CONFORMAL GEOMETRY AND DYNAMICS
An Electronic Journal of the American Mathematical Society
Volume 20, Pages 116–127 (May 4, 2016)
http://dx.doi.org/10.1090/ecgd/293

GROWTH AND MONOTONICITY PROPERTIES FOR ELLIPTICALLY SCHLICHT FUNCTIONS

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Abstract. Let f be a holomorphic function of the unit disc $\mathbb{D}$, with $f(\mathbb{D}) \subset \mathbb{D}$ and $f(0) = 0$. Littlewood's generalization of Schwarz's lemma asserts that for every $w \in f(\mathbb{D})$, we have $|w| \leq \prod_j |z_j|$, where $\{z_j\}_j$ are the pre-images of w. We consider elliptically schlicht functions and we prove an analogous bound involving the elliptic capacity of the image. For these functions, we also study monotonicity theorems involving the elliptic radius and elliptic diameter.

1. Introduction and statement of the results

Let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic self-map of the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ with $f(0) = 0$. By the famous Schwarz lemma

$$|f(z)| \leq |z|, \text{ for every } z \in \mathbb{D}. \tag{1.1}$$

A standard subject in geometric function theory is to obtain variations of Schwarz's lemma. See for example [3], [4], [5], [6], [7], [8], [9], [11], [16, Chapter 4].

The classical theorem of Littlewood[1] [21, Theorem 214] generalizes Schwarz's lemma and asserts that if $f : \mathbb{D} \to \mathbb{D}$ is a holomorphic function with $f(0) = 0$, then for every $w \in f(\mathbb{D})$, we have

$$|w| \leq \prod_j |z_j(w)|, \tag{1.2}$$

where the at most countably infinite set $\{z_1(w), z_2(w), \dots\}$ contains all the pre-images of w repeated as many times as their multiplicity as roots of the function $f(z) - w$. Lehto in [20] proved that equality holds in (1.2) for some $w \in \mathbb{D} \setminus \{0\}$ if and only if f is an inner function.

Multi-point upper bounds for the modulus $|w|,\ w \in f(\mathbb{D})$, involving geometric quantities such as diameter and logarithmic capacity for the image can be found in [3] and [8]. Specifically, let $f : \mathbb{D} \to \mathbb{C}$ be a bounded holomorphic function. Betsakos [3] proved that for every $w \in f(\mathbb{D})$,

$$|w - f(0)| \leq \Psi\Big(\prod_j |z_j(w)|\Big)\mathrm{Diam} f(\mathbb{D}),$$

Received by the editors September 30, 2014 and, in revised form, February 15, 2016.
2010 *Mathematics Subject Classification.* Primary 30C80, 30C85, 31A15.
Key words and phrases. Circular symmetrization, condenser capacity, elliptic capacity, elliptically schlicht functions, Green function, modulus metric, monotonicity theorems, Schwarz lemma, upper bounds.

[1]This result is usually attributed to Littlewood, however it may be found also in Jensen's paper [17].

where Diam$f(\mathbb{D})$ is the diameter of $f(\mathbb{D})$ and

$$\Psi(x) = \frac{x}{1+\sqrt{1-x^2}}, \ 0 \le x < 1.$$

The first-named author proved [8] that for every $w \in f(\mathbb{D}) \setminus \{f(0)\}$,

$$|w - f(0)| \le 4d(f(\mathbb{D}))e^{-\mu(\prod_j |z_j(w)|)},$$

where $d(f(\mathbb{D}))$ is the logarithmic capacity (transfinite diameter) of $f(\mathbb{D})$, μ is defined by

$$\mu(x) = \frac{\pi\mathcal{K}(\sqrt{1-x^2})}{2\mathcal{K}(x)}, \ \ 0 < x < 1, \tag{1.3}$$

and $\mathcal{K}(x)$ is the complete elliptic integral

$$\mathcal{K}(x) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-x^2t^2)}}, \ \ 0 < x < 1.$$

We note that the function μ is strictly decreasing and maps $(0, 1)$ onto $(0, \infty)$; see [1, Chapter 5].

Also, the authors in [9] proved an upper and a lower multi-point bound for holomorphic functions whose image satisfies a measure condition. In the present paper we will give an analogous upper bound involving elliptic capacity for elliptically schlicht functions. First, we give some background for this class of functions.

Let $a \in \mathbb{C} \setminus \{0\}$. We call $a^* = -1/\overline{a}$ the *antipodal point of* a. We also set $0^* = \infty, \ \infty^* = 0$. The projections of two antipodal points on the Riemann sphere are diametrically opposite points. Let $E \subset \mathbb{C}_\infty$. We define its *antipodal set* $E^* = \{a^* : a \in E\}$. The set E is said to be *elliptically schlicht* if $E \cap E^* = \emptyset$. A mapping is said to be *elliptically schlicht* if its image is an elliptically schlicht set. The *pseudoelliptic metric* for points $z, w \in \mathbb{C}$ is defined as follows:

$$[z, w]_e = \left|\frac{z-w}{1+\overline{z}w}\right|, \ \ [z, \infty]_e = \frac{1}{|z|}. \tag{1.4}$$

We define the *elliptic transfinite diameter* of a closed elliptically schlicht set E, denoted by $d_e(E)$, as the limit $d_e(E) = \lim_{n\to\infty} d_{e,n}(E)$, where

$$d_{e,n}(E) = \Big\{ \sup_{z_1,\dots,z_n \in E} \prod_{1\le j<k\le n} [z_j, z_k]_e \Big\}^{\frac{2}{n(n-1)}}, \ n = 2, 3, 4, \dots,$$

with the supremum taken over all n-tuples of points in E. The elliptic transfinite diameter of an elliptically schlicht set is defined to be the elliptic transfinite diameter of its closure and is always a finite quantity. Note that this is equal to the elliptic capacity of the set E; see [12]. For more about elliptically schlicht functions and elliptic capacity, see also [2], [12], [13], [18], [19].

Now let $w_0 \in \mathbb{C} \setminus \{0\}$ and let Ω be the doubly connected domain with complementary components $[0, w_0]$ and $[0, w_0]^*$. The domain Ω can be mapped conformally onto the annulus $\{z : d_0 < |z| < d_0^{-1}\}$, where $d_0 = d_e([0, w_0])$; see [4], [12]. The pre-images of the circles $|z| = r$, where $d_0 < r < d_0^{-1}$, are Jordan curves $\Gamma(r, w_0)$ enclosing the segment $[0, w_0]$. Note that:

(i) $\Gamma(1, w_0)$ is a circle and the sets $[0, w_0]$, $[0, w_0]^*$ are symmetric with respect to $\Gamma(1, w_0)$,

(ii) $d_e(\text{interior}(\Gamma(1, w_0))) = 1$, and

(iii) for $d_0 < r \le 1$, the interior of $\Gamma(r, w_0)$ is elliptically schlicht.

Let $f:\mathbb{D}\to\mathbb{C}$ be a nonconstant, elliptically schlicht holomorphic function with $f(0)=0$. Shah [24] and Betsakos [4] proved respectively that

$$|f(z)|\leq\frac{|z|}{\sqrt{1-|z|^2}},\quad\text{for every } z\in\mathbb{D}, \tag{1.5}$$

$$|f(z)|\leq\varphi\circ\mu^{-1}\Big(\mu(|z|)-\log d_e(f(\mathbb{D}))\Big),\quad\text{for every } z\in\mathbb{D}\setminus\{0\}, \tag{1.6}$$

where μ is the special function in (1.3) and

$$\varphi(x)=\frac{x}{\sqrt{1-x^2}},\quad x\in(0,1). \tag{1.7}$$

Our first purpose in this article is to extend these results, obtaining upper bounds analogous to Littlewood's theorem and without the restriction $f(0)=0$.

Theorem 1. *Let $f:\mathbb{D}\to\mathbb{C}$ be a nonconstant, elliptically schlicht, holomorphic function. Then for every $w\in f(\mathbb{D})\setminus\{f(0)\}$,*

$$[w,f(0)]_e\leq\varphi\circ\mu^{-1}\Big(\mu\big(\prod_j|z_j(w)|\big)-\log d_e(f(\mathbb{D}))\Big). \tag{1.8}$$

Equality holds in (1.8) *for some $w_0\in f(\mathbb{D})\setminus\{f(0)\}$ if and only if*

$$f(z)=\frac{h\circ k(z)+a}{1-h\circ k(z)\overline{a}},$$

where $a\in\mathbb{C}$, k is an inner function with $k(0)=0$ and h is a conformal mapping from $\mathbb{D}$ onto the interior of the curve $\Gamma\left(r,\dfrac{w_0-a}{1+w_0\overline{a}}\right)$ for some r with

$$d_e\left(\left[0,\frac{w_0-a}{1+w_0\overline{a}}\right]\right)<r\leq 1.$$

Since $d_e(f(\mathbb{D}))\leq 1$ (see [12]), μ is decreasing and φ is increasing, we have:

Corollary 1. *If $f:\mathbb{D}\to\mathbb{C}$ is a nonconstant, elliptically schlicht, holomorphic function, then for every $w\in f(\mathbb{D})$,*

$$[w,f(0)]_e\leq\frac{\prod_j|z_j(w)|}{\sqrt{1-\prod_j|z_j(w)|^2}}. \tag{1.9}$$

Equality holds in (1.9) *for some $w_0\in f(\mathbb{D})\setminus\{f(0)\}$ if and only if $f(z)=\dfrac{h\circ k(z)+a}{1-h\circ k(z)\overline{a}}$, where $a\in\mathbb{C}$, k is an inner function with $k(0)=0$, and h is a conformal mapping from $\mathbb{D}$ onto the interior of the circle $\Gamma\left(1,\dfrac{w_0-a}{1+w_0\overline{a}}\right)$. The function h has the form*

$$h(z)=e^{i\theta}\frac{z\sqrt{1-|z_0|^2}}{1-\overline{z_0}z},$$

for some $\theta\in\mathbb{R}$ and $z_0\in\mathbb{D}$ with $z_0=k(z_j(w_0))$, $j\in\mathbb{N}$.

Let us now proceed to our second result which is a monotonicity theorem. In [6] Burckel, Marshall, Minda, Poggi-Corradini, and Ransford presented the monotonicity version of Schwarz's lemma: Let f be holomorphic in the unit disc $\mathbb{D}$ and consider the "radius"

$$\operatorname{Rad} f(r\mathbb{D}) = \sup_{z \in r\mathbb{D}} |f(z) - f(0)|, \quad r \in (0,1),$$

where $r\mathbb{D} = \{z \in \mathbb{C} : |z| < r\}$. Then the function

$$\phi_{\operatorname{Rad}}(r) = \frac{\operatorname{Rad} f(r\mathbb{D})}{r}, \quad r \in (0,1),$$

is increasing.

In the same paper, Burckel et al. proved analogous monotonicity theorems, involving other geometric quantities. The article [6] was a great source of inspiration for several mathematicians to prove monotonicity theorems. Such results can be found in [4], [7], [11], [27] and references therein.

Here we consider as before the class of holomorphic and elliptically schlicht functions in the unit disc. Let $f : \mathbb{D} \to \mathbb{C}$ be such a function. We introduce the elliptic "radius":

$$R_f(r) = \sup_{z \in r\mathbb{D}} [f(z), f(0)]_e, \quad 0 < r < 1,$$

and we prove the following monotonicity theorem:

Theorem 2. *Let $f : \mathbb{D} \to \mathbb{C}$ be an elliptically schlicht, holomorphic function. Then,*

(a) *The function*

$$\varphi_R(r) = \frac{R_f(r)}{r}, \quad 0 < r < 1,$$

is strictly increasing except when f has the form

$$f(z) = \frac{cz + a}{1 - c\overline{a}z}, \tag{1.10}$$

for some $a \in \mathbb{C}$ and $c,\ c\overline{a} \in \overline{\mathbb{D}}$, in which case φ_R is constant. Moreover, $\lim_{r \to 0} \varphi_R(r) = \dfrac{|f'(0)|}{1 + |f(0)|^2}$.

(b) *If $\sup_{z \in \mathbb{D}} [f(z), f(0)]_e = R < \infty$, then*

(i) *$[f(z), f(0)]_e \le R|z|$, for every $z \in \mathbb{D}$,*

(ii) *$\dfrac{|f'(0)|}{1 + |f(0)|^2} \le R$.*

Equality holds in (i) *for some $z \in \mathbb{D} \setminus \{0\}$ or in* (ii) *if and only if f has the form* (1.10).

The article is organized as follows. In section 2 we present the basic tools of our proofs: Green function, circular symmetrization, condensers, and modulus metric. Sections 3 and 4 are devoted to the proof of Theorems 1 and 2, respectively. We close the article with some final remarks in section 5.

2. Preliminaries

2.1. **Green function.** Suppose that D is a domain in the extended complex plane $\mathbb{C}_\infty$ and assume that D has a Green function denoted by $g_D(z, w),\ z, w \in D,\ z \ne w$.

For the unit disc we have the formula

$$g_{\mathbb{D}}(z,w) = \log\left|\frac{1-z\overline{w}}{z-w}\right|,\ z \neq w. \tag{2.1}$$

We present a special case of Lindelöf's Principle (see [22, Ch. III, §4]). If $f : \mathbb{D} \to \mathbb{C}$ is a holomorphic function and $f(\mathbb{D})$ is a domain possessing a Green function, then for every $w \in f(\mathbb{D}) \setminus \{f(0)\}$,

$$\sum_j g_{\mathbb{D}}(z_j(w), 0) \le g_{f(\mathbb{D})}(w, f(0)), \tag{2.2}$$

where $\{z_j(w)\}_j$ is the set of the pre-images of w. For more information about the Green function we refer the reader to [14, p. 26], [22, p. 28], [23, p. 106].

2.2. **Circular symmetrization.** Let $O \subset \mathbb{C}_\infty$ be an open set. We define the circular symmetrized set $S^{\mathbb{R}_+}O$ with respect to the positive real axis $\mathbb{R}_+$ as follows (see e.g. [10]): Let C_r be the circle $\{z \in \mathbb{C} : |z| = r\}$ for $0 < r < \infty$, which degenerates to the point 0 (or ∞) if $r = 0$ (or $r = \infty$ respectively). If, for a given $0 \le r \le \infty$ the intersection of O with the "circle" C_r includes the whole "circle" or is null, then the intersection of $S^{\mathbb{R}_+}O$ with C_r includes also the whole "circle", or is null respectively. In the remaining cases, the set $S^{\mathbb{R}_+}O$ intersects C_r along an open arc with center at the positive real axis $\mathbb{R}_+$, whose linear measure agrees with the measure of the intersection of O with C_r.

The circular symmetrized set $S^\alpha O$ with respect to an arbitrary ray α is defined as the superposition $\phi \circ C_r \circ \phi^{-1}$ where ϕ is a motion in $\mathbb{C}$ mapping the ray α into $\mathbb{R}_+$. For more details, see [10, p. 19] and [16, Chapter 4].

Suppose that D is a domain in $\mathbb{C}$ possessing a Green function. Let $0, w \in D$ and let α be the ray emanating from the origin and passing through w. Suppose also that $S^\alpha D$ possesses a Green function. Then (see [15, Chapter 9] and references therein)

$$g_D(0,w) \le g_{S^\alpha D}(0,w). \tag{2.3}$$

We have equality (see [25]) if and only if $D \overset{n.e}{=} S^\alpha D$; here by $\overset{n.e.}{=}$ we mean that the sets differ on a set of zero logarithmic capacity.

2.3. **Condensers.** A *condenser* is a pair (D,K) where D is a domain and K is a compact subset of D. Here, we will use condensers for which the open set $D \setminus K$ is regular for the Dirichlet problem. The *capacity* $\operatorname{cap}(D,K)$ of the condenser (D,K) is defined to be the Dirichlet integral of its *potential function*; that is, the harmonic function on $D \setminus K$ with boundary values 0 on ∂D and 1 on ∂K. We need that [10]

$$\operatorname{cap}(E\mathbb{D}, \overline{\varepsilon\mathbb{D}}) = \frac{2\pi}{\log(E/\varepsilon)}, \quad E > \varepsilon > 0. \tag{2.4}$$

For more about condensers, see [10], [16], [26].

2.4. **Modulus metric.** Let D be a domain in the complex plane and let $z, w \in D$. We define the modulus metric on D as

$$\mu_D(z,w) = \inf_{l \in \Gamma} \operatorname{cap}(D, l),$$

where Γ is the set of all curves l in D joining z, w. By [3, Lemma 1], there exists a decreasing function Φ on $[0,\infty)$ such that for every simply connected proper subset D of $\mathbb{C}$ and every $z, w \in D,\ z \neq w$,

$$\mu_D(z,w) = \Phi(g_D(z,w)). \tag{2.5}$$

The function $\Phi = \Phi_2 \circ \Phi_1$, where

$$\Phi_1(x) = 2\tanh^{-1}(e^{-x})\ ,\ \Phi_2(x) = \gamma\left(\frac{1}{\tanh(x/2)}\right)$$

and

$$\gamma(x) = \frac{2\pi}{\mu(1/x)},\ x > 1,$$

with the function μ defined as in (1.3). We conclude that

$$\Phi(x) = \frac{2\pi}{\mu(e^{-x})}. \tag{2.6}$$

Let D be a domain which is circularly symmetric with respect to a ray α; that is, $S^\alpha D = D$, and let G be the simply connected domain that we obtain if we fill up all the holes of D. Then G is also circularly symmetric with respect to the ray α, and for every $z, w \in \alpha$,

$$\mu_G(z,w) = \operatorname{cap}(G, [z,w]). \tag{2.7}$$

The relation (2.7) is obtained immediately from the fact that circular symmetrization reduces the capacity of condensers [10]. The analogous relation for Steiner symmetrization is proved in [3].

The basic properties of the modulus metric can be found in [26].

3. Proof of Theorem 1

We will need the following special symmetrization lemma.

Lemma 1 ([4])**.** *Let K be a compact, connected, elliptically schlicht set in $\mathbb{C}$ and let A be an elliptically schlicht domain containing K. Let SK be the closed disk centered at the origin and having radius equal to $d_e(K)$. Let SA be the open disk centered at the origin and having radius equal to $d_e(A)$. Then*

$$\operatorname{cap}(A,K) \geq \operatorname{cap}(SA, SK)$$

with equality if and only if the boundary of A is a level curve for the potential function of the condenser $(\mathbb{C}_\infty \setminus K^*, K)$.

We are now ready to proceed to the proof of Theorem 1.

Proof. Let $f : \mathbb{D} \to \mathbb{C}$ be a holomorphic, elliptically schlicht function. First we assume that $f(0) = 0$. We are going to prove that for every $w \in f(\mathbb{D}) \setminus \{0\}$,

$$|w| \leq \varphi \circ \mu^{-1}\Big(\mu\big(\prod_j |z_j(w)|\big) - \log d_e(f(\mathbb{D}))\Big). \tag{3.1}$$

As $f(0) = 0$, the function f is bounded, because it is elliptically schlicht. Let $w \in f(\mathbb{D}) \setminus \{0\}$. By (2.2) we have

$$\sum_j g_{\mathbb{D}}(z_j(w), 0) \leq g_{f(\mathbb{D})}(w, 0). \tag{3.2}$$

Let α be the ray emanating from the origin and passing through w. Then by (2.3),

$$g_{f(\mathbb{D})}(w,0) \le g_{S^\alpha f(\mathbb{D})}(w,0). \tag{3.3}$$

Let G be the simply connected domain that we obtain if we fill up all the holes of $S^\alpha f(\mathbb{D})$. Then, by the domain monotonicity of the Green function,

$$g_{S^\alpha f(\mathbb{D})}(w,0) \le g_G(w,0). \tag{3.4}$$

From (2.1), we get

$$\sum_j g_{\mathbb{D}}(z_j(w),0) = -\log \prod_j |z_j(w)|. \tag{3.5}$$

By (2.5) and (2.7),

$$g_G(w,0) = \Phi^{-1}(\operatorname{cap}(G,[0,w])). \tag{3.6}$$

Let $E := d_e(f(\mathbb{D}))$, $\varepsilon := d_e([0,w_0])$. Since circular symmetrization reduces the elliptic capacity [2, Theorem 5],

$$d_e(G) = d_e(S^\alpha f(\mathbb{D})) \le E.$$

It follows from Lemma 1 and (2.4) that

$$\operatorname{cap}(G,[0,w]) \ge \operatorname{cap}(E\mathbb{D},\overline{\varepsilon\mathbb{D}}) = \frac{2\pi}{\log(E/\varepsilon)}. \tag{3.7}$$

Combining (3.2) up to (3.7) and the fact that the function Φ is decreasing, we get

$$\Phi\Big(-\log \prod_j |z_j(w)|\Big) \ge \frac{2\pi}{\log(E/\varepsilon)}. \tag{3.8}$$

By (2.6),

$$\Phi\Big(-\log \prod_j |z_j(w)|\Big) = \frac{2\pi}{\mu\Big(\prod_j |z_j|\Big)}. \tag{3.9}$$

By [1, pp. 167, 87]

$$\varepsilon = \exp\Big(-\mu\circ\varphi^{-1}(|w|)\Big). \tag{3.10}$$

Finally from relations (3.8)–(3.10),

$$\mu\Big(\prod_j |z_j(w)|\Big) \le \log E + \mu\circ\varphi^{-1}(|w|),$$

which implies (3.1).

Suppose now that for some $w_0 \in f(\mathbb{D})\setminus\{0\}$ we have equality in (3.1). Then we also have equalities in (3.2), (3.3), (3.4), and (3.7). The equalities in (3.3) and (3.4) imply that $f(\mathbb{D}) \overset{n.e.}{=} S^\alpha f(\mathbb{D})$ and $S^\alpha f(\mathbb{D}) \overset{n.e.}{=} G$. Hence $f(\mathbb{D}) \overset{n.e.}{=} G$, and this implies that necessarily $f(\mathbb{D}) \subset G$; indeed let $z_0 \in f(\mathbb{D})\setminus G$. If $z_0 \notin -\alpha$, where $-\alpha$ is the opposite ray of α, then, as G is circularly symmetric with respect to the ray α, we can find an arc I on the circle $C_{|z_0|}$ with one end point at z_0 such that $I \subset f(\mathbb{D})\setminus G$. As $d(I) > 0$ (where $d(I)$ is the logarithmic capacity of I), this contradicts $G \overset{n.e}{=} f(\mathbb{D})$. If now $z_0 \in -\alpha$, then, using the fact that G is simply connected and circularly symmetric, we can find a segment J on $-\alpha$ with one end point at z_0 such that $J \subset f(\mathbb{D})\setminus G$. But $d(J) > 0$ and this contradicts $G \overset{n.e}{=} f(\mathbb{D})$.

Equality in (3.7) implies by Lemma 1 that the boundary of G is a level curve of the potential function of the condenser

$$(\mathbb{C}_\infty \setminus [0, w_0]^*, [0, w_0]).$$

So the boundary of G is one of the curves $\Gamma(r, w_0)$ with $d_e([0, w_0]) < r \leq 1$. We consider now a conformal mapping h from the unit disc onto G with $h(0) = 0$. As we have the equality in (3.2) and $G \overset{n.e}{=} f(\mathbb{D})$, we have that

$$\sum_j g_{\mathbb{D}}(z_j(w_0), 0) = g_{f(\mathbb{D})}(w_0, 0) = g_G(w_0, 0) = g_{\mathbb{D}}(h^{-1}(w_0), 0). \tag{3.11}$$

And then by also using (2.1) and (3.5), we get

$$|h^{-1}(w_0)| = \prod_j |z_j(w_0)|. \tag{3.12}$$

But now we have equality in Littlewood's theorem for the function

$$k = h^{-1} \circ f : \mathbb{D} \to \mathbb{D},\ k(0) = 0.$$

Then (see [20]) k is an inner function and $f = h \circ k$. So $f(\mathbb{D})$ coincides with G.

Conversely, suppose that $f = h \circ k$, where k is an inner function with $k(0) = 0$ and h is a conformal mapping from $\mathbb{D}$ onto the interior of $\Gamma(r, w_0)$ for some $w_0 \in \mathbb{C} \setminus \{0\}$ and some r with $d_e([0, w_0]) < r \leq 1$ and $h(0) = 0$. It is straightforward to show that the relation (3.1) holds with equality for $w = w_0$.

Let us now proceed to the general case where $f(0)$ is not necessarily 0. We consider the Mobius transformation

$$T(z) = \frac{z - f(0)}{1 + z\overline{f(0)}},\ z \in f(\mathbb{D}).$$

It is easy to observe that $[T(z_1), T(z_2)]_e = [z_1, z_2]_e,\ \forall z_1, z_2 \in \mathbb{C}$. Hence the function $T \circ f : \mathbb{D} \to \mathbb{C}$ is holomorphic, elliptically schlicht, with $T \circ f(0) = 0$ and $d_e(T \circ f(\mathbb{D})) = d_e(f(\mathbb{D}))$. We apply our previous results to this function, and we get immediately the bound (1.8) for the function f, as well as the statement for the equality case. □

4. Proof of Theorem 2

Proof. (a) Let $f : \mathbb{D} \to \mathbb{C}$ be a holomorphic, elliptically schlicht function and $r \in (0, 1)$. Then

$$\begin{aligned} \varphi_R(r) = \frac{R_f(r)}{r} &= \frac{\sup_{z \in r\mathbb{D}} [f(z), f(0)]_e}{r} \\ &= \sup_{z \in r\mathbb{D}} \left| \frac{f(z) - f(0)}{(1 + f(z)\overline{f(0)})z} \right|. \end{aligned} \tag{4.1}$$

We set the function

$$g(z) = \frac{f(z) - f(0)}{(1 + f(z)\overline{f(0)})z}$$

which is holomorphic in $\mathbb{D}$, because f is holomorphic and elliptically schlicht.

By the maximum principle the function φ_R is strictly increasing unless g is a constant $c \in \mathbb{C}$. In this case, we conclude that the function f has the form

$$f(z) = \frac{cz + f(0)}{1 - cz\overline{f(0)}}, \ z \in \mathbb{D}, \tag{4.2}$$

where $|c\overline{f(0)}| \leq 1$ because f is holomorphic, and $|c| \leq 1$ because f is elliptically schlicht.

Moreover, from (4.1) one easily gets that

$$\lim_{r \to 0} \varphi_R(r) = |g(0)| = \frac{|f'(0)|}{1 + |f(0)|^2}.$$

(b) Now let $R := \sup_{z \in \mathbb{D}} [f(z), f(0)]_e$. From (a) we have that

$$\frac{|f'(0)|}{1 + |f(0)|^2} \leq \lim_{\rho \to 1} \varphi_R(\rho) = R. \tag{4.3}$$

Also, for every $z \in \mathbb{D}$

$$[f(z), f(0)]_e \leq \varphi_R(|z|)|z| \leq R|z|. \tag{4.4}$$

The case of equality in any of (4.3), (4.4) holds only when φ_R is not strictly increasing and so when f is of the form (4.2). $\square$

5. Final remarks

In [4] it was proved that if $f : \mathbb{D} \to \mathbb{C}$ is an elliptically schlicht, holomorphic function, then the function

$$\varphi_e(r) = \frac{d_e(f(r\mathbb{D}))}{r}, \quad 0 < r < 1,$$

is increasing.

In view of the elliptic radius and elliptic capacity monotonicity theorems for the elliptically schlicht functions it is natural to wonder whether a similar monotonicity behaviour holds in the context of the elliptic diameter this time (as in [6] and [7]). We present this quantity: Let $f : \mathbb{D} \to \mathbb{C}$ be an elliptically schlicht function and $r \in (0, 1)$. The elliptic diameter of $f(r\mathbb{D})$ is denoted as follows:

$$D_f(r) = \sup_{z, w \in r\mathbb{D}} [f(z), f(w)]_e.$$

We consider also the elliptic diameter of the disc $r\mathbb{D}$,

$$D(r) = \sup_{z, w \in r\mathbb{D}} [z, w]_e.$$

Direct computations and estimates give us

$$D(r) = \frac{2r}{1 - r^2}. \tag{5.1}$$

We define the function

$$\varphi_D(r) = \frac{D_f(r)}{D(r)}, \quad r \in (0, 1).$$

It is reasonable to expect that φ_D will have monotonicity properties similar to those of φ_R and φ_e; but this doesn't hold. As the next examples show, φ_D can behave in many different ways.

Example 1. Let us consider again the class of functions of the form

$$f(z) = \frac{cz + a}{1 - c\overline{a}z}, \; z \in \mathbb{D}, \; a \in \mathbb{C}, \; \text{and } c, c\overline{a} \in \overline{\mathbb{D}}.$$

If $z, w \in \mathbb{D}$, after calculations we get

$$[f(z), f(w)]_e = \left| \frac{c(z - w)}{1 + |c|^2 z\overline{w}} \right|, \tag{5.2}$$

and thus $D_f(r) = \dfrac{2|c|r}{1 - |c|^2 r^2}$ for every $r \in (0, 1)$. Hence, $\varphi_D(r) = \dfrac{|c|(1 - r^2)}{1 - |c|^2 r^2}$.

There are two cases for us to consider:

(i) If $|c| < 1$, the function $\varphi_D(r)$ is decreasing for $r \in (0, 1)$.

(ii) If $|c| = 1$, $\varphi_D(r)$ is constant and equal to 1.

Example 2. Let $f(z) = z^2$, $z \in \mathbb{D}$. By (5.1) $\varphi_D(r) = \dfrac{r}{1 + r^2}$ and therefore is increasing.

Example 3. Let $f(z) = \dfrac{z^2}{2}$, $z \in \mathbb{D}$. One finds that $D_f(r) = \dfrac{4r^2}{4 - r^4}$, and thus

$$\varphi_D(r) = \frac{2r(1 - r^2)}{4 - r^4}, \;\; 0 < r < 1.$$

We obtain that φ_D does not preserve one kind of monotonicity. Here φ_D increases until $r = \sqrt{1 + \sqrt[3]{3} - \sqrt[3]{3}^2}$ and then decreases.

Before we close the article we compute the functions φ_R and φ_e for Examples 1 and 2 (the third example is similar to the second and we omit it). We will need the following identity (see for example [12]):

$$d_e(r\mathbb{D}) = r, \;\; \text{for every } r \in (0, 1). \tag{5.3}$$

Example 1. By (5.2) we have $[f(z), f(0)]_e = |c||z|$ for every $z \in \mathbb{D}$ and so

$$\varphi_R(r) = |c|, \;\; \text{for every } r \in (0, 1).$$

Relation (5.2) leads us to $[f(z), f(w)]_e = [cz, cw]_e$ for every $z, w \in \mathbb{D}$ and then

$$d_{e,n}(f(r\mathbb{D})) = d_{e,n}\big(|c|r\mathbb{D}\big), \;\; \text{for every } n \in \mathbb{N},$$

which combined with (5.3) gives that $\varphi_e(r) = |c|$, for every $r \in (0, 1)$.

Example 2. This time we have that $f(r\mathbb{D}) = r^2\mathbb{D}$, for every $r \in (0, 1)$. We observe that

$$\varphi_R(r) = \varphi_e(r) = r, \;\; \text{for every } r \in (0, 1).$$

Acknowledgements

Some of the results of the paper are part of the first author's Ph.D. thesis which was conducted under the supervision of D. Betsakos. The first-named author thanks the Cyprus State Scholarship Foundation for its support of her Ph.D. studies. The content of the paper was carried out while the second-named author was a postdoctoral associate of the Department of Mathematical Sciences of Aalborg University, Denmark. The authors would like to thank the anonymous referee for the constructive suggestions and especially for pointing their attention to reference [17].

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