THE REALIZATION PROBLEM FOR JØRGENSEN NUMBERS

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ABSTRACT. Let G be a two-generator subgroup of $\mathrm{PSL}(2, \mathbb{C})$. The Jørgensen number $J(G)$ of G is defined by

$$J(G) = \inf\{|\operatorname{tr}^2 A - 4| + |\operatorname{tr}[A, B] - 2| \; ; \; G = \langle A, B \rangle\}.$$

If G is a non-elementary Kleinian group, then $J(G) \geq 1$. This inequality is called Jørgensen's inequality. In this paper, we show that, for any $r \geq 1$, there exists a non-elementary Kleinian group whose Jørgensen number is equal to r. This answers a question posed by Oichi and Sato. We also present our computer generated picture which estimates Jørgensen numbers from above in the diagonal slice of Schottky space.

1. INTRODUCTION

Let G be a two-generator subgroup of $\mathrm{PSL}(2, \mathbb{C})$. Determining whether or not G is discrete is an important problem in Kleinian group theory. In 1976, Jørgensen [7] showed that if $G = \langle X, Y \rangle$ is a non-elementary two-generator discrete subgroup of $\mathrm{PSL}(2, \mathbb{C})$ (Kleinian group), then

$$(1) \qquad\qquad |\operatorname{tr}^2 X - 4| + |\operatorname{tr}[X, Y] - 2| \geq 1,$$

where $[X, Y] = XYX^{-1}Y^{-1}$. The Jørgensen number of an ordered pair (X, Y) is defined as

$$J(X, Y) := |\operatorname{tr}^2 X - 4| + |\operatorname{tr}[X, Y] - 2|$$

and the Jørgensen number of a non-elementary two-generator Kleinian group G is defined as

$$J(G) := \inf\{J(X, Y) \mid \langle X, Y \rangle = G\}.$$

By (1), we have $J(G) \geq 1$. We can think of the Jørgensen number as measuring how far from being indiscrete. Jørgensen's inequality is sharp [7], and if $J(G) = 1$, G is called a Jørgensen group. There are many results on Jørgensen groups in the literature [5, 8, 12–14, 17, 20, 21, 25]. Also, it seems to be a difficult, but also challenging, problem to calculate Jørgensen numbers explicitly [3, 6, 22]. Oichi and Sato [18] asked the following natural question.

Question 1.1 (The realization problem). Let r be a real number with $r \geq 1$. When is there a non-elementary Kleinian group whose Jørgensen number is equal to r?

Oichi and Sato [18] claimed that if $r = 1, 2, 3$ or $r \geq 4$, then there is a non-elementary Kleinian group whose Jørgensen number is equal to r. (See also [3, 25].) Though Jørgensen's inequality is sharp, constructing a Kleinian group with small Jørgensen number (in particular, less than 4) is not easy. To the best of our knowledge, there is no known example of a classical Schottky group whose Jørgensen number is less than 4. In this paper, we solve this realization problem.

Theorem 1.2. *For any real number $r \geq 1$, there is a non-elementary two-generator discrete subgroup of* $\mathrm{PSL}(2, \mathbb{C})$ *whose Jørgensen number is* r.

Jørgensen numbers can be studied by computer experiments. We will present a computer generated picture of the estimates of Jørgensen numbers from above in the diagonal slice of Schottky space. At the end of the paper, we present some Jørgensen groups, and we conjecture that they are counterexamples of Li-Oichi-Sato's conjecture.

In section 2, we give a proof of Theorem 1.2 for the case $4 \leq r$. In section 3, we give a proof for the case $1 \leq r \leq 4$. In section 4, we present our computer generated picture of a one-parameter family of two-generator Kleinian groups, and describe our numerical calculation of the Jørgensen numbers of the groups in this family.

2. The proof for the case $r \geq 4$

Oichi and Sato [18] claimed that, for every real number $r \geq 4$, there is a subgroup G of $\mathrm{PSL}(2, \mathbb{C})$ such that $J(G) = r$. But the proof was not written. For completeness, we give a short proof of this fact by calculating the Jørgensen numbers of the groups in the Riley slice.

First, we recall a classical result of Nielsen. A pair $\{X, Y\}$ of elements in the free group F_2 of rank two are called associated primitives if X and Y generate the group. For any associated primitives $\{A, B\}$ and $\{C, D\}$ of F_2, the commutator $[C, D]$ is conjugate to $[A, B]$ or $[A, B]^{-1}$. (See Theorem 3.9 in [15].) It follows that

Lemma 2.1. *If G is a subgroup of* $\mathrm{PSL}(2, \mathbb{C})$ *isomorphic to F_2, then the trace of the commutator of associated primitives of G does not depend on the choice of associated primitives.*

Let ρ be a complex number, and define $G_\rho = \langle A, B_\rho \rangle$, where

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad B_\rho = \begin{pmatrix} 1 & 0 \\ \rho & 1 \end{pmatrix}.$$

Recall that the Riley slice is defined by

$$\{\rho \in \mathbb{C} \mid G_\rho \text{ is free and Kleinian and } \Omega(G_\rho)/G_\rho \text{ is a 4-times punctured sphere}\},$$

where $\Omega(G_\rho)$ is the regular set of G_ρ. (See [10, 11] for the Riley slice.) If ρ lies in the closure of the Riley slice, then G_ρ is non-elementary, discrete, and free of rank two. Suppose that G_ρ is isomorphic to F_2 and $\{A'_\rho, B'_\rho\}$ are any associated primitives for G_ρ. By Lemma 2.1, we have

$$J(A'_\rho, B'_\rho) = |\operatorname{tr}^2 A'_\rho - 4| + |\operatorname{tr}[A'_\rho, B'_\rho] - 2| \geq |\operatorname{tr}[A, B_\rho] - 2| = J(A, B_\rho)$$

and $J(G_\rho) = J(A, B_\rho) = |\rho^2|$. Hence, the real numbers

$$\{|\rho^2| \in \mathbb{R} \mid \rho \text{ is in the closure of the Riley slice}\}$$

are realized as Jørgensen numbers. In particular, since the closure of the Riley slice contains $\{yi \mid y \geq 2\}$, for any real $r \geq 4$, the Jørgensen number is realized.
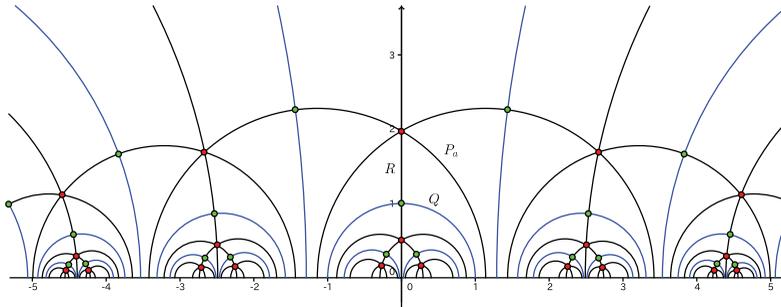
FIGURE 1. The axes of the generators and their orbits.

Remark 2.2. It is possible to improve the value 4 to around 2.467 by calculating the endpoints of the pleating rays of the Riley slice. See [26].

As an example of Kleinian groups that we can calculate the Jørgensen numbers by the same method, we consider the Maskit slice. Recall that, if G is in the *Maskit slice* [9, 16], then it is discrete and isomorphic to F_2 and can be normalized as $G_\mu = \langle A, B_\mu \rangle$ for some complex number μ, where

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \; B_\mu = \begin{pmatrix} -i\mu & -i \\ -i & 0 \end{pmatrix}.$$

Since $|\operatorname{tr}^2 A - 4| = 0$, by the same argument for the Riley slice, we have $J(G_\mu) = J(A, B_\mu) = 4$.

3. THE PROOF FOR THE CASE $1 \leq r \leq 4$

In this section, we give a proof for the case $1 \leq r \leq 4$. For simplicity, we use the same notation for a matrix in $\mathrm{SL}(2, \mathbb{C})$ and its equivalence class in $\mathrm{PSL}(2, \mathbb{C})$.

3.1. **The basic configuration.** Let $z, z' \in \mathbb{C}$, and let $[z, z']$ be the oriented line from z to z' in $\mathbb{H}^3$. Following [4], we define the *line matrix* associated with $[z, z']$ as

$$M([z, z']) := \frac{i}{z' - z} \begin{pmatrix} z + z' & -2zz' \\ 2 & -z - z' \end{pmatrix}.$$

See [4], p. 64, equation (1). It is the order two rotation about $[z, z']$.

For $a \geq 1$, set

$$P_a := M([a, -3a]), \quad Q := M([1, -1]), \quad R := M([0, \infty]) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

Let G_a be a subgroup of $\mathrm{PSL}(2, \mathbb{C})$ generated by P_a, Q, and R. The axes of the generators are contained in the vertical plane $\{(x, 0, z) | z > 0\} \cong \mathbb{H}^2$. Since we assumed that $a \geq 1$, $[a, -3a]$ and $[1, -1]$ do not intersect in $\mathbb{H}^3$. The angle between $[1, -1]$ and $[0, \infty]$ is $\pi/2$, and the angle between $[0, \infty]$ and $[a, -3a]$ is $\pi/3$. See Figure 1. Hence, as an abstract group, G_a has a presentation

$$\langle P_a, Q, R \mid P_a^2 = Q^2 = R^2 = (QR)^2 = (RP_a)^3 = id \rangle$$

for any $a \geq 1$ and is isomorphic to a hyperbolic (full) triangle group in $\mathbb{H}^2$. (Note that hyperbolic (full) triangle groups are defined as groups generated by reflections, but G_a is generated by rotations in $\mathbb{H}^3$.) Hence, G_a is non-elementary and discrete.

Remark 3.1. Groups conjugate to $\langle P_a, Q, R \rangle$ were studied by Series, Tan, and the first author in [23].

For later purpose, we classify non-trivial elements of G_a into three types:

  (i) elements conjugate to P_a, Q, R, QR, or RP_a,
  (ii) elements conjugate to $(P_a Q)^n$ for some $n \in \mathbb{Z} \setminus \{0\}$,
 (iii) others.

First, we consider the case $a = 1$.

Lemma 3.2. *Let E_1 be a non-trivial element in G_1. Then*

  (i) *E_1 is of type* (i) *if and only if* $\operatorname{tr} E_1 \in \{0, \pm 1\}$.
  (ii) *E_1 is of type* (ii) *if and only if* $\operatorname{tr} E_1 = \pm 2$.
 (iii) *E_1 is of type* (iii) *if and only if* $\operatorname{tr} E_1 \in \{\pm 3, \pm 4, \ldots\} \cup \{\pm i, \pm 2i, \ldots\}$.

Proof. By construction, if E_1 is of type (i) (resp., (ii)), then $\operatorname{tr} E_1 \in \{0, \pm 1\}$ (resp., $\operatorname{tr} E_1 = \pm 2$).

For the converse, set

$$M = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \sqrt{2} \end{pmatrix}.$$

Then, we have

$$MP_1M^{-1} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, MQM^{-1} = \begin{pmatrix} -i & 0 \\ -2i & i \end{pmatrix}, MRM^{-1} = \begin{pmatrix} i & -i \\ 0 & -i \end{pmatrix}.$$

Hence, for any element $E_1 \in G_1$, we have

$$(2) \qquad \operatorname{tr} E_1 \in \{0, \pm 1, \pm 2, \pm 3, \ldots\} \cup \{\pm i, \pm 2i, \pm 3i, \ldots\}.$$

Note that G_1 acts on $\{(x, 0, z) | z > 0\} \cong \mathbb{H}^2$ as the hyperbolic (full) triangle group of type $(2, 3, \infty)$ (extended modular group). Let F_1 be the (ideal) triangle of type $(2, 3, \infty)$ bounded by the axes of P_1, Q, and R. Then, F_1 is a fundamental region for the action of G_1. For each point in the interior of F_1, the stabilizer is trivial. For each point in the interior of the edges, the stabilizer is $\langle P_1 \rangle$, $\langle Q \rangle$, or $\langle R \rangle$. For the (non-ideal) vertices, the stabilizers are $\langle Q, R \rangle$ and $\langle R, P_1 \rangle$. For the ideal vertex, the stabilizer is $\langle P_1, Q \rangle$.

If $\operatorname{tr} E_1 \in \{0, \pm 1\}$, then E_1 is elliptic and has a fixed point in $\mathbb{H}^2$, and this point has non-trivial stabilizer. The observation in the previous paragraph shows that E_1 is of type (i). If $\operatorname{tr} E_1 = \pm 2$, then E_1 is parabolic and has a parabolic fixed point on $\partial \mathbb{H}^2$. The observation in the previous paragraph shows that E_1 is of type (ii).

Since we have shown the equivalences in (i) and (ii), the equivalence in (iii) follows from (2). □

Remark 3.3. By this proof, we see that G_1 is a conjugate to a subgroup of the Picard group. Sato [22] showed that the Picard group is a Jorgensen group. González-Acuña and Ramírez [6] described all Jørgensen subgroups of the Picard group.

Since the abstract group structure will not change for any $a \geq 1$, essentially the same type of arguments as in the proof of the previous lemma shows the following.

Lemma 3.4. *Let E_a be a non-trivial element in G_a for $a \geq 1$. Then E_a is of type* (i) *if and only if* $\operatorname{tr} E_a \in \{0, \pm 1\}$.

Next, we consider type (iii) for $a \geq 1$. First, we prove a technical lemma. Let a' be a number with $1 \leq a' < a$. Let F_a (resp., $F_{a'}$) be the region bounded by the axes of P_a (resp., $P_{a'}$), Q, and R. We define a map f from $(\partial F_a) \cap \mathbb{H}^2$ to $(\partial F_{a'}) \cap \mathbb{H}^2$ as follows:

$$f(p) = \begin{cases} \frac{a'}{a} \cdot p & (\text{if } |p| > \frac{a}{a'}), \\ \frac{p}{|p|} & (\text{if } |p| \leq \frac{a}{a'}). \end{cases}$$

We denote the hyperbolic distance in $\mathbb{H}^2$ by d.

Lemma 3.5. *For any $p, p' \in (\partial F_a) \cap \mathbb{H}^2$, we have $d(f(p), f(p')) \leq d(p, p')$.*

Proof. Set $q = f(p), q' = f(p')$. We denote the axes of P_a, Q, and R by $A(P_a), A(Q)$, and $A(R)$, respectively.

Suppose that $p \in A(P_a)$ and $p' \in A(R)$. We have $|p| > a > \frac{a}{a'}$. If $|p'| > \frac{a}{a'}$, then we have

$$d(f(p), f(p')) = d\left(\frac{a'}{a} \cdot p, \ \frac{a'}{a} \cdot p'\right) = d(p, p').$$

Thus, it suffices to consider the case $|p'| \leq \frac{a}{a'}$. To compare $d(p, p')$ and $d(q, q')$, we consider two hyperbolic triangles given by p, p', p'' and q, q', q'', where p'' (resp., q'') is the intersection of the axes of P_a (resp., $P_{a'}$) and R. That is,

$$p'' = \sqrt{3}a \cdot i, \quad q'' = \sqrt{3}a' \cdot i \quad (i = \sqrt{-1}).$$

By the hyperbolic low of cosines,

$$\cosh d(p, p') = g(d(p, p''), d(p', p'')), \quad \cosh d(q, q') = g(d(q, q''), d(q', q'')),$$

where

$$g(x, y) = \cosh x \cdot \cosh y - \sinh x \cdot \sinh y \cdot \frac{1}{2}.$$

(Recall that the angle between the axes of P_a (or $P_{a'}$) and R is $\pi/3$, so its cosine is $\frac{1}{2}$.) Since $d(q, q'') = d(p, p'')$ and $d(q', q'') < d(p', p'')$, if we can show that $g(x, y)$ is monotonically increasing in y for each fixed x, we have $d(q, q') < d(p, p')$. Since we have

$$\frac{d}{dy} g(x, y) = \cosh x \cdot \sinh y - \sinh x \cdot \cosh y \cdot \frac{1}{2}$$

$$> \cosh x \cdot \sinh y - \cosh x \cdot \cosh y \cdot \frac{1}{2}$$

$$= \cosh x \left(\sinh y - \cosh y \cdot \frac{1}{2}\right),$$

and $\cosh x$ is always positive and the second factor is positive if $y > \frac{\log 3}{2}$, then $g(x, y)$ is monotonically increasing in y if $y > \frac{\log 3}{2}$. Our conditions $|p'| < \frac{a}{a'}$ and $1 < a'$ imply that

$$d(p', p'') = \log \frac{|p''|}{|p'|} > \log \frac{\sqrt{3}a}{\frac{a}{a'}} > \frac{\log 3}{2},$$

$$d(q', q'') = \log \frac{|q''|}{|q'|} = \log \frac{\sqrt{3}a'}{1} > \frac{\log 3}{2}.$$

Hence, $d(q, q') < d(p, p')$, as required.
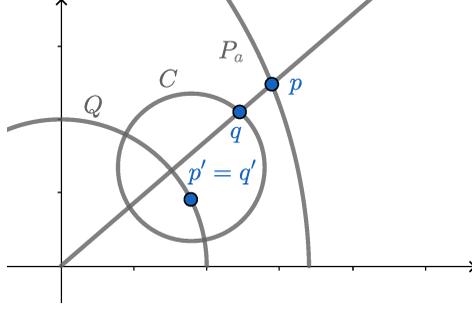
FIGURE 2. The case $p \in A(P_a)$ and $p' \in A(Q)$.

Next, suppose that $p \in A(P_a)$ and $p' \in A(Q)$. As in the previous case, $q = \frac{a'}{a} \cdot p$. Since $p' \in A(Q)$ is on the unit circle, $q' = f(p') = \frac{p'}{|p'|} = p'$, and q' is on $A(Q)$. Let C be the hyperbolic circle with center q' that passes through q. C is a Euclidean circle and orthogonal to $A(Q)$. See Figure 2. Hence, any Euclidean line passing through the origin meets C once inside $A(Q)$ and once outside $A(Q)$. Thus, we see that the point p is outside of C, and $d(q, q') < d(p, p')$.

Other cases $((p, p') \in A(Q) \times A(R), A(P_a) \times A(P_a), A(Q) \times A(Q), A(R) \times A(R))$ are easy and omitted. $\qquad\square$

Lemma 3.6. *Let E_a be an element of type* (iii) *in G_a. Then, we have*
$$\operatorname{tr} E_a \in \{\pm x \mid x \in \mathbb{R}, x \geq 3\} \cup \{\pm yi \mid y \in \mathbb{R}, y \geq 1\}.$$

Proof. First, we show that $|\operatorname{tr} E_a|$ is (weakly) increasing in a. Let a' be a number with $1 \leq a' \leq a$, and suppose that E_a and $E_{a'}$ are loxodromic. Since E_a preserves the vertical plane $\mathbb{H}^2$ in $\mathbb{H}^3$, E_a or E_a^{-1} is conjugate to
$$\begin{pmatrix} \sqrt{l_a} & 0 \\ 0 & 1/\sqrt{l_a} \end{pmatrix} \text{ or } \begin{pmatrix} \sqrt{l_a}i & 0 \\ 0 & 1/\sqrt{l_a}i \end{pmatrix},$$
where l_a is the translation length of E_a. Thus, we have
$$(3) \qquad \operatorname{tr} E_a = \pm(\sqrt{l_a} + 1/\sqrt{l_a}) \text{ or } \pm(\sqrt{l_a} - 1/\sqrt{l_a})i.$$
We claim that l_a is (weakly) increasing in a.

To see this, let $m_a \in \mathbb{H}^2$ be the axis of E_a. Since F_a is a fundamental region for G_a in $\mathbb{H}^2$, we can map m_a into F_a by the elements in G_a. The image is a closed trajectory of the hyperbolic billiard of billiard table F_a. Let $p_1, \ldots, p_n$ be the points on the trajectory where it hits the sides (i.e., the boundary) of the table F_a so that
$$l_a = \sum_{i=1}^{n} d(p_i, p_{i+1}).$$

Here, the indices are understood mod n. See Figure 3. Let f be the function defined before Lemma 3.5, and put $q_i = f(p_i)$ for $i = 1, \ldots, n$. By Lemma 3.5, $d(q_i, q_{i+1}) \leq d(p_i, p_{i+1})$ for $i = 1, \ldots, n$.

Let t be the closed piecewise geodesic given by connecting $q_1, q_2, \ldots, q_n, q_1$. Let $\widehat{t}$ be the "unfolding" of t starting from a segment in t connecting, say, q_1 and q_2. See Figure 4. $\widehat{t}$ is a bi-infinite piecewise geodesic in $\mathbb{H}^2$. Let $\ldots, \widehat{q_{-2}}, \widehat{q_{-1}}, \widehat{q_0}, \widehat{q_1}, \widehat{q_2}, \widehat{q_3}, \ldots$ be the vertices of $\widehat{t}$ with $\widehat{q_1} = q_1$ and $\widehat{q_2} = q_2$. The vertex $\widehat{q_i}$ in $\widehat{t}$ is mapped to
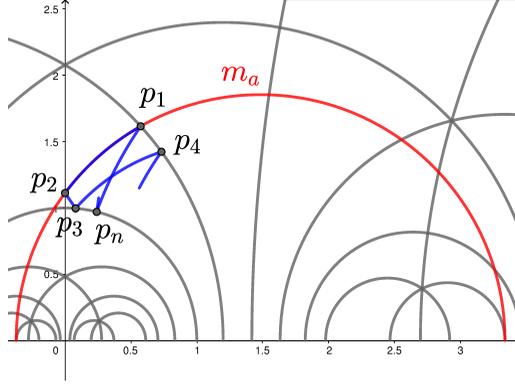
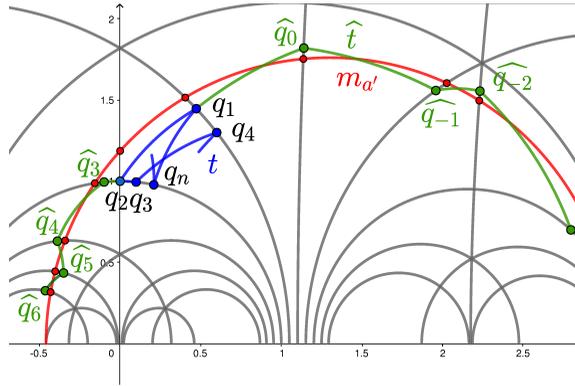FIGURE 3. The axis m_a of a type (iii) element and (a part of) the hyperbolic billiard.

FIGURE 4. The unfolding $\widehat{t}$ of t and the axis $m_{a'}$.

q_j in t with $i \equiv j \pmod{n}$ by the action of $G_{a'}$. In particular, we have either $E_{a'}(\widehat{q_1}) = \widehat{q_{n+1}}$ or $E_{a'}(\widehat{q_{n+1}}) = \widehat{q_1}$ depending on the choice of numbering. Thus, we see that

$$l_{a'} \leq d(\widehat{q_1}, \widehat{q_{n+1}}) \leq \sum_{i=1}^{n} d(\widehat{q_i}, \widehat{q_{i+1}}) = \sum_{i=1}^{n} d(q_i, q_{i+1}).$$

Thus, we have

$$l_{a'} \leq \sum_{i=1}^{n} d(q_i, q_{i+1}) \leq \sum_{i=1}^{n} d(p_i, p_{i+1}) = l_a,$$

as required. Hence, by formula (3), we have $|\operatorname{tr} E_{a'}| \leq |\operatorname{tr} E_a|$.

Since all the entries of $P_a, Q,$ and R are pure imaginary numbers, $\operatorname{tr} E_a$ is always real if the word length of E_a is even, and pure imaginary if it is odd. By Lemma 3.2(iii) and the arguments in the previous paragraphs, $\operatorname{tr} E_a$ cannot be in $\{\pm x \mid x \in \mathbb{R}, 2 < x < 3\} \cup \{\pm yi \mid y \in \mathbb{R}, 0 < y < 1\}$. (Recall that, at the beginning of the first paragraph of this proof, we assumed that E_a is loxodromic. For a', we can take $a' = 1$.) Since $\operatorname{tr} E_a$ is continuous on a, E_a cannot be in $\{\pm x \mid x \in \mathbb{R}, 0 \leq x \leq 2\}$, too. $\qquad\square$

3.2. **A generating pair.** We define

$$A_a := P_a Q = \begin{pmatrix} -3a/2 & 1/2 \\ -1/2 & -1/2a \end{pmatrix}, \quad B := R = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

Since $Q = A_a B A_a^{-1} B^{-1} A_a B$, (A_a, B) is a generating pair of G_a, and G_a is a two-generator group. As an abstract group, by Tietze transformation, we have

$$G_a = \langle P_a, Q, B, A_a \mid P_a^2 = Q^2 = B^2 = (QB)^2 = (BP_a)^3 = id, A_a = P_a Q \rangle$$

$$= \left\langle A_a, B \;\middle|\; \begin{matrix} (A_a B A_a^{-1} B A_a B A_a^{-1})^2 = (A_a B A_a^{-1} B^{-1} A_a B)^2 = B^2 \\ = (A_a B A_a^{-1} B^{-1} A_a B \cdot B)^2 = (B \cdot A_a B A_a^{-1} B A_a B A_a^{-1})^3 = id \end{matrix} \right\rangle$$

$$(4) \quad = \langle A_a, B \mid (A_a B A_a^{-1} B^{-1} A_a B)^2 = B^2 = (A_a B A_a^{-1} B^{-1} A_a)^2 = id \rangle.$$

Remark 3.7. A conjugate to this generating pair A_a and B was studied in [23] and called the *singular solid torus*.

Lemma 3.8. *In G_a, any element of order three cannot be a part of a minimal generating system.*

Proof. Let τ be the map from the free group $F_2 = \langle \alpha, \beta \rangle$ of rank two to $\{0,1\}^2$ that counts the exponents of α and β mod 2 in any word of F_2. Since

$$\tau((\alpha\beta\alpha^{-1}\beta^{-1}\alpha\beta)^2) = \tau(\beta^2) = \tau((\alpha\beta\alpha^{-1}\beta^{-1}\alpha)^2) = (0,0),$$

τ is well-defined on G_a. We denote the set $\{X \in G_a \mid \tau(X) = (i,j)\}$ by $G_{(i,j)}$. $G_{(i,j)}$ is not an empty set for $(i,j) = (0,0), (0,1), (1,0), (1,1)$.

Let X be an element of G such that $X^3 = id$. Then, $\tau(X) = (0,0)$. (Otherwise, we have $\tau(X^3) \equiv \tau(X^2) + \tau(X) \equiv \tau(X) \not\equiv (0,0) = \tau(id)$ (mod 2), which is a contradiction.) Hence, for any $Y \in G_a$, we have

$$\langle X, Y \rangle \subset G_{(0,0)} \cup G_{\tau(Y)} \neq G_{(0,0)} \cup G_{(1,0)} \cup G_{(0,1)} \cup G_{(1,1)} = G_a$$

and the lemma is proved. □

3.3. **The Jørgensen number.** Finally, we calculate the Jørgensen number.

Proposition 3.9. *For any $1 \leq a \leq a_0$, where $a_0 = (\sqrt{7}+2)/3$, we have*

$$J(G_a) = J(A_a, B) = \frac{(3a^2 - 1)^2}{4a^2}.$$

In particular, $J(G_1) = 1$ and $J(G_{a_0}) = 4$. For any $1 \leq r \leq 4$, the Jørgensen number is realized.

Proof. Since $\operatorname{tr}^2 A_a = (3a^2 + 1)^2/4a^2$ and $\operatorname{tr}[A_a, B] = 1$, for $a \geq 1$, we have

$$J(A_a, B) = \left| \frac{(3a^2 + 1)^2}{4a^2} - 4 \right| + 1 = \frac{(3a^2 - 1)^2}{4a^2}.$$

Since a_0 satisfies the equation $(3a_0^2 - 1)^2/4a_0^2 = 4$, if $1 \leq a \leq a_0$, we have $1 \leq J(A_a, B) \leq 4$. To prove this proposition by contradiction, suppose that there exist $C, D \in G_a$ such that $J(C, D) < J(A_a, B)$ and $G_a = \langle C, D \rangle$. Since G_a is non-elementary, $|\operatorname{tr}[C, D] - 2| > 0$. (See p. 68 in [1].)

Case 1: C is of type (i). If $\operatorname{tr} C = 0$, then

$$J(C, D) = |0^2 - 4| + |\operatorname{tr}[C, D] - 2| > 4 \geq J(A_a, B),$$

and this cannot happen. If $\operatorname{tr} C = \pm 1$, then C is an elliptic element of order three. Since C cannot be a part of a minimal generating system by Lemma 3.8, this cannot happen.

Case 2: C is of type (ii). C can be written as $K(P_a Q)^n K^{-1}$ for some $K \in G_a$ and $n \neq 0$.

Case 2-1: $|n| > 1$. We claim that $G_a \neq \langle C, D \rangle$. To see this, consider the Coxeter group

$$H_n := \langle P, Q, R \mid P^2 = Q^2 = R^2 = (PR)^3 = (QR)^2 = (PQ)^n = id \rangle$$

and let $\pi : G_a \to H_n$ be the natural surjection. Since $\pi(C) = \pi(K(P_a Q)^n K^{-1}) = id$, we have

$$\pi(\langle C, D \rangle) = \langle \pi(D) \rangle \neq H_n,$$

because H_n is not cyclic when $|n| > 1$. It follows that $G_a \neq \langle C, D \rangle$ and the claim is proved.

Case 2-2: $|n| = 1$. In this case, $\operatorname{tr}^2 C = \operatorname{tr}^2 A_a$.

Case 2-2-1: $[C, D]$ is of type (i). If $\operatorname{tr}[C, D] = 0$, then

$$J(C, D) = |\operatorname{tr}^2 A_a - 4| + |0 - 2| > |\operatorname{tr}^2 A_a - 4| + |1 - 2| = J(A_a, B).$$

If $\operatorname{tr}[C, D] = \pm 1$,

$$J(C, D) = |\operatorname{tr}^2 C - 4| + |\operatorname{tr}[C, D] - 2| = |\operatorname{tr}^2 A_a - 4| + |\pm 1 - 2| \geq J(A_a, B).$$

Case 2-2-2: $[C, D]$ is of type (ii). In this case, $\operatorname{tr}[C, D] = \pm \operatorname{tr} A_a^m$ for some $m \neq 0$. Since $\operatorname{tr} A_a = \pm(-3a/2 - 1/2a)$, A_a is hyperbolic for any $a > 1$ and parabolic when $a = 1$. The same is true for A_a^m. Let us denote C and D in G_a by C_a and D_a. Since $\operatorname{tr}[C_a, D_a]$ is continuous on a, we have either $\operatorname{tr}[C_a, D_a] \leq -2$ for any $a \geq 1$ or $2 \leq \operatorname{tr}[C_a, D_a]$ for any $a \geq 1$.

If $\operatorname{tr}[C_a, D_a] \leq -2$, then

$$J(C, D) \geq |\operatorname{tr}^2 A_a - 4| + |-2 - 2| \geq J(A_a, B).$$

If $2 \leq \operatorname{tr}[C_a, D_a]$, then $\lim_{b \to 1} \operatorname{tr}[C_b, D_b] = 2$, and $J(C_b, D_b)$ is close to 0 when b is close to 1, which contradicts Jørgensen's inequality, because G_b is non-elementary and discrete.
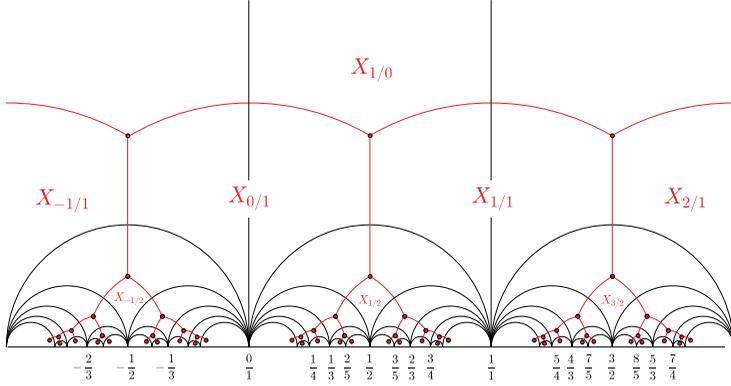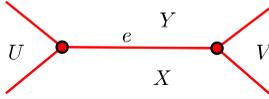
Case 2-2-3: $[C, D]$ is of type (iii). By Lemma 3.6, we have

$$J(C, D) = |\operatorname{tr}^2 C - 4| + |\operatorname{tr}[C, D] - 2| \geq |\operatorname{tr}^2 A_a - 4| + 1 = J(A_a, B).$$

Case 3: C is of type (iii). By Lemma 3.6, we have

$$J(C, D) = |\operatorname{tr}^2 C - 4| + |\operatorname{tr}[C, D] - 2| \geq |\operatorname{tr}^2 C - 4| \geq 5 \geq J(A_a, B).$$

Hence, $J(G_a) = J(A_a, B)$ for $1 \leq a \leq a_0$, and the proposition is proved. $\square$

FIGURE 5. Farey triangulation and the dual graph Σ (red).

FIGURE 6. $e = (X, Y; U, V)$.

4. THE DIAGONAL SLICE OF SCHOTTKY SPACE

Jørgensen numbers can be studied by computer experiments. In this section, we present a computer generated picture (Figure 7) which estimates Jørgensen numbers in the diagonal slice of Schottky space.

4.1. **Markoff maps.** First, we recall Bowditch, Tan-Wong-Zhang theory [2, 24] on Markoff maps very briefly. See section 3 in [24] for details.

We denote the set $\mathbb{Q} \cup \{1/0\}$ by $\widehat{\mathbb{Q}}$. Let $\mathcal{F}$ be the Farey triangulation of the upper half-plane $\mathbb{H}^2$. Recall that the vertex set of $\mathcal{F}$ is $\widehat{\mathbb{Q}}$, and two vertices p/q and r/s are connected in $\mathcal{F}$ by the hyperbolic geodesic connecting these two rational numbers if $ps - qr = \pm 1$. Let Σ be the binary tree dual to $\mathcal{F}$. See Figure 5. A *complementary region* of Σ is the closure of a connected component of the complement of Σ. The set of complementary regions of Σ is denoted by Ω. For each $X \in \Omega$, $\partial X \cap \mathbb{R}$ consists of a unique rational number, say p/q, and we denote this complementary region by $X_{p/q}$. See Figure 5. Thus, we identify Ω with $\widehat{\mathbb{Q}}$. Let e be an edge of Σ with endpoints u and v. Then, there exist $X, Y, U, V \in \Omega$ such that $e = X \cap Y$, $u = X \cap Y \cap U$, and $v = X \cap Y \cap V$. See Figure 6. We write $e = (X, Y; U, V)$ to indicate these regions. A *Markoff map* is a map ψ from Ω to $\mathbb{C}$ such that, for every edge $e = (X, Y; U, V)$ in Σ, we have

$$\psi(U) + \psi(V) = \psi(X)\psi(Y).$$

This condition is called the *edge relation*. Given $k \geq 0$, the set $\Omega_\psi(k)$ is defined by $\Omega_\psi(k) = \{X \in \Omega \mid |\psi(X)| \leq k\}$. We will need the following lemma.

Lemma 4.1 (Theorem 3.1(2), [24]). *Let ψ be a Markoff map. For any $k \geq 2$, the union $\bigcup_{X \in \Omega_\psi(k)} X$ is connected as a subset of $\mathbb{H}^2$.*

Let $F_2 = \langle A, B \rangle$ be the free group on A and B. An element W of F_2 is called *primitive* if there exists an element V such that $F_2 = \langle W, V \rangle$, and W and V are called *associated primitives*. Let α be the abelianization homomorphism from F_2 onto F_2/F_2' ($\cong \mathbb{Z}^2$). Let ρ be an $SL(2, \mathbb{C})$ representation of F_2. Then, the next map $\psi_\rho : \Omega \to \mathbb{C}$ is well-defined:

$$\psi_\rho(X_{n/m}) = \operatorname{tr} \rho(W_{n/m}),$$

where $W_{n/m}$ is a primitive element such that $\alpha(W_{n/m}) = A^m B^n$. See Corollary 3.2 in [19]. For example, we have

$$\psi_\rho(X_{0/1}) = \operatorname{tr}_\rho(A), \quad \psi_\rho(X_{1/0}) = \operatorname{tr}_\rho(B), \quad \psi_\rho(X_{1/1}) = \operatorname{tr}_\rho(AB).$$

By Theorems 1.2 and 1.3 in [19] and the trace identity in $SL(2, \mathbb{C})$,

$$(5) \qquad \operatorname{tr} AB + \operatorname{tr} AB^{-1} = \operatorname{tr} A \operatorname{tr} B,$$

we have the next lemma.

Lemma 4.2. ψ_ρ *is a Markoff map.*

4.2. **The diagonal slice of Schottky space.** Let us go back to the singular solid torus in section 2. Recall that

$$(6) \qquad A_a = \begin{pmatrix} -3a/2 & 1/2 \\ -1/2 & -1/2a \end{pmatrix}, \quad B = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

In this section, we consider that $A_a, B \in SL(2, \mathbb{C})$. We denote $\langle A_a, B \rangle$ by G_a. Set

$$(7) \qquad x = 2 - \operatorname{tr} A_a^2.$$

Then, we have

$$(8) \qquad \operatorname{tr} A_a = \sqrt{-x + 2}, \quad \operatorname{tr} B = 0, \quad \operatorname{tr} A_a B = \sqrt{x + 1}, \quad \operatorname{tr}[A_a, B] = 1.$$

See section 5.0.2 (in particular, Remark 5.4 for the sign of the square roots) in [23]. From now on, we consider that x is a complex parameter. The locus $\mathcal{D}$ in the x-plane of discrete and faithful representations was fully determined by computing Keen-Series *pleating rays* [23]. (Here, "faithful" means that $\langle A_a, B \rangle$ is isomorphic to $\langle A_1, B \rangle$ as an abstract group.) $\mathcal{D}$ is called *the diagonal slice of Schottky space.* See Figure 7. The outside of the *center black eye* corresponds to $\mathcal{D}$. $\mathcal{D}$ is foliated by pleating rays. We will describe pleating rays briefly in the next subsection.

Let a be a complex number such that $x = 2 - \operatorname{tr} A_a^2 \in \mathcal{D}$. We denote the representation from $F_2 = \langle \mathcal{A}, \mathcal{B} \rangle$ to $SL(2, \mathbb{C})$ which sends $\mathcal{A}$ to A_a and $\mathcal{B}$ to B by ρ_x. Let $\psi_x : \widehat{\mathbb{Q}} \to \mathbb{C}$ be the Markoff map associated with $\langle A_a, B \rangle$:

$$\psi_x(0/1) = \operatorname{tr} A_a, \ \psi_x(1/0) = \operatorname{tr} B, \ \psi_x(1/1) = \operatorname{tr} A_a B.$$

If W and V are associated primitives of F_2, then their images in G_a generate G_a. By Lemma 2.1 and equation (8), $\operatorname{tr} \rho_x([W, V]) = \operatorname{tr}[A_a, B] = 1$. Hence,

$$J(\rho_x(W), \rho_x(V)) = |\psi_x^2(n/m) - 4| + 1 \geq J(G_a),$$

where n/m is determined by the abelianization $\alpha(W) = \mathcal{A}^m \mathcal{B}^n$. Hence,

$$\Psi(x) := \inf_{q \in \widehat{\mathbb{Q}}} |\psi_x^2(q) - 4| + 1$$
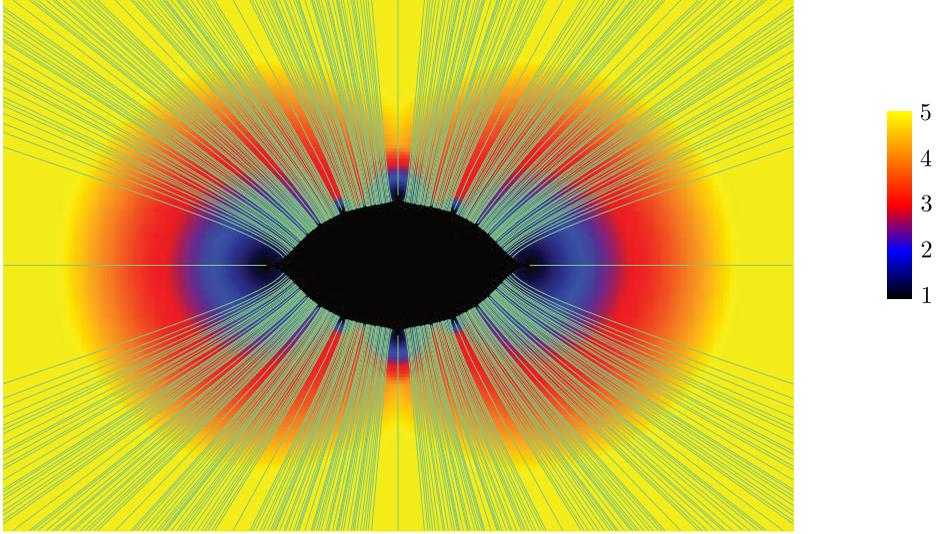
FIGURE 7. The diagonal slice and Jørgensen number: $-7 \leq \Re x \leq 8, -5 \leq \Im x \leq 5$. The color indicates the estimate of the Jørgensen number from above.

gives an estimate of $J(G_a)$ from above. We calculate $\Psi(x)$ by computer for $\{x \in \mathbb{C} \mid -7 \leq \Re(x) \leq 8, -5 \leq \Im(x) \leq 5\}$. The colors outside the "black eye" in Figure 7 indicate the values of $\Psi(x)$. Since these groups are discrete and faithful (in particular non-elementary), $\Psi(x) \geq 1$. Since $|\psi_x^2(1/0) - 4| + 1 = 5$, we have $\Psi(x) \leq 5$. Figure 7 suggests that for each $1 \leq r \leq 5$, there are many non-elementary Kleinian groups in the diagonal slice with Jørgensen number less than or equal to r.

Remark 4.3. In practice, we cannot calculate $\psi_x^2(q)$ for all $q \in \widehat{\mathbb{Q}}$, and Figure 7 is an approximation of $\Psi(x)$ by calculating $\psi_x^2(q)$ for many $q \in \widehat{\mathbb{Q}}$. But, if x is in the *Bowditch set*, in principle, we can calculate $\Psi(x)$. The key is Lemma 3.24 in [24]. The diagonal slice of Schottky space seems to coincide with the Bowditch set. See Section 2 in [23].

Question 4.4. Does the equation $J(G_a) = \Psi(x)$ hold for each $x \in \mathcal{D}$? Note that, since G_a is not free, there might be generating pairs of G_a which do not come from associated primitives in F_2.

4.3. **Jørgensen groups on the boundary of the diagonal slice.** We consider the case $\Psi(x) = 1$ on $\partial\mathcal{D}$. We begin by describing pleating rays very briefly. See [23] for details.

For $p/q \in \widehat{\mathbb{Q}}$, the *real trace locus* $\mathbb{R}_{p/q}$ of p/q is $\{x \in \mathbb{C} \mid \psi_x(p/q) \in (-\infty, -2] \cup [2, \infty)\}$. The rational pleating ray $\mathcal{P}_{p/q}$ is a union of connected non-singular branches of $\mathbb{R}_{p/q}$ (Corollary 4.11, [23]). The rational pleating rays are indexed by $\widehat{\mathbb{Q}}/\sim$, where $p/q \sim p'/q'$ if and only if $p'/q' = \pm p/q + 2k, k \in \mathbb{Z}$ (Proposition 4.8, [23]). (If $p/q \sim p'/q'$, then we have $\psi_x(p/q) = \pm\psi_x(p'/q')$.)

If p/q is not an integer, $\mathcal{P}_{p/q}$ consists of two components (branches). One is in the upper half-plane, and the other in the lower half-plane. Let $e_{p/q}$ be the endpoint of $\mathcal{P}_{p/q}$ in the upperhalf-plane, or on the real axis if p/q is an integer. By construction, we have $\psi^2_{e_{p/q}}(p/q) - 4 = 0$. Hence, $\Psi(e_{p/q}) = 1$, and the corresponding representation is a Jørgensen group of parabolic type. (If $\langle A, B \rangle$ is a Jørgensen group with $J(A, B) = 1$ and A parabolic, it is called a Jørgensen group of parabolic type [13].)

In order to describe these groups, we recall the following Li-Oichi-Sato normalization.

Lemma 4.5 (Lemma 3.1, [13]). *Let M and N be elements of $\mathrm{PSL}(2, \mathbb{C})$ such that M is parabolic and N is elliptic or loxodromic. Then, M, N can be normalized as*

$$M = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad N_{\sigma,\mu} = \begin{pmatrix} \mu\sigma & \mu^2\sigma - 1/\sigma \\ \sigma & \mu\sigma \end{pmatrix},$$

where $\sigma \in \mathbb{C} \setminus \{0\}$ and $\mu \in \mathbb{C}$.

Remark 4.6. Li-Oichi-Sato [13], [12], [14] considered the case $\mu = ik$, $k \in \mathbb{R}$. They conjectured that, for any Jørgensen group G of parabolic type, there exists a marked group $G_{\sigma,ik} = \langle M, N_{\sigma,ik} \rangle$, $\sigma \in \mathbb{C} \setminus \{0\}$, $k \in \mathbb{R}$, such that $G_{\sigma,ik}$ is conjugate to G. Later, Callahan [3] found counterexamples for this conjecture.

Now, we compute some examples. We begin by calculating some Markoff maps and the endpoints of pleating rays:

$$\psi^2_x(0/1) - 4 = (-x + 2) - 4, \qquad e_{0/1} = -2,$$
$$\psi^2_x(1/3) - 4 = (-x + 1)^2(x + 1) - 4, \qquad e_{1/3} \approx -0.5652 + 1.0434i,$$
$$\psi^2_x(3/8) - 4 = (-x + 2)(x + 1)(x^3 - x^2 - 1)^2 - 4, \quad e_{3/8} \approx -0.2992 + 1.0726i,$$
$$\psi^2_x(2/5) - 4 = x^4(-x + 2) - 4, \qquad e_{2/5} \approx -0.1372 + 1.1260i,$$
$$\psi^2_x(1/2) - 4 = (-x + 2)(x + 1) - 4, \qquad e_{1/2} = (1 + \sqrt{7}i)/2.$$

(Note that the equation $\psi^2_x(p/q) = 4$ has many roots, and in order to make the right choice for $e_{p/q}$, we need [23].) Let $A_{p/q}$ denote the matrix in (6) such that $e_{p/q} = 2 - \mathrm{tr}\, A_a^2$. Then, for example,

$$(A_{0/1}, B), (A_{1/3}^3 B, A_{1/3}), (A_{3/8}^3 B A_{3/8}^3 B A_{3/8}^2 B, A_{3/8}^3 B),$$
$$(A_{2/5}^3 B A_{2/5}^2 B, A_{2/5}^3 B), (A_{1/2}^2 B, A_{1/2})$$

generate Jørgensen groups of parabolic type. The Li-Oichi-Sato parameters for these groups are as follows:

$$(\sigma, \mu) = (i, 0) \qquad \text{for } (A_{0/1}, B),$$
$$(\sigma, \mu) \approx (-i, 0.1597 + 0.8166i) \qquad \text{for } (A_{1/3}^2 B, A_{1/3}),$$
$$(\sigma, \mu) \approx (i, 0.1839 + 0.9356i) \qquad \text{for } (A_{3/8}^3 B A_{3/8}^3 B A_{3/8}^2 B, A_{3/8}^3 B),$$
$$(\sigma, \mu) \approx (i, 0.3016 + 0.9041i) \qquad \text{for } (A_{2/5}^3 B A_{2/5}^2 B, A_{2/5}^3 B),$$
$$(\sigma, \mu) = (i, (1 + \sqrt{7}i)/4) \qquad \text{for } (A_{1/2}^2 B, A_{1/2}).$$

We conjecture that groups which correspond to the endpoints of the rational pleating rays (except -2 and 3 on the real axis) are counterexamples for Li-Oichi-Sato's conjecture in Remark 4.6.

Remark 4.7. The limit set of $\langle M, N_{\sigma,\mu} \rangle$ with $(\sigma, \mu) = (i, 0)$ is the real line. The limit sets of other examples are complicated and beautiful. For example, see Figure 8 for the limit set of the third case $(\sigma, \mu) \approx (i, 0.1839 + 0.9356i)$.
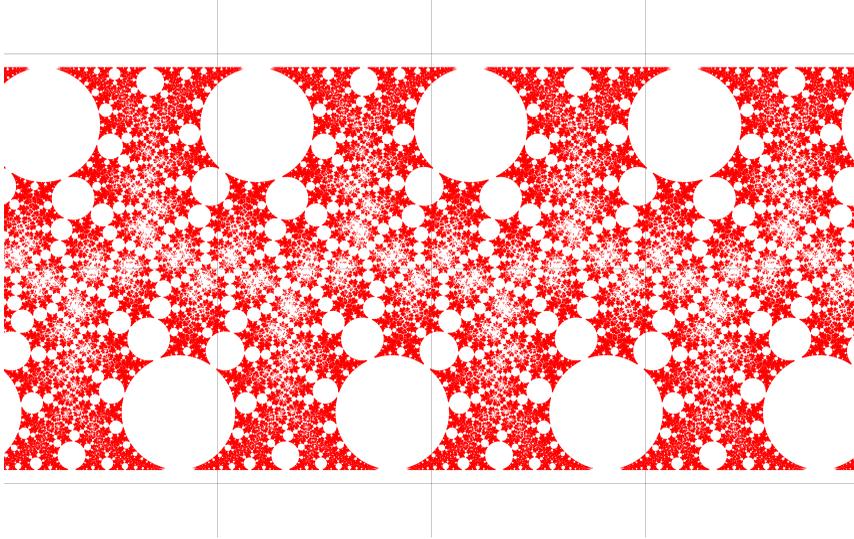

FIGURE 8. The limit set of $\langle M, N_{\sigma,\mu} \rangle$ with $(\sigma, \mu) \approx (i, 0.1839 + 0.9356i)$.

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