THE HEISENBERG GROUP ACTS ON A
STRICTLY CONVEX DOMAIN

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Abstract. This paper gives the first example of a unipotent group that is not virtually abelian and preserves a strictly convex domain.

Every linear group acts by isometries on some properly convex domain in real projective space. This follows from the fact that action of $\text{SL}(n, \mathbb{R})$ on the space of quadratic forms in $n$ variables preserves the projectivization, $\text{Pos}(n)$, of the properly convex cone consisting of positive definite forms. If $\Gamma$ is the holonomy of a properly convex orbifold of finite volume, then every virtually nilpotent subgroup of $\Gamma$ is virtually abelian; moreover, every unipotent subgroup of $\Gamma$ is conjugate into $\text{PO}(n, 1)$. A reference for all of this is [1]. This paper gives the first example of a unipotent group that is not virtually abelian and preserves a strictly convex domain. It answers a question asked by Misha Kapovich.

The Heisenberg group is the subgroup $H \subset \text{SL}(3, \mathbb{R})$ of unipotent upper-triangular matrices. Define $\theta : H \rightarrow \text{SL}(10, \mathbb{R})$ and $G = \theta(H)$ where

$$
\begin{pmatrix}
1 & a & c \\
0 & 1 & b \\
0 & 0 & 1
\end{pmatrix}
= 
\begin{pmatrix}
1 & 2a & 2c & a^2/2 & a^3/6 & b & 2a^2 + b^2/2 & b^3/6 + 2ac & (a^4 + b^4)/24 + c^2 \\
0 & 1 & b & 0 & 0 & 0 & 0 & 2a & ab + c & bc \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & a & c & d \\
0 & 0 & 0 & 1 & a & a^2/2 & 0 & 0 & 0 & a^3/6 \\
0 & 0 & 0 & 0 & 1 & a & 0 & 0 & 0 & a^2/2 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & a \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & b & b^2/2 & b^3/6 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & b & b^2/2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & b \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
$$

It is clear that $\theta$ is injective, and it is easy to check that it is a homomorphism.
Theorem 0.1. There is a strictly convex domain $\Omega \subset \mathbb{RP}^9$ that is preserved by $G$. This is an effective action of the Heisenberg group on $\Omega$ by parabolic isometries that are unipotent.

Proof. The group $G$ acts affinely on the affine patch $[x_1 : x_2 : x_3 : x_4 : x_5 : x_6 : x_7 : x_8 : x_9 : x_{10} = 1]$ that we identify with $\mathbb{R}^9$. Let $p \in \mathbb{R}^9$ be the origin. Then $G \cdot p$ is

$$(a^4 + b^4)/24 + c^2, bc, c, a^3/6, a^2/2, a, b^3/6, b^2/2, b)$$

This orbit is an algebraic embedding $\mathbb{R}^3 \to \mathbb{R}^9$ which limits on the single point

$$q = [1 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0] \in \mathbb{RP}^9$$

in the hyperplane at infinity, $P_\infty$, given by $x_{10} = 0$. This follows from the fact that $(a^4 + b^4)/24 + c^2$ dominates all the other entries whenever at least one of $|a|, |b|, |c|$ is large.

Let $S \subset \mathbb{R}^9$ be this orbit. Choose 10 random points on $S \subset \mathbb{RP}^9$ and compute the determinant, $d$, of the corresponding 10 vectors in $\mathbb{R}^{10}$. Then $d \neq 0$ therefore the interior $\Omega^+ \subset \mathbb{R}^9$ of the convex hull of $S$ has dimension 9.

Moreover, the closure, $\Omega'$, of $\Omega^+$ in $\mathbb{RP}^9$ is disjoint from the closure of the affine hyperplane $x_1 = -1$ in $\mathbb{R}^9$, hence $\Omega^+$ is properly convex. Since $\Omega' \cap \text{cl}(P_\infty) = q$, and $G$ preserves $q$ and $P_\infty$, and $G$ is unipotent, it follows from (5.8) in [1] that $G$ preserves some strictly convex domain $\Omega \subset \Omega'$. Moreover, $P_\infty$ is a supporting hyperplane to $\Omega$ at $q$.

We sketch an argument that the extrinsic curvature of $\partial \Omega \setminus q$ is positive. A properly convex domain $\Omega_0 \subset \mathbb{RP}^n$ is the projectivization of a cone $\mathcal{C} \subset \mathbb{R}^{n+1}$. There is a Vinberg characteristic function $\phi : \mathcal{C} \to \mathbb{R}$ and the level sets have extrinsic positive curvature. Applying this to $\Omega_0 = \Omega^+$ in the construction of $\Omega$ in (5.8) of [1] implies the positive curvature of $\partial \Omega \setminus q$.

Let $K$ be the 1-parameter subgroup of $G$ given by $a = c = 0$. Assume, for simplicity only, that $p \in \text{cl}(\Omega)$. Then $K \cdot p$ is the curve $[b^4/24 : 0 : 0 : 0 : 0 : b^3/6 : b^2/2 : b : 1]$. Omitting the 0-coordinates, and rescaling the remainder to remove the constants, this is $\gamma(t) = [1 : t : t^2 : t^3 : t^4]$ where $t = b^{-1}$. It follows that the height of $\gamma(t)$ above $P_\infty$ is $t^4$, so $\gamma(t)$ has third-order contact with $P_\infty$ at $q = \gamma(0)$, and therefore $\partial \Omega$ does not have strictly positive extrinsic curvature at $q$.

Replacing $p$ by a generic point gives a similar result.

Define $\phi(x_1, \cdots, x_9) = x_5$ then $\phi(G \cdot p) = a^2/2$ hence $\Omega^+$ is disjoint from the affine hyperplane $P' = \phi^{-1}(-1) \subset \mathbb{R}^9$. Moreover, $q$ is in the closure of $P'$ in $\mathbb{RP}^9$. Since $\Omega \subset \Omega^+$, it follows that the closures in $\mathbb{RP}^9$ of $P_\infty$ and $P'$ are distinct supporting projective hyperplanes to $\Omega$ at $q$. Thus $q$ is not a $C^1$ point of $\partial \Omega$.

Corollary 0.2. There is a strictly convex real projective manifold $\Omega/\Gamma$ of dimension 9 with nilpotent fundamental group $\Gamma \cong \langle \alpha, \beta : \{\alpha, [\alpha, \beta], [\beta, [\alpha, \beta]]\} \rangle$ that is not virtually abelian. Moreover, $\Gamma$ is unipotent.

Proof. If $\Gamma$ is a lattice in $G$, then $\Omega/\Gamma$ is a strictly convex manifold with unipotent holonomy and $\Gamma$ is nilpotent but not virtually abelian. \qed
The genesis of this example is as follows. The image of $H$ in $\text{SL}(6, \mathbb{R})$ under the irreducible representation $\text{SL}(3, \mathbb{R}) \to \text{SL}(6, \mathbb{R})$ is

$$
\begin{pmatrix}
1 & 2a & a^2 & 2c & 2ac & c^2 \\
0 & 1 & a & b & ab + c & bc \\
0 & 0 & 1 & 0 & 2b & b^2 \\
0 & 0 & 0 & 1 & a & c \\
0 & 0 & 0 & 0 & 1 & b \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
$$

and preserves $Q := \text{Pos}(3) \subset \mathbb{RP}^5$. The boundary of the closure of $Q$ consists of semidefinite forms and contains flats, so $Q$ is not strictly convex. Let $A, B, C \in \text{SL}(6, \mathbb{R})$ be the elements corresponding to one of $a, b, c$ being 1 and the others 0. Each of $A, B, C$ has a parabolic fixed point in $\partial Q$ corresponding to a rank 1 quadratic form. Every point in $Q$ converges to this parabolic fixed point under iteration by the given group element. The fixed points for $A$ and $B$ are distinct and lie in a flat in $\partial Q$.

The idea is to increase the dimension of the representation and use the extra dimensions to add parabolic blocks of size 5 onto $A$ (row 1 and rows 7-10) and onto $B$ (row 1 and rows 11-14) that commute, and the parabolic fixed point of each block is the rank-1 form that is a fixed point of $C$. This gives a 14-dimensional representation of $H$:

$$
\begin{pmatrix}
1 & 2a & a^2 & 2c & 2ac & c^2 & a & a^2/2 & a^3/6 & a^4/24 & b & b^2/2 & b^3/6 & b^4/24 \\
0 & 1 & a & b & ab + c & bc & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 2b & b^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & a & c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & a & a^2/2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & a & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & b & b^2/2 & b^3/6 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & b \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
$$

The top-left $6 \times 6$ block is the image of $H$ in $\text{SL}(6, \mathbb{R})$. The entries in $A^n$ and $B^n$ grow like $n^2$. This is less than the growth of some entries in the added blocks of size 5 which grow like $n^4$. The orbit of

$$
[0 : 0 : 0 : 0 : 1 : 0 : 0 : 1 : 0 : 0 : 0 : 1]
$$

is

$$
$$

so there is a codimension-4 projective hyperplane that is preserved, and which is defined by

$$
x_6 = x_{10} = x_{14} \quad x_5 = x_{13} \quad x_3 = 2x_{12}
$$

The restriction to this hyperplane gives $\theta$. 

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