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CONICAL LIMIT POINTS AND THE CANNON-THURSTON MAP

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Abstract. Let G be a non-elementary word-hyperbolic group acting as a convergence group on a compact metrizable space Z so that there exists a continuous G-equivariant map $i : \partial G \to Z$, which we call a *Cannon-Thurston map*. We obtain two characterizations (a dynamical one and a geometric one) of conical limit points in Z in terms of their pre-images under the Cannon-Thurston map i. As an application we prove, under the extra assumption that the action of G on Z has no accidental parabolics, that if the map i is not injective, then there exists a non-conical limit point $z \in Z$ with $|i^{-1}(z)| = 1$. This result applies to most natural contexts where the Cannon-Thurston map is known to exist, including subgroups of word-hyperbolic groups and Kleinian representations of surface groups. As another application, we prove that if G is a non-elementary torsion-free word-hyperbolic group, then there exists $x \in \partial G$ such that x is not a "controlled concentration point" for the action of G on ∂G.

1. Introduction

Let G be a Kleinian group, that is, a discrete subgroup of the isometry group of hyperbolic space $G \leq \mathrm{Isom}^+(\mathbb{H}^n)$. In [6], Beardon and Maskit defined the notion of a *conical limit point* (also called a *point of approximation* or *radial limit point*), and used this to provide an alternative characterization of geometric finiteness. Gehring and Martin [42] abstracted the notion of Kleinian group to that of a *convergence group* acting on $\mathbb{S}^{n-1}$, which was then further generalized, for example by Tukia in [86], to actions on more general compact metric spaces (see Definition 2.1). This generalization includes, for example, the action of a discrete group of isometries of a proper, Gromov-hyperbolic, metric space on its boundary at infinity; see [36, 86]. Conical limit points can be defined in this level of generality (see Definition 2.4), and play a key role in the convergence group characterization of word-hyperbolic groups by Bowditch [16], of relatively hyperbolic groups by Yaman [88], and of quasiconvex subgroups of word-hyperbolic groups by Swenson [83], and arise in numerous other results in topology, geometry, and dynamics; see, for example,

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[2, 26, 34, 37, 38, 58, 59] for other results involving convergence groups and conical limit points.

In their 1984 preprint, published in 2007 [25], Cannon and Thurston proved the following remarkable result. If M is a closed hyperbolic 3-manifold fibering over a circle with fiber a closed surface Σ, then the inclusion $\mathbb{H}^2 = \widetilde{\Sigma} \subset \widetilde{M} = \mathbb{H}^3$ extends to a continuous surjective map $\mathbb{S}^1 = \partial\mathbb{H}^2 \to \partial\mathbb{H}^3 = \mathbb{S}^2$, equivariant with respect to $\pi_1(\Sigma)$ which is acting as a convergence group on both. Based on this we make the following general, abstract definition.

Definition 1.1. When G is a word-hyperbolic group acting as a convergence group on a compact metrizable space Z, a map $i : \partial G \to Z$ is called a *Cannon-Thurston map* if i is continuous and G-equivariant.

Under some mild assumptions, it is known that if a Cannon-Thurston map $i : \partial G \to Z$ exists, then it is unique; see Proposition 2.11 below. Of particular interest is the case that a non-elementary word-hyperbolic group G acts on a proper, Gromov hyperbolic, geodesic metric space Y, properly discontinuously by isometries, and without accidental parabolics (see Definition 2.8). In this case, if there exists a Cannon-Thurston map $i\colon \partial G \to \partial Y$, then it is known to be unique and to extend to a G-equivariant continuous map $G\cup\partial G \to Y\cup\partial Y$ (see Proposition 2.12). A special subcase of interest is when G_1, G_2 are non-elementary word-hyperbolic groups, with $G_1 \leq G_2$ acting on (the Cayley graph of) G_2 by restriction of the left action of G_2 on itself. Here a *Cannon-Thurston map* is classically defined as a continuous extension $G_1 \cup \partial G_1 \to G_2 \cup \partial G_2$ of the inclusion of $G_1 \to G_2$. By Proposition 2.12, the existence of such a map is equivalent to the existence of a Cannon-Thurston map in the sense of Definition 1.1 for the induced action of G_1 on ∂G_2. Quasi-isometrically embedded subgroups $G_1 \leq G_2$ of word-hyperbolic groups provide examples where Cannon-Thurston maps exist. However, Cannon-Thurston's original result [25] described above implies that for the word-hyperbolic groups $G_1 = \pi_1(\Sigma) \leq \pi_1(M) = G_2$, there is a Cannon-Thurston map $\partial G_1 \to \partial G_2$, but here G_1 is exponentially distorted in G_2. Subsequent work of Mitra [68, 69] showed that there are many other interesting situations where G_1 is not quasiconvex in G_2 but where the Cannon-Thurston map nevertheless does exist (see also [5]). On the other hand, a recent remarkable result of Baker and Riley [4] proves that there exists a word-hyperbolic group G_2 and a word-hyperbolic (in fact, non-abelian free) subgroup $G_1 \leq G_2$ such that the Cannon-Thurston map $i : \partial G_1 \to \partial G_2$ does not exist.

Generalizing the Cannon-Thurston example from [25] in another direction, one can consider other actions of $G = \pi_1(\Sigma)$, the fundamental group of a closed, orientable surface of genus at least 2, acting properly discontinuously by isometries on $\mathbb{H}^3$, i.e. as a classical Kleinian surface group. The first partial results beyond those in [25] about the existence of Cannon-Thurston maps for such actions of G on $\mathbb{H}^3$ are due to Minsky [65]. Extending beyond the case $G = \pi_1(\Sigma)$, there have been numerous results on the existence of Cannon-Thurston maps of various types (not necessarily fitting into Definition 1.1), especially for Kleinian groups [18, 19, 35, 60, 64, 70, 72–75, 81]. Recently, Mj [77] has shown that for *any* properly discontinuous action on $\mathbb{H}^3$ without accidental parabolics, there exists a Cannon-Thurston map, using the theory of model manifolds which were developed by Minsky. There are extensions of the Cannon-Thurston maps also for subgroups of mapping class groups [62], and in other related contexts [39, 41].

Mj has also shown [71] that in the case of classical Kleinian surface groups without parabolics, the non-injective points of a Cannon-Thurston map are exactly the endpoints of the lifts of the ending laminations to the domains of discontinuity. This characterization of non-injective points of Cannon-Thurston maps has some applications: for instance, the first and the fourth authors have used this to prove *the measurable rigidity* for Kleinian groups (see [46]), which is a generalization of the results by Sullivan [82] and Tukia [84]. Also, using the same kind of characterization for free classical Kleinian groups, Jeon-Kim-Ohshika-Lecuire [45] gave a criterion for points on the boundary of the Schottky space to be primitive stable.

Another reason to be interested in understanding injective points of Cannon-Thurston maps comes from the study of dynamics and geometry of fully irreducible elements of $\mathrm{Out}(F_N)$. If $\varphi \in \mathrm{Out}(F_N)$ is an atoroidal fully irreducible element, then the mapping torus group $G_\varphi = F_N \rtimes_\varphi \mathbb{Z}$ is word-hyperbolic and the Cannon-Thurston map $i : \partial F_N \to \partial G_\varphi$ exists by the result of [68]. In this case, if $T_\pm$ are the "attracting" and "repelling" $\mathbb{R}$-trees for φ, there are associated $\mathcal{Q}$-maps (defined in [32])

$$\mathcal{Q}_+ : \partial F_N \to \widehat{T}_+ = \overline{T_+} \cup \partial T_+$$

and

$$\mathcal{Q}_- : \partial F_N \to \widehat{T}_- = \overline{T_-} \cup \partial T_-$$

(here $\overline{T_\pm}$ denotes the metric completion of $T_\pm$). These maps play an important role in the index theory of free group automorphisms, particularly for the notion of $\mathcal{Q}$-index; see [28–30, 32]. It is shown in [53] that a point $x \in \partial F_N$ is injective for the Cannon-Thurston map i if and only if x is injective for both $\mathcal{Q}_+$ and $\mathcal{Q}_-$.

There are a number of results in the literature which prove in various situations where the Cannon-Thurston map exists that every conical limit point is "injective", that is, has exactly one pre-image under the Cannon-Thurston map; see, for example, [39, 56, 61]. We discuss some of these facts in more detail after the statement of Theorem B below. These results naturally raise the question of whether the converse holds, that is, whether a point with exactly one pre-image under the Cannon-Thurston map must be a conical limit point. (The only result in the literature dealing with this converse direction is Theorem 8.6 in [56], which incorrectly claims that every "injective" limit point is conical in the original setting of a closed hyperbolic 3-manifold fibering over a circle.) We show in Theorem C below that the converse statement fails in great generality and prove that, under rather mild assumptions, if a Cannon-Thurston map exists and is not injective, then there *always* exists a non-conical limit point with exactly one pre-image under the Cannon-Thurston map.

In this paper, given a non-elementary convergence action of a word-hyperbolic group G on a compact metrizable space Z, such that the Cannon-Thurston map $i : \partial G \to Z$ exists, we give two characterizations (a dynamical one and a geometric one) of conical limit points $z \in Z$ in terms of their pre-images under the map i.

To state these characterizations we need to introduce some definitions. Under the above assumptions, denote $L_i = \{(x, y) | x, y \in \partial G, i(x) = i(y), \text{ and } x \neq y\}$. We say that a point $x \in \partial G$ is *asymptotic to* L_i if for *every* conical sequence $\{g_n\}_{n=1}^\infty$ for x with pole pair (x_-, x_+), we have $(x_-, x_+) \in L_i$, that is, $i(x_-) = i(x_+)$. (See Definition 2.4 below for the notions of a conical sequence and pole pair.)

The following result provides a dynamical characterization of conical limit points in Z:

Theorem A. *Suppose G is word-hyperbolic and acts on the compact, metrizable space Z as a non-elementary convergence group, and suppose $i\colon \partial G \to Z$ is a Cannon-Thurston map. Let $z \in i(\partial G)$. Then:*

(1) *The point $z \in Z$ is not a conical limit point for the action of G on Z if and only if some point $x \in i^{-1}(z)$ is asymptotic to L_i.*
(2) *If $|i^{-1}(z)| > 1$, then any $x \in i^{-1}(z)$ is asymptotic to L_i, and hence z is non-conical.*

We also provide a geometric counterpart of Theorem A:

Theorem B. *Let G be a word-hyperbolic group and let Z be a compact metrizable space equipped with a non-elementary convergence action of G such that the Cannon-Thurston map $i : \partial G \to Z$ exists and such that i is not injective. Let X be a δ-hyperbolic (where $\delta \geq 1$) proper geodesic metric space equipped with a properly discontinuous cocompact isometric action of G (so that ∂G is naturally identified with ∂X).*

Let $x \in \partial G$, let $z = i(x) \in Z$ and let ρ be a geodesic ray in X limiting to x.

Then the following are equivalent:

(1) *The point z is a conical limit point for the action of G on Z.*
(2) *There exist a geodesic segment $\tau = [a, b]$ in X of length $\geq 100\delta$ and an infinite sequence of distinct elements $g_n \in G$ such that the 20δ-truncation τ' of τ is not a coarse X-leaf segment of L_i and such that for each $n \geq 1$ the segment $g_n\tau$ is contained in a 6δ-neighborhood of ρ, and that* $\lim_{n\to\infty} g_n a = \lim_{n\to\infty} g_n b = x$.

See the definition of a "coarse leaf segment" and of other relevant terms in Section 5.

Theorem A is used in the proof of Theorem C, as discussed in more detail below. Theorem B is used as a key ingredient in the proof of Theorem 6.6 of [33].

Theorems A and B are partially motived by the result of M. Kapovich [56], who proved that in the setting of Cannon and Thurston's original construction [25] of a Cannon-Thurston map $i\colon \partial G \to \mathbb{S}^2$, for $G = \pi_1(\Sigma)$, if $z \in \partial \mathbb{S}^2 = \partial G$ has $|i^{-1}(z)| \geq 2$, then z is not a conical limit point for the action of G on $\mathbb{S}^2$. This result was extended by Leininger, Long and Reid [61], who proved that the same result for any doubly degenerate Kleinian representation (where i exists from [77]), and later by Gerasimov [39], for arbitrary "$\times$-actions". In fact, part (2) of Theorem A follows from a general result of Gerasimov [39, Proposition 7.5.2] about conical limit points for $\times$-actions. Gerasimov also explained to us how one can derive part (1) of Theorem A from the results of [39] using a result of Bowditch. We provide a short direct proof of Theorem A here.

It is known by results of Swenson [83] and Mj [76] that in the geometric context, where the Cannon-Thurston map $i : \partial G \to \partial Y$ arises from a properly discontinuous isometric action of a word-hyperbolic G on a proper Gromov-hyperbolic space Y, if $i : \partial G \to \partial Y$ is injective, then the orbit-map $G \to Y$ is a quasi-isometric embedding; see Proposition 2.13 below for a precise statement. In this case every limit point

$z \in \partial Y$ is conical and has exactly one pre-image under i. Therefore Theorem A implies:

Corollary 1.2. *Let G be a non-elementary word-hyperbolic group equipped with a properly discontinuous isometric action on a proper geodesic Gromov-hyperbolic space Y without accidental parabolics. Suppose that the Cannon-Thurston map $i : \partial G \to \partial Y$ exists.*

Then there exists $z \in i(\partial G)$ such that z is a non-conical limit point for the action of G on ∂Y if and only if i is not injective.

Proof. If i is not injective and $x_1, x_2 \in \partial G$ are points such that $x_1 \neq x_2$ and that $i(x_1) = i(x_2)$, then, by part (2) of Theorem A, $z = i(x_1) = i(x_2)$ is not conical. If the map i is injective, then, since ∂G and $i(\partial G) \subseteq \partial Y$ are compact and Hausdorff, the map i is a G-equivariant homeomorphism between ∂G and $i(\partial G)$. Since G is hyperbolic, every point of ∂G is conical for the action of G on ∂G; see [86]. Therefore every $z \in i(\partial G)$ is conical for the action of G on $i(\partial G)$ and hence, by Lemma 2.6, also for the action of G on ∂Y. □

The main result of this paper is:

Theorem C. *Suppose a word-hyperbolic group G acts on a compact metrizable space Z as a non-elementary convergence group without accidental parabolics, and suppose that there exists a non-injective Cannon-Thurston map $i\colon \partial G \to Z$. Then there exists a non-conical limit point $z \in Z$ with $|i^{-1}(z)| = 1$.*

Theorem C applies whenever G_1 is a non-elementary non-quasiconvex word-hyperbolic subgroup of a word-hyperbolic group G_2 such that the Cannon-Thurston map $\partial G_1 \to \partial G_2$ exists. Similarly, Theorem C applies whenever Σ is a closed hyperbolic surface and $\pi_1(\Sigma)$ is equipped with a properly discontinuous isometric action on $\mathbb{H}^3$ without accidental parabolics, assuming that the Cannon-Thurston map $\mathbb{S}^1 = \partial\pi_1(\Sigma) \to \partial\mathbb{H}^3 = \mathbb{S}^2$ exists and is non-injective.

Brian Bowditch (private communication) showed us another argument for obtaining the conclusion of Theorem C for a large class of Kleinian groups, including the original case of a closed hyperbolic 3-manifold fibering over a circle. His argument is different from the proof presented in this paper and relies on the Kleinian groups and 3-manifold methods.

Another application of our results concerns "controlled concentration points". Originally, the notion of a controlled concentration point was defined for a properly discontinuous isometric action of a torsion-free group G on $\mathbb{H}^n$. A point x in $\partial\mathbb{H}^n$ is called a controlled concentration point of G when x has a neighborhood V such that for any neighborhood U of x there is $g \in G$ with $gU \subset V$ and $x \in g(V)$. This is equivalent to saying that there is a sequence of elements $(g_n)_{n\geq 1} \subset G$ such that $g_n(x) \to x$ and $(g_n|_{\partial\mathbb{H}^n \setminus \{x\}})$ converges locally uniformly to a constant map to some point $y \neq x$. Aebischer, Hong and McCullough [3] showed that a limit point $x \in \partial\mathbb{H}^n$ is a controlled concentration point if and only if it is an endpoint of a lift of a recurrent geodesic ray in $M := \mathbb{H}^n/G$. A geodesic ray $\alpha(t)$ in M is called *recurrent* if for any t_0, there exists a sequence $\{t_i\}$ with $t_i \to \infty$ such that $\lim_{i\to\infty} \alpha'(t_i) = \alpha'(t_0)$ in the unit tangent bundle of M. They also showed there exist non-controlled concentration points in the limit set of a rank-2 Schottky group.

We generalize the notion of controlled concentration points to points at infinity of general word-hyperbolic groups by adopting the latter condition above as its definition; see Definition 7.4 below. As an application of Theorem C, we get the following existence theorem of non-controlled concentration points:

Theorem D. *Let G be a non-elementary torsion-free word-hyperbolic group. Then there exists $x \in \partial G$ which is not a controlled concentration point.*

In Appendix A we discuss several specific situations where the Cannon-Thurston map $i : \partial G \to Z$ is known to exist and where a more detailed description of the lamination L_i is known.

2. Definitions and basic facts

2.1. Convergence groups.

Definition 2.1 (Convergence action)**.** An action of a group G on a compact metrizable space Z by homeomorphisms is called a *convergence action* (in which case we also say that G acts on Z as a *convergence group*) if for any infinite sequence $(g_n)_{n\geq 1}$ of distinct elements of G there exist $a, b \in Z$ and a subsequence $(g_{n_k})_{k\geq 1}$ of (g_n), called a *convergence subsequence*, such that the sequence of maps $\{g_{n_k}|_{Z\setminus\{a\}}\}$ converges uniformly on compact subsets to the constant map $c_b \colon Z \setminus \{a\} \to Z$ sending $Z \setminus \{a\}$ to b. In this case we call $(a, b, (g_{n_k}))$ the *convergence subsequence data.* The action is called *elementary* if either G is finite or G preserves a subset of Z of cardinality ≤ 2, and it is called *non-elementary* otherwise.

Note that if G acts as a convergence group on Z and if $Z' \subseteq Z$ is a non-empty closed G-invariant subset, then the restricted action of G on Z' is also a convergence action.

For a group G acting on a set Z and for $g \in G$ denote $Fix_Z(g) := \{z \in Z | gz = z\}$. The following is a basic fact about convergence groups; see [17, Lemma 3.1] and [86].

Proposition 2.2. *Suppose G acts as a convergence group on a compact metrizable space Z and let $g \in G$. Then exactly one of the following occurs:*

1. *The element g has finite order in G; in this case g is said to be* elliptic.
2. *The element g has infinite order in G and the fixed set $Fix_Z(g)$ consists of a single point; in this case g is called* parabolic.
3. *The element g has infinite order in G and the fixed set $Fix_Z(g)$ consists of two distinct points; in this case g is called* loxodromic.

Moreover, for every $k \neq 0$ the elements g and g^k have the same type; also in cases (2) and (3) we have $Fix_Z(g) = Fix_Z(g^k)$ and the group $\langle g\rangle$ acts properly discontinuously on $Z \setminus Fix_Z(g)$. Additionally, if $g \in G$ is loxodromic, then $\langle g\rangle$ acts properly discontinuously and cocompactly.

It is also known that if $g \in G$ is parabolic with a fixed point $a \in Z$, then for every $z \in Z$ we have $\lim_{n\to\infty} g^n z = \lim_{n\to-\infty} g^n z = a$. Also, if $g \in G$ is loxodromic, then we can write $Fix_Z(g) = \{a_-, a_+\}$, and for every $z \in Z \setminus \{a_+\}$ we have $\lim_{n\to\infty} g^n z = a_-$, and for every $z \in Z \setminus \{a_-\}$ we have $\lim_{n\to-\infty} g^n z = a_+$, and these convergences are uniform on compact subsets of $Z \setminus \{a_-, a_+\}$.

Definition 2.3 (Limit set)**.** If G acts on Z as a non-elementary convergence group, there exists a unique minimal non-empty closed G-invariant subset $\Lambda(G) \subseteq Z$ called the *limit set* of G in Z. In this case $\Lambda(G)$ is perfect and hence $\Lambda(G)$ is infinite [86]. If $\Lambda(G) = Z$, then we say that the action of G on Z is *minimal*.

Definition 2.4 (Conical limit point)**.** Let G act on Z as a convergence group. A point $z \in Z$ is called a *conical limit point* for the action of G on Z if there exist an infinite sequence $(g_n)_{n\geq 1}$ of distinct elements of G and a pair of distinct points $z_-, z_+ \in Z$ such that $\lim_{n\to\infty} g_n z = z_+$ and that $(g_n|_{Z\setminus\{z\}})$ converges uniformly on compact subsets to the constant map $c_{z_-} : Z \setminus \{z\} \to Z$ sending $Z \setminus \{z\}$ to z_-. We call such a sequence g_n a *conical sequence* for z, and the pair (z_-, z_+) the *pole pair* corresponding to z and $(g_n)_{n\geq 1}$. If every point of Z is a conical limit point, then the action is called a *uniform convergence action*. In particular, in this case the action is minimal.

Note that if $g \in G$ is loxodromic with $Fix_Z(g) = \{a_-, a_+\}$, then both a_+, a_- are conical limit points for the action of G on Z, and one can use $(g^n)_{n\geq 1}$ as the conical sequence with pole pair (a_-, a_+).

As usual, for $\delta \geq 0$, a *δ-hyperbolic space* is a geodesic metric space X such that for every geodesic triangle in X each side of this triangle is contained in the δ-neighborhood of the union of two other sides. A metric space X is *Gromov-hyperbolic* if there exists $\delta \geq 0$ such that X is δ-hyperbolic. A finitely generated group G is called *word-hyperbolic* if for some (equivalently, any) finite generating set $\mathcal{S}$ of G, the Cayley graph of G with respect to $\mathcal{S}$ is Gromov-hyperbolic. See [21] for basic background information about Gromov-hyperbolic spaces and word-hyperbolic groups; also see [49] for the background regarding boundaries of hyperbolic spaces and of word-hyperbolic groups.

Example 2.5. Let G be an infinite word-hyperbolic group, and write ∂G to denote the Gromov boundary. Then the action of G on ∂G is a uniform convergence action. In fact, according to a result of Bowditch [16], if the action of a group G on a compact metrizable space Z is a uniform convergence action, then G is a word-hyperbolic and there is a G-equivariant homeomorphism between ∂G and Z.

Lemma 2.6. *Let a group G act as a convergence group on Z, let $Z' \subseteq Z$ be a non-empty infinite closed G-invariant subset, and let $z \in Z'$. Then z is a conical limit point for the action of G on Z if and only if z is a conical limit point for the action of G on Z'.*

Proof. The "only if" direction is obvious from the definition of a conical limit point. Thus suppose that $z \in Z'$ is a conical limit point for the action of G on Z'. Let $(g_n)_n$ be a conical sequence for z for the action of G on Z' and let (z_-, z_+) be the corresponding pole pair (also for the action of G on Z'). Since G acts on Z as a convergence group, there exist convergence subsequence data $(a, b, (g_{n_k}))$. By assumption Z' is infinite and hence for every $x \in Z' \setminus \{a, z\}$ we have $\lim_{n\to\infty} g_n x = z_-$ and $\lim_{n\to\infty} g_n x = b$; it follows that $z_- = b$.

We claim that $a = z$. Indeed, suppose that $a \neq z$. Since $z \in Z \setminus \{a\}$, it follows that $\lim_{n\to\infty} g_n z = b = z_-$. On the other hand, by assumption about (z_-, z_+) being the pole pair for z and $(g_n)_n$, it follows that $\lim_{n\to\infty} g_n z = z_+$. By definition $z_- \neq z_+$, which gives a contradiction. Thus indeed $a = z$. Hence z is a conical limit point for the action of G on Z, as claimed. □

The following basic fact is well known; see, for example, [17].

Proposition 2.7. *Let G be a word-hyperbolic group acting as a non-elementary convergence group on a compact metrizable space Z.*

Then a G-limit point $z \in Z$ is a conical limit point for the action of G on Z if and only if for every point $s \in Z$ such that $s \neq z$ there exists an infinite sequence $g_n \in G$ of distinct elements of G and points $s_\infty, z_\infty \in Z$ such that $s_\infty \neq z_\infty$ and such that $\lim_{n\to\infty} g_n z = z_\infty$ and $\lim_{n\to\infty} g_n s = s_\infty$.

Definition 2.8 (Accidental parabolic)**.** Let G be an infinite word-hyperbolic group acting as a non-elementary convergence group on a compact metrizable space Z. An *accidental parabolic* for this action is an infinite order element $g \in G$ such that g acts parabolically on Z.

2.2. **Cannon-Thurston map.**

Definition 2.9 (Cannon-Thurston map)**.** Let G be a word-hyperbolic group acting as a non-elementary convergence group on a compact metrizable space Z. A map $i\colon \partial G \to Z$ is called a *Cannon-Thurston map* if i is continuous and G-equivariant.

Lemma 2.10. *Let G be a word-hyperbolic group acting as a non-elementary convergence group on a compact metrizable space Z and suppose $i\colon \partial G \to Z$ is a Cannon-Thurston map. Then:*

1. *If g acts as a loxodromic on Z, then the attracting and repelling fixed points in ∂G of g, respectively, are sent by i to the attracting and repelling fixed points of g in Z, respectively.*
2. *If g is an accidental parabolic, then there is exactly one fixed point for g on Z, which is the i-image of the two fixed points in ∂G.*

Proof. Let $g \in G$ be an element of infinite order. Denote by g^∞ and $g^{-\infty}$ the attracting and repelling points for g in ∂G respectively. Since i is G-equivariant and the points $g^{\pm\infty} \in \partial G$ are fixed by g, it follows that $i(\{g^\infty, g^{-\infty}\}) \subseteq Fix_Z(g)$. If g is parabolic and $Fix_Z(g) = \{a\}$, it follows that $i(g^{\pm\infty}) = a$.

Suppose now that g acts on Z loxodromically. Since by assumption G acts on Z and hence on $i(\partial G)$ as a non-elementary convergence group, the set $i(\partial G)$ is infinite. Hence there exists $x \in \partial G$ such that $i(x) \notin Fix_Z(g)$ (and hence $x \notin \{g^\infty, g^{-\infty}\}$). If g acts loxodromically on Z with $Fix_Z(g) = \{a_+, a_-\}$, then $\lim_{n\to\infty} g^n x = g^\infty$ and hence, by continuity and g-equivariance of i, $\lim_{n\to\infty} g^n i(x) = i(g^\infty)$. On the other hand, by definition of a loxodromic element, since $i(x) \neq a_-$ we have $\lim_{n\to\infty} g^n i(x) = a_+$. Thus $i(g^\infty) = a_+$. Replacing g by g^{-1} we get $i(g^{-\infty}) = a_-$. □

Proposition 2.11 (Cannon-Thurston map unique)**.** *Let G be a word-hyperbolic group acting as a non-elementary convergence group on a compact metrizable space Z. Then any two Cannon-Thurston maps $i, j\colon \partial G \to Z$, if they exist, must be equal.*

Proof. Since i, j are continuous, they are determined by what they do to a dense set of points. The set of attracting endpoints of any infinite order element $g \in G$ and its conjugates $\{g^h\}_{h\in G}$ forms such a dense set. By Lemma 2.10, i and j must agree on this set, hence must be equal. □

In the situation where Proposition 2.11 applies, if a Cannon-Thurston map $i : \partial G \to Z$ exists, we will refer to i as *the Cannon-Thurston map.*

There is a particular geometric situation where the Cannon-Thurston map has a more natural geometric meaning:

Proposition 2.12. *Let G be a non-elementary word-hyperbolic group equipped with a properly discontinuous (but not necessarily cocompact) isometric action of a proper Gromov-hyperbolic geodesic metric space Y, so that every element of infinite order acts as a loxodromic isometry of Y. Then the following hold:*

1. *∂Y is compact and G acts on ∂Y as a convergence group without accidental parabolics. (Thus Proposition 2.11 applies.)*
2. *Suppose the Cannon-Thurston map $i : \partial G \to \partial Y$ exists. Then for every $p \in Y$ the map*
$$f : G \cup \partial G \to Y \cup \partial Y$$
given by $f(g) = gp$ for $g \in G$, and $f(x) = i(x)$ for $x \in \partial G$, is continuous for the hyperbolic compactification topologies on $G \cup \partial G$ and $Y \cup \partial Y$.

Proof. Part (1) is well known and due to Tukia [85].

For part (2), note that the topology on G is discrete. Thus we only need to check continuity of f at points of ∂G. Since i is assumed to be continuous, it suffices to establish the following:

Claim. If $x \in \partial G$ and $(g_n)_{n\geq 1} \subset G$ is an infinite sequence of distinct elements of G such that $\lim_{n\to\infty} g_n = x$ in $G \cup \partial G$, then $\lim_{n\to\infty} g_n p = i(x)$ in $Y \cup \partial Y$ for every $p \in Y$.

Assume that x and (g_n) are as in the Claim, but that the sequence $g_n p$ does not converge to $i(x)$ in $Y \cup \partial Y$. Since G acts properly discontinuously on Y, it follows that, after replacing g_n by a subsequence, we have $\lim_{n\to\infty} g_n p = z$ for some $z \in \partial Y$ such that $z \neq i(x)$. Then there exist a subsequence g_{n_k} and points $a, b \in \partial G$ and $c, d \in \partial Y$ such that $(g_{n_k}|_{\partial G\setminus\{a\}})$ converges uniformly on compact sets to the constant map to b, and $g_{n_k}|_{\partial Y\setminus\{c\}}$ converges uniformly on compact sets to the constant map to d. Moreover, the fact that $\lim_{n\to\infty} g_n = x$ implies that $x = b$ and, similarly, the fact that $\lim_{n\to\infty} g_n p = z$ implies that $z = d$ (see [85]). Since the set $i(\partial G)$ is infinite, we can find $y \in \partial G$ such that $y \neq a$ and $i(y) \neq c$. Then, on one hand, we have $\lim_{k\to\infty} g_{n_k} i(y) = d = z$. On the other hand, $\lim_{k\to\infty} g_{n_k} y = b = x$ and therefore, by continuity of i, we have $\lim_{k\to\infty} g_{n_k} i(y) = i(x)$. This contradicts $z \neq i(x)$. □

A general result of Mj shows that in this situation injectivity of the Cannon-Thurston map is equivalent to the orbit map $X \to Y$ being a quasi-isometric embedding [76, Lemma 2.5]:

Proposition 2.13. *Let G and Y be as in Proposition 2.12 and let $p \in Y$. Then the following conditions are equivalent:*

1. *The Cannon-Thurston map $i : \partial G \to \partial Y$ exists and is injective.*
2. *The orbit map $G \to Y$, $g \mapsto gp$, is a quasi-isometric embedding.*

Proof. As noted above, this proposition holds by [76, Lemma 2.5]. The proposition also follows directly from the older result of Swenson [83]. Indeed, (2) obviously implies (1). Thus assume that (1) holds and that the Cannon-Thurston map $i : \partial G \to \partial Y$ exists and is injective. Since both ∂G and ∂Y are compact and Hausdorff, the map i is a G-equivariant homeomorphism between ∂G and $Z' = i(\partial G)$. Since G is word-hyperbolic, every point of ∂G is conical. Therefore every point of Z' is conical for the G-action on Z' and hence, by Lemma 2.6, also for the action of G on ∂Y. The main result of Swenson [83] then implies that the orbit map $G \to Y$, $g \mapsto gp$, is a quasi-isometric embedding. □

Proposition 2.13 implies, in particular, that if G_1 is a word-hyperbolic subgroup of a word-hyperbolic group G_2 and if the Cannon-Thurston map $i : \partial G_1 \to \partial G_2$ exists and is injective, then G_1 is quasiconvex in G_2.

3. Algebraic laminations

If G is a word-hyperbolic group, we denote $\partial^2 G := \{(z, s) \in \partial G \times \partial G | z \neq s\}$. The set $\partial^2 G$ is equipped with the subspace topology from the product topology on $\partial G \times \partial G$. The group G has a natural diagonal action on $\partial^2 G$: for $g \in G$ and $(z, s) \in \partial^2 G$ we have $g(z, s) := (gz, gs)$. Let $\partial G \times \partial G \to \partial G \times \partial G$ be the "flip" map given by $j : (x, y) \mapsto (y, x)$ for $(x, y) \in \partial G$.

Definition 3.1 (Algebraic lamination)**.** Let G be a word-hyperbolic group. An *algebraic lamination* on G is a subset $L \subseteq \partial^2 G$ such that L is closed in $\partial^2 G$, flip-invariant and G-invariant. A pair $(x, y) \in L$ is called a *leaf* of L. An element $x \in \partial G$ is called an *end* of L if there exists $y \in \partial G$, $y \neq x$, such that $(x, y) \in L$.

For an algebraic lamination L on G denote by $End(L)$ the set of all ends of L. Note that $End(L)$ is a G-invariant subset of ∂G.

Definition 3.2 (Lamination and relation associated to a Cannon-Thurston map)**.** Let G be a word-hyperbolic group and let Z be a compact metrizable space equipped with a convergence action of G such that the Cannon-Thurston map $i : \partial G \to Z$ exists. Denote

$$L_i := \{(x, y) \in \partial G \times \partial G | i(x) = i(y), x \neq y\}.$$

Since i is continuous and G-equivariant, L_i is a closed G-invariant and flip-invariant subset of $\partial^2 G$. Thus L_i is an algebraic lamination on G.

4. Dynamical characterization

Suppose G is a word-hyperbolic group acting as a non-elementary convergence group on Z, and let $i\colon \partial G \to Z$ be a Cannon-Thurston map. We say that a point $x \in \partial G$ is *asymptotic to* L_i if for *every* conical sequence $\{g_n\}_{n=1}^{\infty}$ for x with pole pair (x_-, x_+) we have $(x_-, x_+) \in L_i$, that is, $i(x_-) = i(x_+)$.

In this section, we prove the first theorem from the introduction.

Theorem A. *Suppose G is word-hyperbolic and acts on the compact, metrizable space Z as a non-elementary convergence group, and suppose $i\colon \partial G \to Z$ is a Cannon-Thurston map. Let $z \in i(\partial G)$. Then:*

1. *The point $z \in Z$ is not a conical limit point for the action of G on Z if and only if some point $x \in i^{-1}(z)$ is asymptotic to L_i.*
2. *If $|i^{-1}(z)| > 1$, then any $x \in i^{-1}(z)$ is asymptotic to L_i, and hence z is non-conical.*

In the rest of this section, we assume the hypotheses of the theorem. Our first lemma shows that, up to subsequences, conical sequences in G for Z must come from conical sequences for ∂G.

Lemma 4.1. *Suppose that $z \in Z$ is a conical limit point and that $(g_n)_{n\geq 1}$ is a conical sequence for z. Then there exist $x \in i^{-1}(z)$ and a subsequence $(g_{n_k})_{k\geq 1}$ which is a conical sequence for x. Moreover, if (x_-, x_+) is the pole pair for (g_{n_k}) and x, then $(i(x_-), i(x_+))$ is the pole pair for (g_{n_k}) and z, and, in particular, $i(x_-) \neq i(x_+)$.*

Proof. Without loss of generality, we may assume that the action of G on Z is minimal, so $i(\partial G) = Z$. Let (z_-, z_+) be the pole pair for z and (g_n). This is also a pole pair for any subsequence of (g_n).

By the convergence property, there exist subsequence data (x, x_-, g_{n_k}) such that $(g_{n_k}|_{\partial G\setminus\{x\}})_{k\geq 1}$ converges locally uniformly to the constant map to x_-. By passing to a further subsequence, we may assume that $\lim_{k\to\infty} g_{n_k}(x) = x_+$, for some $x_+ \in \partial G$ (possibly equal to x_-).

Since i is continuous, it follows that for any $y \in \partial G$, we have

$$i(\lim_{k\to\infty} g_{n_k}(y)) = \lim_{k\to\infty} g_{n_k}(i(y)).$$

From this, we see that if $y \in \partial G \setminus (i^{-1}(z) \cup \{x\})$, then $i(x_-) = i(\lim_{k\to\infty} g_{n_k}(y)) = z_-$. Furthermore, since any $y \in \partial G \setminus \{x\}$ has $\lim_{k\to\infty} g_{n_k}(i(y)) = i(x_-) = z_-$, it follows that $i(\partial G \setminus \{x\}) \subset Z \setminus \{z\}$; that is, $i(x) = z$. Finally, we have

$$i(x_+) = i(\lim_{k\to\infty} g_{n_k}(x)) = \lim_{k\to\infty} g_{n_k}(z) = z_+.$$

Therefore $i(x_+) = z_+ \neq z_- = i(x_-)$, and so $x_+ \neq x_-$ and $\{g_{n_k}\}$ is a conical sequence for x with pole pair (x_-, x_+). □

Proof of Theorem A. To prove part (1), first suppose that $z \in Z$ is conical. Let $(g_n)_{n\geq 1}$ be a conical sequence for z with pole pair (z_-, z_+). According to Lemma 4.1, there exist $x \in i^{-1}(z)$ and a subsequence $(g_{n_k})_{k\geq 1}$ which is a conical sequence for x with pole pair (x_-, x_+). Since $i(x_-) = z_- \neq z_+ = i(x_+)$, it follows that x is not asymptotic to L_i.

Now suppose x is not asymptotic to L_i and let $\{g_n\}_{n=1}^{\infty}$ be any conical sequence for x with pole pair (x_-, x_+) such that $i(x_-) \neq i(x_+)$. Because the action of G on Z is a convergence action, there exist a subsequence $(g_{n_k})_{k\geq 1}$ and $z, z_- \in Z$ such that on $Z \setminus \{z\}$, g_{n_k} converges locally uniformly to z_-. For any $y \in \partial G \setminus \{x\}$ we have $\lim_{k\to\infty} g_{n_k}(i(y)) = i(x_-)$. Thus taking $y \notin i^{-1}(z)$, this implies $i(x_-) = z_-$. On the other hand, $\lim_{k\to\infty} i(g_{n_k}(x)) = i(x_+) \neq i(x_-) = z_-$ by assumption. It follows that $i(x) = z$ (since anything else must converge to z_- on applying g_{n_k}). Therefore, setting $z_+ = i(x_+)$, it follows that $\{g_{n_k}\}$ is a conical sequence for z with pole pair (z_-, z_+), and z is a conical limit point. Thus part (1) of Theorem A is proved.

For part (2) of Theorem A we suppose $|i^{-1}(z)| > 1$, and prove that any $x \in i^{-1}(z)$ is asymptotic to L_i. For this, let $y \in i^{-1}(z)$ be any other point with $y \neq x$. Let $(g_n)_{n\geq 1}$ be a conical sequence for x with pole pair (x_-, x_+). Then $\lim_{n\to\infty} g_n(x) = x_+$ and $\lim_{n\to\infty} g_n(y) = x_-$. Since L_i is G-invariant, $i(g_n(x)) = i(g_n(y))$ and since L_i is closed (or equivalently, the algebraic lamination L_i is closed), it follows that $i(x_-) = i(x_+)$. Since (g_n) was an arbitrary conical sequence for x, the point x is asymptotic to L_i, as required. Hence, by part (1), z is not a conical limit point for the action of G on Z. □

5. Geometric characterization

Definition 5.1 (Coarse leaf segments)**.** Let G be a word-hyperbolic group and let $L \subseteq \partial^2 G$ be an algebraic lamination on G.

Let X be a δ-hyperbolic (where $\delta \geq 1$) proper geodesic metric space equipped with a properly discontinuous cocompact isometric action of G, so that ∂G is naturally identified with ∂X.

For an algebraic lamination L on G, a geodesic segment $\tau = [a, b]$ in X is called a *coarse X-leaf segment of L* if there exist a pair $(x, y) \in L$ and a bi-infinite geodesic γ from x to y in X such that τ is contained in the 2δ-neighborhood of γ.

If $C \geq 0$, for a geodesic segment $\tau = [a, b]$ of length $\geq 2C$, the *C-truncation* of τ is defined as $[a', b'] \subseteq [a, b]$ where $a', b' \in [a, b]$ are such that $d(a, a') = d(b, b') = C$.

Theorem B. *Let G be a word-hyperbolic group and let Z be a compact metrizable space equipped with a non-elementary convergence action of G such that the Cannon-Thurston map $i : \partial G \to Z$ exists and such that i is not injective. Let X be a δ-hyperbolic (where $\delta \geq 1$) proper geodesic metric space equipped with a properly discontinuous cocompact isometric action of G (so that ∂G is naturally identified with ∂X).*

Let $x \in \partial G$, let $z = i(x) \in Z$ and let ρ be a geodesic ray in X limiting to x.

Then the following are equivalent:

(1) *The point z is a conical limit point for the action of G on Z.*
(2) *There exist a geodesic segment $\tau = [a, b]$ in X of length $\geq 100\delta$ and an infinite sequence of distinct elements $g_n \in G$ such that the 20δ-truncation τ' of τ is not a coarse X-leaf segment of L_i and such that for each $n \geq 1$ the segment $g_n\tau$ is contained in a 6δ-neighborhood of ρ. [Note that this condition automatically implies that $\lim_{n\to\infty} g_n a = \lim_{n\to\infty} g_n b = x$.]*

Proof. Suppose first that (1) holds and that z is a conical limit point for the action of G on Z. Since by assumption i is not injective, there exists a pair $(y', y) \in L_i$ such that $i(y) = i(y')$. Denote $s = i(y) = i(y')$. By translating by an element of g if necessary, we may also assume that $s \neq z$.

Since $i(x) = z$ and $z \neq s$, we have $x \neq y$. Note that $y \in End(L_i)$.

Consider a geodesic γ from y to x in X. Since z is conical, by Proposition 2.7 there exists an infinite sequence of distinct elements $h_n \in G$ such that $\lim_{n\to\infty} h_n(s, z) = (s_\infty, z_\infty)$ for some $s_\infty, z_\infty \in Z$ such that $s_\infty \neq z_\infty$. After passing to a further subsequence, we may assume that $\lim_{n\to\infty} h_n x = x_\infty$ and $\lim_{n\to\infty} h_n y = y_\infty$ for some $x_\infty, y_\infty \in \partial G = \partial X$. By continuity of i we have $i(x_\infty) = z_\infty$ and $i(y_\infty) = s_\infty$. In particular, this means that $x_\infty \neq y_\infty$ and that $\lim_{n\to\infty} h_n(y, x) = (y_\infty, x_\infty)$ in $\partial^2 G$. Let γ_∞ be a geodesic in X from y_∞ to x_∞.

Then there exists a sequence of finite subsegments $\tau_n = [q_n, r_n]$ of γ and a sequence of subsegments $[a_n, b_n]$ of γ_∞ with the following properties:

a) We have $\lim_{n\to\infty} a_n = y_\infty$, $\lim_{n\to\infty} b_n = x_\infty$ and $[a_n, b_n]$ is a subsegment of $[a_{n+1}, b_{n+1}]$.
b) We have either $\lim_{n\to\infty} q_n = \lim_{n\to\infty} r_n = x$ or $\lim_{n\to\infty} q_n = \lim_{n\to\infty} r_n = y$.
c) For all $n \geq 1$ the paths $h_n[q_n, r_n]$ and $[a_n, b_n]$ are 4δ-close.
d) We have $h_n q_n \to_{n\to\infty} y_\infty$ and $h_n r_n \to_{n\to\infty} x_\infty$.

If $\lim_{n\to\infty} q_n = \lim_{n\to\infty} r_n = y$, then, since $y \in End(L_i)$ and since $L_i \subseteq \partial^2 G$ is closed, it follows that $(y_\infty, x_\infty) \in L_i$. Therefore $z_\infty = i(x_\infty) = i(y_\infty) = s_\infty$, which contradicts the fact that $s_\infty \neq p_\infty$. Therefore $\lim_{n\to\infty} q_n = \lim_{n\to\infty} r_n = x$. Since $s_\infty \neq z_\infty$, it follows that $(y_\infty, x_\infty) \notin L_i$. Then there exists $m \geq 1$ such that $d(a_m, b_m) \geq 100\delta$ and such that for $\tau := [a_m, b_m]$ the 20δ-truncation $\tau' = [a'_m, b'_m] \subseteq \gamma_\infty$ of τ is not a coarse X-leaf segment of L_i. By construction, for every $n \geq m$, $h_n^{-1}\tau$ is contained in a 4δ-neighborhood of $[q_n, r_n]$ and hence, for all sufficiently large n, in a 6δ-neighborhood of ρ. Thus we have verified that (1) implies (2).

Suppose now that (2) holds and that there exist a geodesic segment $\tau = [a, b]$ in X of length $\geq 100\delta$ and an infinite sequence of distinct elements $g_n \in G$ such that the 10δ-truncation $\tau' = [a', b']$ of τ is not an X-leaf segment of L_i and such that for each $n \geq 1$ the segment $g_n\tau$ is contained in a 6δ-neighborhood of ρ. We claim that z is a conical limit point for the action of G on Z. In view of Lemma 2.6, we may assume that $i(\partial G) = Z$.

Indeed, let $s \in Z$ be arbitrary such that $s \neq z$. Recall that $i(x) = z$. Choose $y \in \partial G$ such that $i(y) = s$. Thus $x \neq y$. Consider the bi-infinite geodesic γ from y to x in X. Recall that ρ is a geodesic ray in X limiting to x.

After chopping-off a finite initial segment of ρ if necessary, we may assume that there is a point $w \in \gamma$ such that the ray ρ' from w to x contained in γ is 2δ-close to ρ. By assumption, for every $n \geq 1$ the geodesic $g_n^{-1}\gamma$ from $g_n^{-1}y$ to $g_n^{-1}x$ contains a subsegment which is 8δ-close to τ. By compactness, after passing to a further subsequence, we may assume that $\lim_{n\to\infty} g_n^{-1}y = y_\infty$ and $\lim_{n\to\infty} g_n^{-1}x = x_\infty$ for some distinct points $x_\infty, y_\infty \in \partial G$. Let γ_∞ be a geodesic from y_∞ to x_∞ in X.

We have $\tau' = [a', b'] \subseteq [a, b] = \tau$ with $d(a, a') = d(b, b') = 20\delta$. Since τ is contained in the 8δ-neighborhood of $g_n^{-1}\gamma$, the segment τ' is contained in a 2δ-neighborhood of γ_∞, and τ' has length $\geq 50\delta$. Since by assumption τ' is not a coarse X-leaf segment of L_i, it follows that $(y_\infty, x_\infty) \notin L_i$ and hence $i(x_\infty) \neq i(y_\infty)$. Denote $z_\infty = i(x_\infty)$ and $s_\infty = i(y_\infty)$. Since $i(x) = z$, $i(y) = s$ and since $\lim_{n\to\infty} g_n^{-1}y = y_\infty$ and $\lim_{n\to\infty} g_n^{-1}x = x_\infty$, the continuity of i implies that $\lim_{n\to\infty} g_n^{-1}(s, z) = (s_\infty, z_\infty)$. Since $s_\infty \neq z_\infty$, Proposition 2.7 implies that z is indeed a conical limit point for the action of G on Z, as required. $\square$

6. Injective, non-conical limit points

Here we prove that injective non-conical limit points occur quite often.

Theorem C. *Suppose a word-hyperbolic group G acts on a compact metrizable space Z as a non-elementary convergence group without accidental parabolics, and suppose that there exists a non-injective Cannon-Thurston map $i\colon \partial G \to Z$. Then there exists a non-conical limit point $z \in Z$ with $|i^{-1}(z)| = 1$.*

Suppose that G is a hyperbolic group acting as a non-elementary convergence group on Z as in the statement of the theorem, from which it follows that G is also non-elementary. Fix a finite generating set $\mathcal{S}$ for G, such that $\mathcal{S} = \mathcal{S}^{-1}$, and let X be the Cayley graph of G with respect to $\mathcal{S}$, endowed with the usual geodesic metric in which every edge has length 1. Then X is δ-hyperbolic for some $\delta > 0$. We denote the length of a geodesic segment σ in X as $|\sigma|$. Recall that for $r > 0$

an *r-local geodesic* in X is a path α parameterized by arclength such that every subsegment of α of length r is a geodesic. There exist integers $r > 0$ and $D > 0$ such that any r-local geodesic in X is quasigeodesic (with constants depending only on r and δ), and such that the Hausdorff distance between an r-local geodesic and the geodesic with the same endpoints is at most D; see e.g. [21, Part III, Chapter 1].

Given an algebraic lamination $L \subset \partial^2 G$, define the *geodesic realization of L with respect to $\mathcal{S}$*, denoted $\mathcal{L}$, as the set of all $\ell \subset X$ such that there exist $x, y \in \partial G$ with $(x, y) \in L$ such that ℓ is a bi-infinite geodesic in X from x to y.

Convention 6.1. For the remainder of this section, we assume G, Z, i are as in the statement of the theorem, $\mathcal{S}, X, \delta, r, D$ are as above, L_i is the algebraic lamination associated to i as in Definition 3.2, and $\mathcal{L}_i$ denotes the geodesic realization of L_i.

Given integers $p \geq 1$, a *p-periodic, r-local geodesic in X* is a bi-infinite r-local geodesic γ in X for which some element $g \in G$ acts on γ translating a distance p along γ. As γ is a quasigeodesic, it follows that g has infinite order (and γ is a quasigeodesic axis for g in X).

We will use the following lemma in the proof of the theorem.

Lemma 6.2. *For any $p \geq 1$, there exists $c(p) \geq 1$ with the following property. If γ is a p-periodic, r-local geodesic in X and $\ell \in \mathcal{L}_i$ contains a segment $\sigma \subset \gamma$ in its $(\delta + D)$-neighborhood, then $|\sigma| < c(p)$.*

Proof. It suffices to prove this statement for any fixed p-periodic, r-local geodesic γ in X (since, for a given p, there are only finitely many G-orbits of such γ). Translating such γ if necessary, we may assume that γ passes through the identity 1 in $G \subset X$. Let $g \in G$ be a translation of length p along γ.

Now if the requisite $c(p)$ does not exist, then there exists a sequence $\{\ell_n\}_{n \geq 1}$ of elements of $\mathcal{L}_i$ so that each ℓ_n contains a segment $\sigma_n \subseteq \gamma$ of length at least n in its $(\delta + D)$-neighborhood. Since $\mathcal{L}_i$ is G-invariant, after applying an appropriate power of g to ℓ_n if necessary, we can assume that the midpoint of σ_n lies within distance p of $1 \in G$. In particular, for $n > p$, $1 \in \sigma_n$ and ℓ_n is within $\delta + D$ of 1. Passing to a subsequence, we can assume that $\ell_n \to \ell \in \mathcal{L}_i$ as $n \to \infty$ (since L_i is closed). On the other hand, since $\sigma_n \to \gamma$, as $n \to \infty$, we see that γ is within $\delta + D$ of ℓ. Therefore, ℓ and γ have the same endpoints on ∂G. Since the endpoints of γ are the fixed points of g, and the endpoints of ℓ are identified by i, it follows that g is an accidental parabolic for the action on Z, yielding a contradiction. □

We are now ready for the proof of the theorem.

Proof of Theorem C. Let $\ell \in \mathcal{L}_i$ be a bi-infinite geodesic in $\mathcal{L}_i$ and let $\ell_+ \subset \ell$ be a geodesic ray contained in ℓ. We can view ℓ_+ as a semi-infinite word over the alphabet $\mathcal{S}$.

For any $m \geq r$, let $v_m \in \mathcal{S}^*$ be a word of length m which occurs (positively) infinitely often in ℓ_+. Such a word exists, for every m, by the pigeonhole principle. Now we define several additional families of subwords of ℓ_+. These subwords will serve as the building blocks for a new r-local geodesic (hence quasigeodesic) infinite ray.

For each $m \geq r$:

(1) Let u_m be any subword of ℓ_+ of length at least m so that $v_m u_m v_m$ occurs in ℓ_+. Such u_m exists because v_m occurs in ℓ_+ infinitely often.
(2) Let t_m be any non-empty word so that $v_m t_m v_{m+1}$ occurs in ℓ_+. These exist for the same reason as u_m.
(3) Put $\alpha_m = v_m u_m$.

Let $p_m = |\alpha_m|$, and let $\kappa_m > 0$ be an integer such that $\kappa_m p_m > c(p_m)$. Note that, since $v_m u_m v_m$ is a subword of a geodesic ray ℓ_+ with $|u_m|, |v_m| \geq m$, it follows that the word $\alpha_m = v_m u_m$ is cyclically reduced and that for every $k \geq 1$ every subword of length m in α_m^k is a geodesic and occurs as a subword of $v_m u_m v_m$ and thus of ℓ_+.

Now consider the following semi-infinite word (which we also view as a semi-infinite path in X with origin $1 \in G$):

$$w_\infty := \alpha_r^{\kappa_r} v_r t_r \alpha_{r+1}^{\kappa_{r+1}} v_{r+1} t_{r+1} \alpha_{r+2}^{\kappa_{r+2}} \cdots .$$

This word w_∞ is naturally a union of subwords of the following forms:

(1) α_m, which is a subword of ℓ_+.
(2) $v_m t_m v_{m+1}$, which is a subword of ℓ_+.
(3) $\alpha_m^{\kappa_m}$, which is a word of length $p_m \kappa_m > c(p_m)$, is contained in a p_m-periodic, r-local geodesic. As such, the word $\alpha_m^{\kappa_m}$ is not contained in a $(D+\delta)$-neighborhood of any $\ell' \in \mathcal{L}_i$, by Lemma 6.2. However, any subword of length m of $\alpha_m^{\kappa_m}$ occurs in ℓ_+.

Moreover, any subword v of w_∞ of length r is contained in at least one such word, and thus v occurs as a subword of ℓ_+. Therefore w_∞ is an r-local geodesic in X and hence a global quasigeodesic in X. Furthermore, note that as m tends toward infinity, the lengths of the words α_m and $v_m t_m v_{m+1}$ tend to infinity. Denote the endpoint of w_∞ in ∂G by x.

First, we claim that $|i^{-1}(i(x))| = 1$. If this were not the case, then the ray w_∞ would be asymptotic to (i.e. have a finite Hausdorff distance to) an infinite ray $\ell'_+ \subset \ell'$ for some geodesic $\ell' \in \mathcal{L}_i$. In this case, a subray $w'_\infty \subset w_\infty$ would be contained in the $(\delta + D)$-neighborhood of ℓ'_+. Since this ray contains arcs labeled $\alpha_m^{\kappa_m}$ for m sufficiently large, this contradicts Lemma 6.2.

Second, we claim that $i(x)$ is non-conical. To prove this, let γ be an r-local geodesic containing w_∞ as a subray. For example, let γ be the concatenation of the ray which is $\alpha_r^{-\infty}$ with w_∞. One endpoint of γ is x, and we denote the other by y. Let $(g_n)_{n\geq 1}$ be any convergence sequence for x with pole pair (x_-, x_+). Then $g_n(x) \to x_+$ and $g_n(y) \to x_-$. Since $x_- \neq x_+$, after passing to a subsequence (g_{n_k}), the r-local geodesics $g_{n_k}\gamma$ must converge to an r-local geodesic with endpoints x_-, x_+. After passing to a further subsequence (still denoted (g_{n_k})), it follows that $g_{n_k}\gamma$ converges to an r-local geodesic with endpoints x_-, x_+. Since (g_{n_k}) is a convergence sequence for x, if k is sufficiently large, any closest point $h_k \in g_{n_k}\gamma$ to 1 must have $g_{n_k}^{-1}(h_k) \in w_\infty$ with distance to the initial point of w_∞ tending toward infinity. Passing to yet a further subsequence if necessary, we can assume that the subword $w_k \subset w_\infty$ of length $2k$ centered on $g_{n_k}^{-1}h_k$ is a subword of $\ell_+ \subset \ell$. Thus for all k there exists $\ell_k \in \mathcal{L}_i$ (a translate of ℓ) so that the *segment* w_k is contained in ℓ_k.

Now observe that $g_{n_k}(w_k)$ is a segment of $g_{n_k}(\ell_k) \in \mathcal{L}_i$. Because $g_{n_k}(w_k)$ is a geodesic of length $2k$ centered on h_k, it follows that $g_{n_k}(w_k)$, and hence $g_{n_k}(\ell_k)$, converges to a geodesic with endpoints (x_-, x_+) as $k \to \infty$. However, $g_{n_k}(\ell_k)$ must converge to a leaf of $\mathcal{L}_i$ since L_i is closed. Since (g_n) was an arbitrary convergence sequence for x, x is asymptotic to L_i, and by Theorem A, $i(x)$ is non-conical. □

7. Controlled concentration points

Definition 7.1. Let G be a non-elementary torsion-free discrete subgroup of hyperbolic isometries acting on $\mathbb{H}^n$ and let S^{n-1}_∞ be the ideal boundary of $\mathbb{H}^n$. A neighborhood $U \subset S^{n-1}_\infty$ of $x \in \Lambda(G)$ is called *concentrated* at x if for every neighborhood V of x, there exists an element $g \in G$ such that $x \in g(U)$ and $g(U) \subset V$. If such g can always be chosen so that $x \in g(V)$, then we say U is *concentrated with control*. A limit point x in $\Lambda(G)$ is called a *controlled concentration point* if it has a neighborhood which is concentrated with control.

A geodesic ray in $\mathbb{H}^n$ is called *recurrent with respect to* G if its image α in $M = \mathbb{H}^n/G$ by the covering projection is recurrent. Recall that a geodesic ray α parameterized by $[0, \infty)$ in M is called *recurrent* if for any tangent vector $v = \alpha'(t_0), t_0 > 0$, in the unit tangent bundle $UT(M)$ of M, there exists an infinite sequence of times $\{t_i\}$ such that $\alpha'(t_i)$ converges to v in $UT(M)$. The main result of [3] is that controlled concentration points correspond to the endpoints of recurrent geodesic rays.

Theorem 7.2 (Aebischer, Hong and McCullough [3])**.** *Let G be as in Definition* 7.1*. Then for a limit point $x \in \Lambda(G)$, the following are equivalent:*

(1) *There is a recurrent geodesic ray whose endpoint is x.*
(2) *x is a controlled concentration point.*
(3) *There exists a sequence $\{g_n\}$ of distinct elements of G such that for any geodesic ray β whose endpoint $\beta(\infty)$ is x, $g_n(\beta)$ converges to some geodesic ray whose endpoint is again x up to taking a subsequence.*
(4) *There exists a sequence $\{g_n\}$ of distinct elements of G and $y \in S^{n-1}_\infty$ with $y \neq x$ such that $g_n x \to x$ and $g_n|_{S^n_\infty \setminus \{x\}}$ converges uniformly on compact subsets to the constant map to y.*

From the last characterization of a controlled concentration point, it is clear that every controlled concentration point is conical, but the converse is not true in general. In fact [3, Prop. 5.1] gives an example of a conical limit point which is not a controlled concentration point in the case of a rank-2 Schottky group. It is also known that the set of controlled concentration points has full Patterson-Sullivan measure in $\Lambda(G)$ if G is of divergence type. Note that for a geodesic lamination λ in a hyperbolic surface S, a leaf of λ is always a recurrent geodesic, and hence its endpoints are controlled concentration points.

The following proposition follows easily from condition (1) of Theorem 7.2.

Proposition 7.3. *A limit point x in $\Lambda(G) \subset S^{n-1}_\infty$ is a controlled concentration point if there exists a geodesic ray β in $\mathbb{H}^n$ which limits to x, and the ω-limit set of β in the geodesic foliation on the unit tangent bundle $UT(M)$ of $M = \mathbb{H}^n/G$ has only one minimal component.*

We extend the notion of controlled concentration points to the case of hyperbolic groups.

Definition 7.4. Let G be a non-elementary hyperbolic group. Then we say $x \in \partial G$ is a *controlled concentration point* if there exists a sequence $\{g_n\}$ of distinct elements of G and $y \in \partial G$ with $y \neq x$ such that $g_n x \to x$ and $g_n|_{\partial G \setminus \{x\}}$ converges locally uniformly to the constant map to y.

Proposition 7.5. *Suppose G is word-hyperbolic and acts on the compact, metrizable space Z as a non-elementary convergence group, and suppose $i \colon \partial G \to Z$ is a Cannon-Thurston map. If a controlled concentrated point $x \in \partial G$ satisfies $|i^{-1}(i(x))| = 1$, then $i(x) \in Z$ is conical.*

Proof. Since x is a controlled concentrated point, there exist $y \in \partial G$ with $y \neq x$ and a sequence $\{g_n\}$ of distinct elements in G such that $\lim_{n\to\infty} g_n x = x$ and $(g_n|_{\partial G\setminus\{x\}})$ locally uniformly converges to the constant map to y. Note that $|i^{-1}(i(x))| = 1$ implies $i(x) \neq i(y)$. Suppose $i(x) \in Z$ is not conical. Then by Proposition 2.7 there exists a subsequence (g_{n_k}) of (g_n) such that $\lim_{k\to\infty} g_{n_k} i(x) = \lim_{k\to\infty} g_{n_k} i(y)$ and hence by continuity

$$(i(x), i(y)) = \lim_{k\to\infty} (i(g_{n_k} x), i(g_{n_k} y)) = \lim_{k\to\infty} (g_{n_k} i(x), g_{n_k} i(y)) = (z, z)$$

for some $z \in Z$, and hence $i(x) = i(y)$. This is a contradiction. □

We can now prove the last theorem from the Introduction:

Theorem D. *Let G be a non-elementary torsion-free word-hyperbolic group. Then there exists $x \in \partial G$ which is not a controlled concentration point.*

Proof. Kapovich [47] proved that, given a non-elementary torsion-free word-hyperbolic group G, there exists a word-hyperbolic group G_* containing G as a non-quasiconvex subgroup. Moreover, G_* is constructed in [47] as an HNN-extension

$$G_* = \langle G, t | t^{-1} K t = K_1 \rangle$$

where $K \leq G$ is a quasiconvex free subgroup of rank 2 and where $K_1 \leq K$ is also free of rank 2 (and hence K_1 is also quasiconvex in G). Therefore, by a general result of Mitra [69] (see also [78]) about graphs of groups with hyperbolic edge and vertex groups, there does exist a Cannon-Thurston map $i : \partial G \to \partial G_*$. Since $G \leq G_*$ is not quasiconvex, Proposition 2.13 implies that the map i is not injective. Therefore, by Theorem C, there exists a non-conical limit point $z \in i(\partial G)$ with $|i^{-1}(z)| = 1$. By Proposition 7.5, $x = i^{-1}(z) \in \partial G$ is not a controlled concentration point. □

Appendix A. Descriptions of L_i

There are several situations where the Cannon-Thurston map $i : \partial G \to Z$ is known to exist and where a more detailed description of the lamination L_i is known. Theorem A and Theorem B may be useful in these contexts. The proof of Theorem 6.6 in [33] uses Theorem B as a key ingredient in this way.

A.1. **Kleinian representations of surface groups.** Let G be the fundamental group of a closed, orientable hyperbolic surface S. The universal covering of S is isometric to the hyperbolic plane $\mathbb{H}^2$ with G acting cocompactly by isometries, and so we can identify the Gromov boundary of G with the circle at infinity $\partial G \cong S^1_\infty$. A faithful *Kleinian representation* $\rho \colon G \to \mathrm{PSL}(2, \mathbb{C}) \cong \mathrm{Isom}^+(\mathbb{H}^3)$ is an injective homomorphism with discrete image. This determines a convergence action of G on S^2_∞, and hence also on the limit set $\Lambda(G)$. The existence of a Cannon-Thurston map for such groups was first proved in the special case when $\rho(G)$ is the fiber subgroup of a hyperbolic 3-manifold fibering over the circle by Cannon and Thurston [25]. This was extended to include other classes of Kleinian representations of G in [65,69] and then arbitrary faithful, Kleinian representations of G in [77].

The hyperbolic 3-manifold $M = \mathbb{H}^3/\rho(G)$ is homeomorphic to $S \times (-\infty, \infty)$ by the Tameness Theorem ([15], and [1,24] in more general settings) and thus M has only two ends, E_+ and E_-. Assume that $\rho(G)$ has no parabolics. Associated to each end is a (possibly empty) *ending lamination* λ_+ and λ_-, which is a geodesic lamination on S, that is, a closed union of pairwise disjoint complete geodesics; see [27] for more on geodesic laminations and [23,66,87] for more on the ending laminations associated to ends of 3-manifolds. The pre-images $\widetilde{\lambda}_\pm \subset \mathbb{H}^2$ of the ending laminations in $\mathbb{H}^2$ are geodesic laminations in $\mathbb{H}^2$, and the endpoints of the leaves determine a pair of algebraic laminations $L_\pm \subset \partial^2 G$. Set $\mathcal{R}_+, \mathcal{R}_- \subset \partial G \times \partial G$ to be the reflexive and transitive closures of the pair L_+, L_-, respectively. Then for the Cannon-Thurston map i has $i(x) = i(y)$ if and only if $(x, y) \in \mathcal{R}_+ \cup \mathcal{R}_-$ according to [25] in the original setting, [65] for the cases treated there, and in general in [71]. Furthermore, the transitive closure adds only endpoints of finitely many G-orbits of leaves, and thus L_i is equal to $L_+ \cup L_-$, together with finitely many additional G-orbits of leaves (which correspond to the "diagonals" of the complementary components of $\widetilde{\lambda}_+$ and $\widetilde{\lambda}_-$).

A.2. **Short exact sequences of hyperbolic groups.** Let

$$1 \to G_1 \to G_2 \to Q \qquad (\ddagger)$$

be a short exact sequence of three word-hyperbolic groups, such that G_1 is non-elementary. In this case G_1 acts on $Z = \partial G_2$ as a non-elementary convergence group without accidental parabolics. Mitra [68] proved that in this case the Cannon-Thurston map $i : \partial G_1 \to \partial G_2$ does exist. Therefore the results of this paper, including Theorem B, do apply. In [67] Mitra also obtained a general geometric description of L_i in this case in terms of the so-called "ending laminations".

We give here a brief description of the results of [67].

Given every $\xi \in \partial Q$, Mitra defines an "ending lamination" $\Lambda_\xi \subseteq \partial^2 G_1$. To define Λ_ξ, Mitra starts with choosing a quasi-isometric section $r : Q \to G_2$ (he later proves that the specific choice of r does not matter). Then given any $\xi \in \partial Q$, take a geodesic ray in Q towards ξ and let ξ_n be the point at distance n from the origin on that ray. Lift ξ_n to G_2 by the section r to get an element $g_n = r(\xi_n) \in G_2$. Conjugation by g_n gives an automorphism φ_n of G_1 defined as $\varphi_n(h) = g_n h g_n^{-1}$, $h \in G_1$. Now pick any non-torsion element $h \in G_1$. Then look at all $(x, y) \in \partial^2 G_1$ such that there exists a sequence of integers $k_n \to \infty$ and of conjugates (with respect to conjugation in G_1) w_n of $\varphi_{k_n}(h)$ in G_1 such that $\lim_{n\to\infty}(w_n^{-\infty}, w_n^{\infty}) = (x, y)$ in $\partial^2 G_1$. For a fixed non-torsion $h \in G_1$, the collection of all such $(x, y) \in \partial^2 G_1$ is denoted $\Lambda_{\xi,h}$. Denote by A the set of all elements of infinite order in G_1. Finally, put $\Lambda_\xi = \bigcup_{h\in A} \Lambda_{\xi,h}$. The main result of Mitra in [67] says that, in this case

$$L_i = \bigcup_{\xi \in \partial Q} \Lambda_\xi.$$

For every $\xi \in \partial Q$ the subset $\Lambda_\xi \subseteq \partial^2 G_1$ is an algebraic lamination on G_1, and Mitra refers to Λ_ξ as the "ending lamination" on G_1 corresponding to ξ. Moreover, the arguments of Mitra actually imply that if $\xi_1, \xi_2 \in \partial Q$ are distinct, then $End(\Lambda_{\xi_1}) \cap End(\Lambda_{\xi_2}) = \varnothing$. Mitra also notes that for any $\xi \in \partial Q$ there exists a finite subset $B \subseteq A$ such that $\Lambda_\xi = \bigcup_{h\in B} \Lambda_{\xi,h}$.

In general, for a short exact sequence (‡) and $\xi \in \partial Q$, the "ending lamination" $\Lambda_\xi \subseteq \partial^2 G_1$ can, at least a priori, be quite large and difficult to understand. This is the case even if $Q = \langle t \rangle \cong \mathbb{Z}$ is infinite cyclic, so that $\partial Q = \{t^{-\infty}, t^{\infty}\}$ consists of just two points. However, in some situations the laminations Λ_ξ are well understood.

A.3. **Hyperbolic extensions of free groups.** In particular, let $N \geq 3$, let $\varphi \in \mathrm{Out}(F_N)$ be a fully irreducible atoroidal element and let $\Phi \in \mathrm{Aut}(F_N)$ be a representative of the outer automorphism class of φ (see [8–10, 28, 43, 44, 48, 63] for the relevant background). Then

$$G = F_N \rtimes_\Phi \mathbb{Z} = \langle F_N, t | twt^{-1} = \Phi(w), w \in F_N \rangle$$

is word-hyperbolic and we have a short exact sequence $1 \to F_N \to G \to \langle t \rangle \to 1$. Thus, by [68], there does exist a Cannon-Thurston map $i : \partial F_N \to \partial G$. Using the results of Mitra [67] mentioned above as a starting point, Kapovich and Lustig proved in [53] that $\Lambda_{t^\infty} = diag(L_{BFH}(\varphi)) = L(T_-)$ and, similarly, $\Lambda_{t^{-\infty}} = diag(L_{BFH}(\varphi^{-1})) = L(T_+)$. Here $L_{BFH}(\varphi) \subseteq \partial^2 F_N$ is the "stable" lamination of φ, introduced by Bestvina, Feighn and Handel in [9], and $diag(L_{BFH}(\varphi^{-1}))$ is the "diagonal extension" of $L_{BFH}(\varphi)$, that is, the intersection of $\partial^2 F_N$ with the equivalence relation on ∂F_N generated by the relation $L_{BFH}(\varphi) \subseteq \partial^2 F_N$ on ∂F_N. Also, here $L(T_-)$ is the "dual algebraic lamination" (in the sense of [30, 32, 51]) corresponding to the "repelling" $\mathbb{R}$-tree T_- for φ (the tree T_- is constructed using a train-track representative for φ^{-1} and the projective class of T_- is the unique repelling fixed point for the right action of φ on the compactified Outer space. Thus, in view of the discussion above, we have

$$L_i = diag(L_{BFH}(\varphi)) \cup diag(L_{BFH}(\varphi^{-1})) = L(T_-) \cup L(T_+)$$

in this case. The stable lamination $L_{BFH}(\varphi)$ of φ is defined quite explicitly in terms of a train-track representative $f : \Gamma \to \Gamma$ of φ. Thus a pair $(x, y) \in \partial^2 F_N$ belongs to $L_{BFH}(\varphi)$ if and only if for every finite subpath $\tilde{v}$ of the geodesic from x to y in $\widetilde{\Gamma}$, the projection v of $\tilde{v}$ to Γ has the property that for some edge e of Γ and some $n \geq 1$ the path v is a subpath of $f^n(e)$. Kapovich and Lustig also proved in [52] that $diag(L_{BFH}(\varphi))$ is obtained from $L_{BFH}(\varphi)$ by adding finitely many F_N orbits of " diagonal" leaves (x, y) of a special kind. These extra "diagonal leaves" play a similar role to the diagonals of ideal polygons given by complementary regions for the lift to $\mathbb{H}^2$ of the stable geodesic lamination of a pseudo-anosov homeomorphism of a closed hyperbolic surface.

In [33] Dowdall, Kapovich and Taylor generalize the above description of L_i to the case of word-hyperbolic extensions E_Γ of F_N determined by purely atoroidal "convex cocompact" subgroups $\Gamma \leq Out(F_N)$. See [33, Corollary 5.3] for details.

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