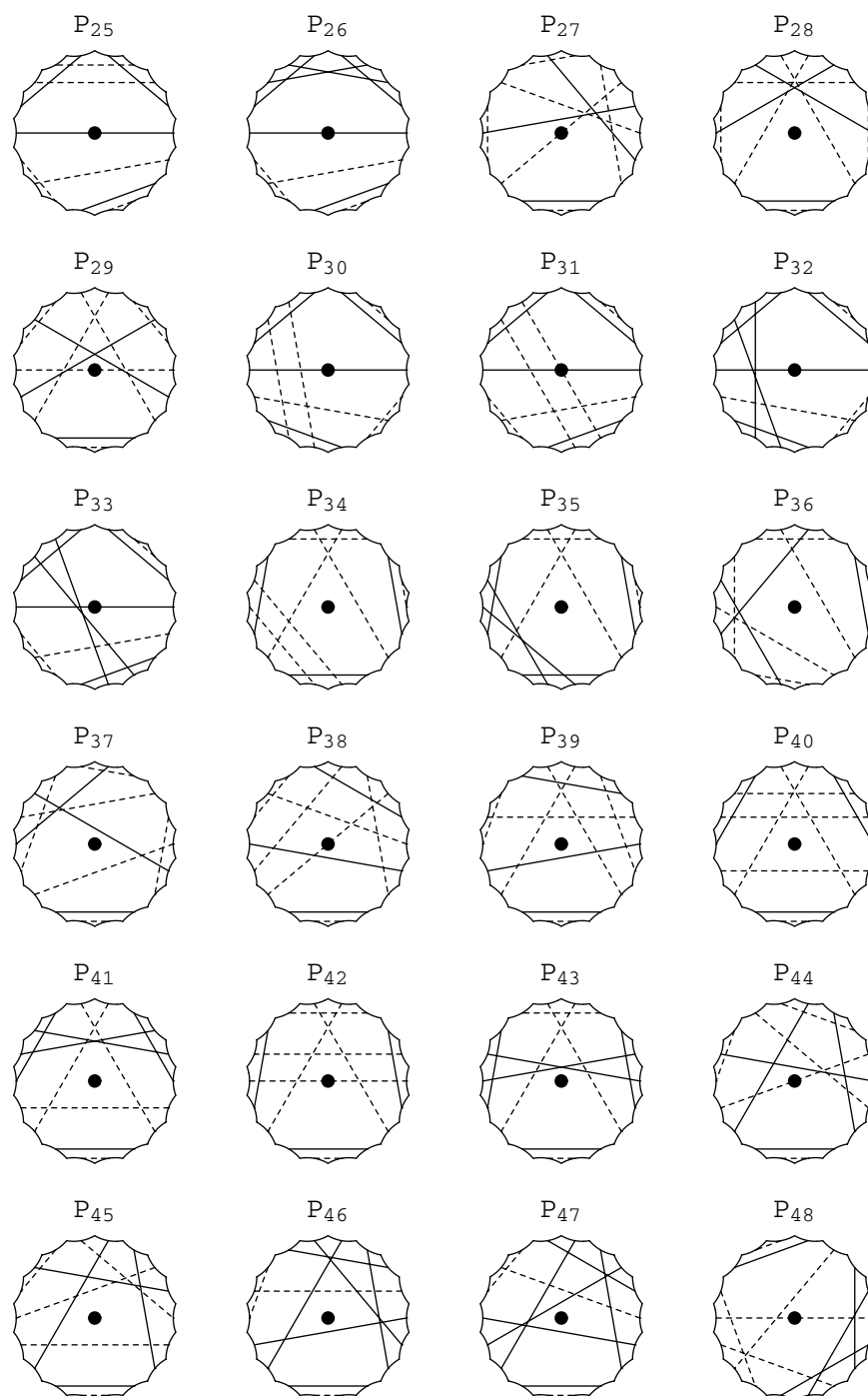Figure 3. Side-pairing patterns and the centers of extremal discs

P_{25} P_{26} P_{27} P_{28}

P_{29} P_{30} P_{31} P_{32}

P_{33} P_{34} P_{35} P_{36}

P_{37} P_{38} P_{39} P_{40}

P_{41} P_{42} P_{43} P_{44}

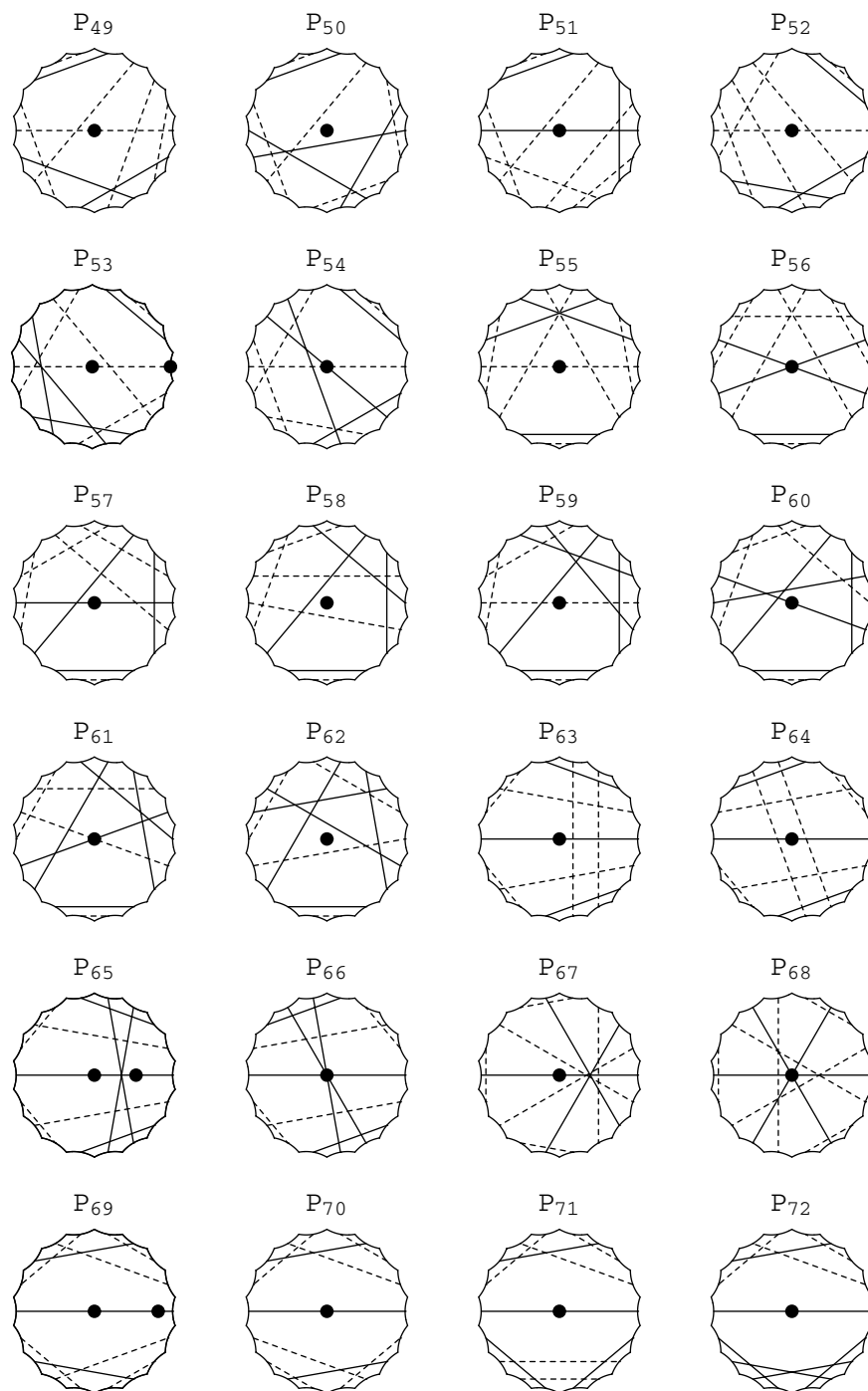P_{45} P_{46} P_{47} P_{48}

FIGURE 4. Side-pairing patterns and the centers of extremal discs

FIGURE 5. Side-pairing patterns and the centers of extremal discs

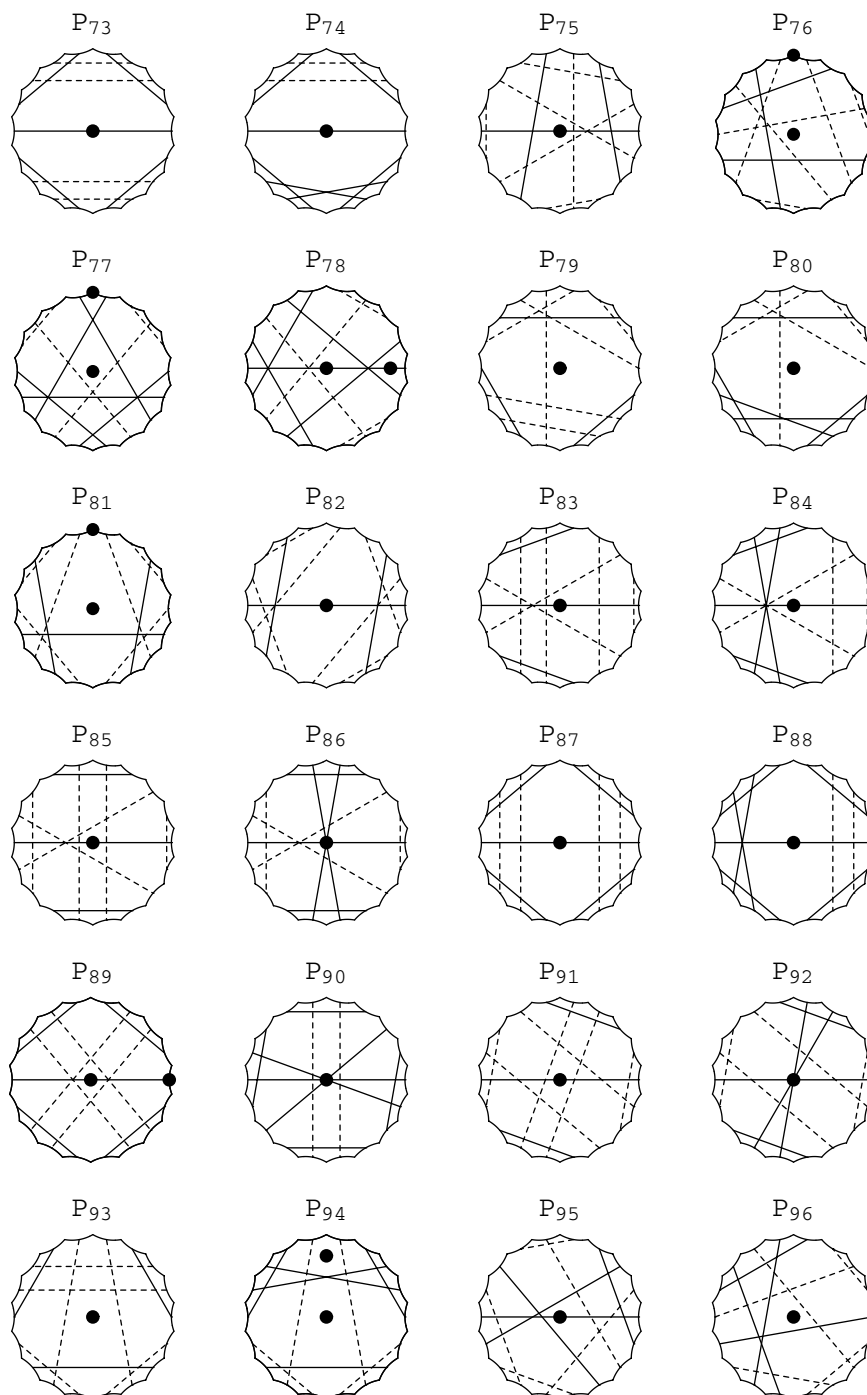FIGURE 6. Side-pairing patterns and the centers of extremal discs

P_{97} P_{98} P_{99} P_{100}

P_{101} P_{102} P_{103} P_{104}

P_{105} P_{106} P_{107} P_{108}

P_{109} P_{110} P_{111} P_{112}

P_{113} P_{114} P_{115} P_{116}

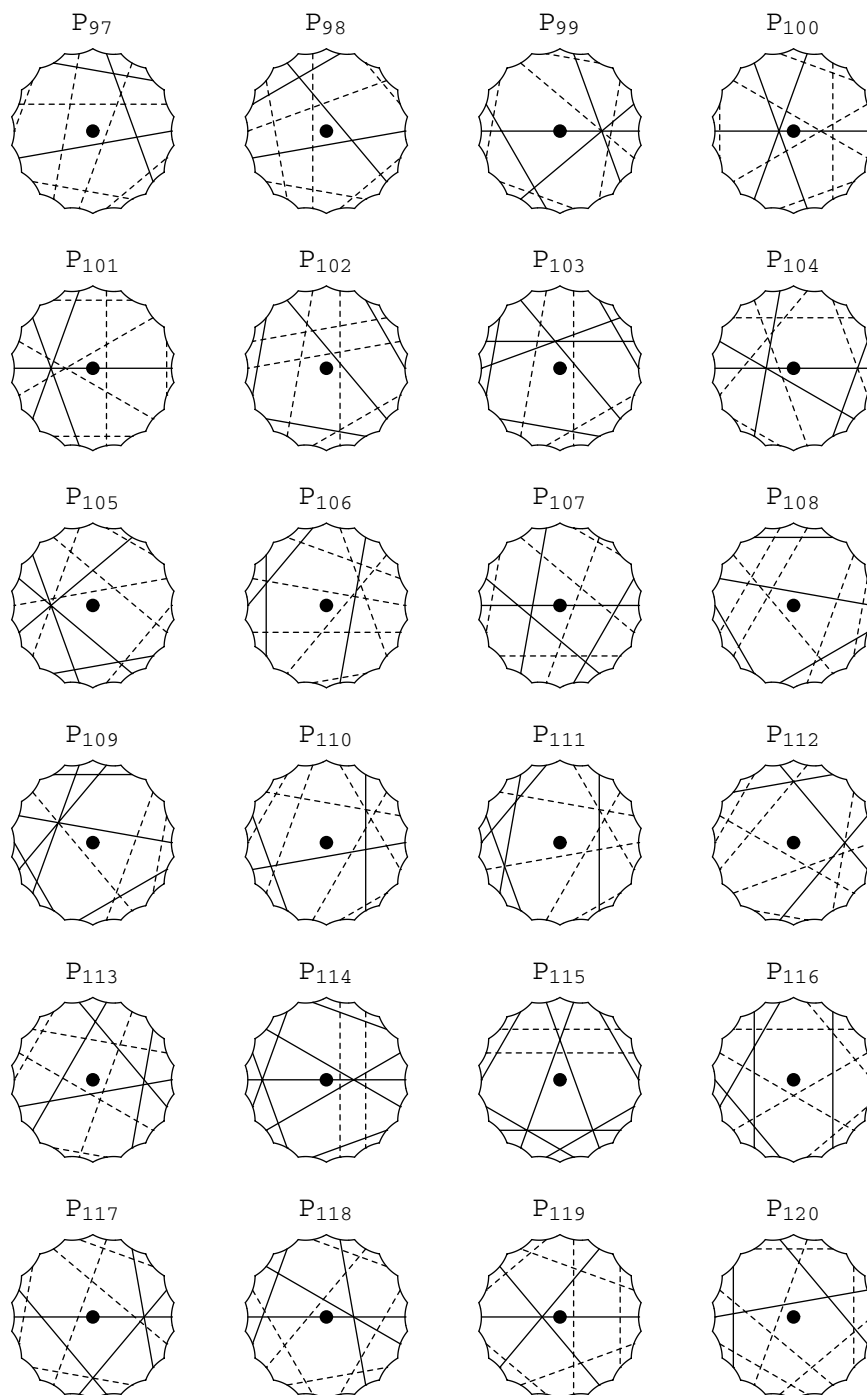P_{117} P_{118} P_{119} P_{120}

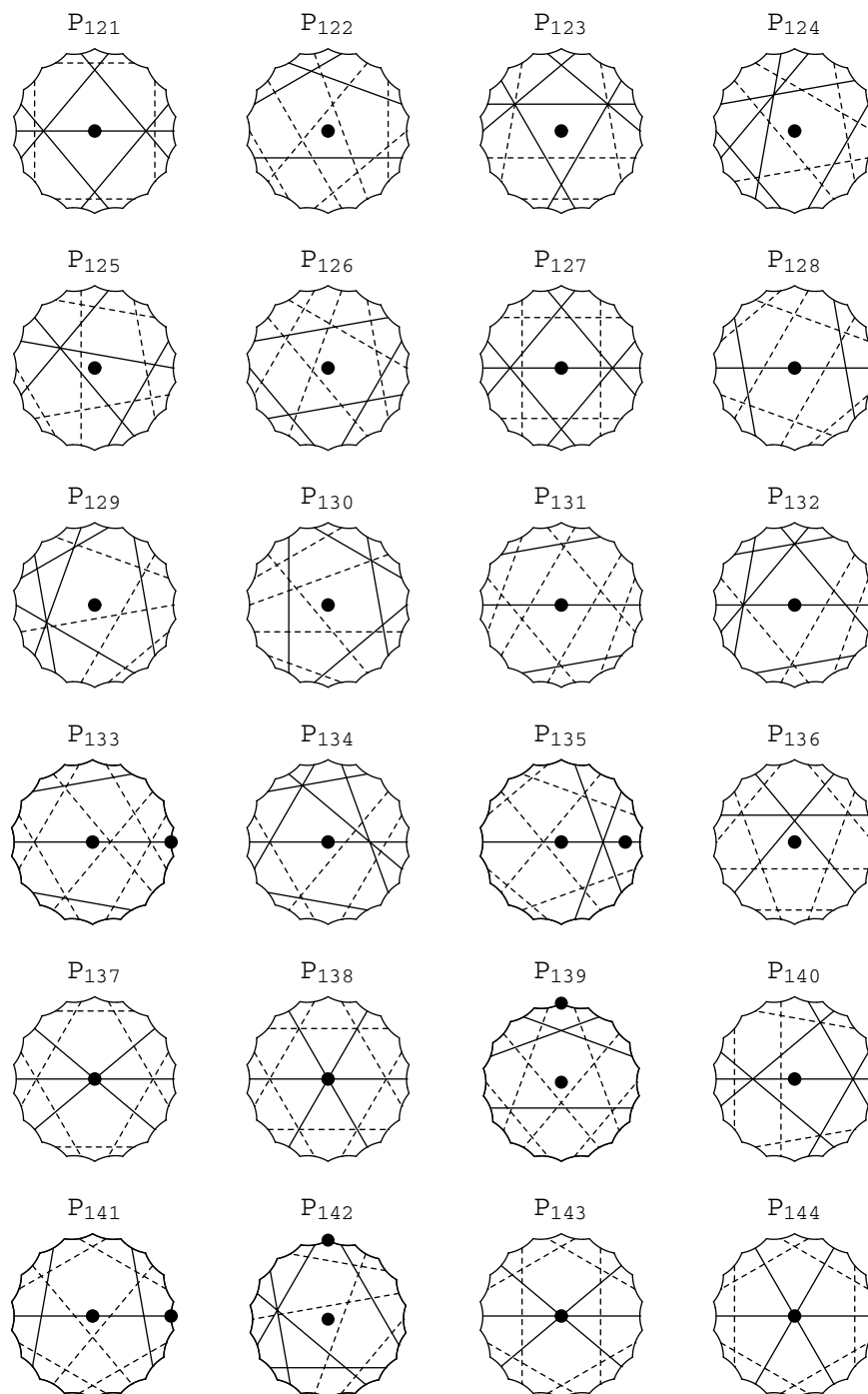FIGURE 7. Side-pairing patterns and the centers of extremal discs

FIGURE 8. Side-pairing patterns and the centers of extremal discs

In order to accomplish our purpose, we shall refer to a process to find the center of an extremal disc.

(1) According to a side-pairing pattern P_j, draw L_n, M_n (or L'_n, M'_n) on K_n for every n;
(2) find intersections of these curves;
(3) select every point ζ in the intersections such that $d(\zeta, t_k(\zeta))$ is in the list of Lemma 3.1 for every side-pairing mapping t_k of P_j.

Applying this process to P_j $(j = 1, \ldots, 144)$, we see that 22 side-pairing patterns admit two points and that the others admit a unique point. We can show that the two points (one is the origin) arising from the 22 patterns are transitive by some isometry $f \in \mathrm{Aut}^{\pm}(\mathbb{D})$ compatible with the side-pairing mappings. Since the origin projects to the center of an extremal disc, so does the other point.

Example 3.3. As an example we shall consider P_1. Through the process above we get two points $\zeta = \tanh(R/2)$ and the origin. Put $f(z) = (\zeta - z)/(1 - \overline{\zeta}z)$, then we can verify that f is compatible with the side-pairing mappings of P_1:

$$\begin{aligned}
&f \; \alpha_{1,4} \; f^{-1} = \alpha_{14,17}, && f \; \gamma_{2,3} \; f^{-1} = \gamma_{15,16}, \\
&f \; \alpha_{5,8} \; f^{-1} = \alpha_{9,0} \; \alpha_{17,14}, && f \; \gamma_{6,7} \; f^{-1} = \alpha_{9,0} \; \gamma_{12,11} \; \alpha_{17,14}, \\
&f \; \alpha_{9,18} \; f^{-1} = \alpha_{18,9}, && f \; \alpha_{10,13} \; f^{-1} = \alpha_{4,1} \; \alpha_{0,9}, \\
&f \; \gamma_{11,12} \; f^{-1} = \alpha_{4,1} \; \gamma_{7,6} \; \alpha_{0,9}, && f \; \alpha_{14,17} \; f^{-1} = \alpha_{1,4}, \\
&f \; \gamma_{15,16} \; f^{-1} = \gamma_{2,3},
\end{aligned}$$

where $\alpha_{n,m}$ and $\gamma_{n,m}$ denote the orientation preserving and reversing side-pairing mapping from C_n to C_m, respectively. Consequently, the projection $\pi(\zeta)$ is the center of an extremal disc.

The surfaces derived from P_j $(j = 1, \ldots, 144)$ are not isomorphic to each other (see [4, Remark 5.9]). Furthermore it is not difficult to determine the full group of automorphisms by the fact that the centers of extremal discs are fixed or interchanged by automorphisms (cf. [4, Corollary 5.10]).

For example, $\mathrm{Aut}^{\pm}(S_1) = D_2 \times \mathbb{Z}_2 = \langle \sigma, J_0 \rangle \times \langle T \rangle (= \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$, where σ, J_0, and T are automorphisms induced by $z \mapsto -z$, $z \mapsto \bar{z}$, and $z \mapsto f(z)$, respectively.

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