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HODGE THEORY IN THE SOBOLEV TOPOLOGY FOR THE DE RHAM COMPLEX ON A SMOOTHLY BOUNDED DOMAIN IN EUCLIDEAN SPACE

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ABSTRACT. The Hodge theory of the de Rham complex in the setting of the Sobolev topology is studied. As a result, a new elliptic boundary value problem is obtained. Next, the Hodge theory of the $\overline{\partial}$-Neumann problem in the Sobolev topology is studied. A new $\overline{\partial}$-Neumann boundary condition is obtained, and the corresponding subelliptic estimate derived.

The classical Hodge theory on a domain in $\Omega \subseteq \mathbb{R}^{N+1}$ (or, more generally, on a real manifold) is based on the complex

$$\bigwedge^0 \xrightarrow{d} \bigwedge^1 \xrightarrow{d} \bigwedge^2 \xrightarrow{d} \cdots$$

In the topology of $L^2(\Omega)$, one can calculate both the domain of existence and the actual form of d^*, the Hilbert space adjoint of d. It turns out that d^* equals d', the formal adjoint of d *on the domain of* d^*. See [FOK] for details.

Of course these facts are well known. They lead to the study of the second order, self-adjoint operator

$$\Box \equiv d\,d^* + d^*\,d.$$

Said operator $\Box$ makes sense precisely on those forms ϕ that lie both in the domain of d and in the domain of d^*. Calculated on a domain or region Ω in space, the operator $\Box$ turns out to be the (negative of the) ordinary Laplacian $\triangle$. The Hodge theory of $\Box$, which by today's standards is rather straightforward, shows that $\Box$ has closed range in $L^2(\Omega)$. The orthogonal complement of the range is of course the kernel of the adjoint of $\Box$ (which is nothing other than $\Box$ itself). The *Neumann operator* N for the d-complex is a right inverse for $\Box$.

It turns out that the operator N is, essentially, a pseudodifferential operator of degree -2. The regularity theory for the operator $\Box$, and also for the operator d, may be read off from the mapping properties of N. However, because the operator d has a large kernel, one must choose a solution to the equation $du = f$ carefully.

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The form

$$u_c \equiv d^* N f \tag{1}$$

will always be a solution to $du = f$ (so long as f is orthogonal to the harmonic space) and it follows from formal considerations that $u_c \perp \operatorname{Ker} d$. We call u_c the *canonical solution* to $du = f$. In many applications, u_c is the "good" solution that we seek.

It is by now well known (see [SWE]) that u_c satisfies standard elliptic regularity estimates for a first order (uniformly) elliptic linear partial differential operator: measured in either the Sobolev topology, the Lipschitz topology, or in fact in any Triebel-Lizorkin space topology, the solution u_c exhibits a gain in smoothness of order 1 when compared to the smoothness of the data.

The purpose of the present work is to develop the Hodge theory for the d operator in a *Sobolev space* W^1 topology. While the general W^s-case, $s = 1, 2, \ldots$ is of considerable intrinsic interest, the case of the topology of W^1 is of particular interest in geometric applications. This work on the d operator also serves as a step towards working out the theory of the $\overline{\partial}$-Neumann problem in a Sobolev space topology. We shall spend the remainder of this announcement doing two things: **(1)** summarizing the key features of the Hodge theory for d in the Sobolev space W^1 topology, and **(2)** describing progress on our program to study the $\overline{\partial}$-Neumann problem in the Sobolev space W^1 topology.

1. The Formalism of the de Rham Complex in the Sobolev Topology

We consider the complex

$$\bigwedge^0 \xrightarrow{d} \bigwedge^1 \xrightarrow{d} \bigwedge^2 \xrightarrow{d} \cdots .$$

This is the same complex as considered above but now, when we turn our attention to the operator $\Box \equiv dd^* + d^*d$, we consider the adjoints in the W^s topology.

Fix now a smoothly bounded domain $\Omega \subseteq \mathbb{R}^{N+1}$ (or the half space $\mathbb{R}^{N+1}_+$).

Following notation of [FOK], we let $\bigwedge^r_0(\overline{\Omega})$ denote those r-forms with $C^\infty_0(\mathbb{R}^{N+1})$ coefficients and such that the intersection of the support of the form with $\overline{\Omega}$ is *compact in* $\overline{\Omega}$.

Proposition 1.1. *With Ω as above, and with $q = 0, 1, 2, \ldots, N$, we have*

$$\operatorname{dom} d^* \cap \bigwedge\nolimits_0^{q+1}(\overline{\Omega}) = \left\{ \phi \in \bigwedge\nolimits_0^{q+1}(\overline{\Omega}) : \nabla_{\vec{n}}\phi \,\llcorner\, \vec{n} \Big|_{b\Omega} = 0 \right\}.$$

Here we denote by $\nabla_{\vec{n}}\phi$ the covariant derivative in the normal direction of the form ϕ, and by $\llcorner$ the standard contraction operator.

Proposition 1.2. *Let Ω be a smoothly bounded domain. Then, on $\operatorname{dom} d^*$,*

$$d^* = d' + \mathcal{K},$$

where d' is the formal adjoint of d, and $\mathcal{K}$ is an operator sending $(q+1)$-forms to q-forms. The components of $\mathcal{K}\phi$ are solutions of the following boundary value problem

$$\begin{cases} (-\triangle + I)(\mathcal{K}\phi)_I = 0 & \text{on } \Omega \\ \dfrac{\partial}{\partial n}(\mathcal{K}\phi)_I = T_2 \phi \llcorner \vec{n} & \text{on } b\Omega, \end{cases}$$

where T_2 is a second order tangential differential operator on forms whose top order terms equal those of the Laplacian on the boundary.

Definition 1.3. *For a smoothly bounded domain Ω we set*

$$G_\Omega = \mathcal{K}d + d\mathcal{K}.$$

Notice that now $\Box \equiv dd^* + d^*d = -\triangle + G_\Omega$.

In the language just introduced, our boundary value problem becomes

$$\begin{cases} (-\triangle + G_\Omega)\phi = \alpha \quad \text{on} \quad \Omega \\ \phi \in \operatorname{dom} d^* \\ d\phi \in \operatorname{dom} d^* \end{cases} . \tag{2}$$

We are now ready to state our results about existence and regularity of the boundary value problem.

THEOREM 1.4. *Let Ω be a smoothly bounded domain. Consider the boundary value problem (2). Let $s > 1/2$. Then there exists a finite dimensional subspace (the harmonic space) $\mathcal{H}_q$ of $\bigwedge^q(\overline{\Omega})$ and a constant $c = c_s > 0$ such that if $\alpha \in W^s_q(\Omega)$ is orthogonal (in the W^1-sense) to $\mathcal{H}_q$, then the boundary value problem (2) has a unique solution ϕ orthogonal to $\mathcal{H}_q$ in the W^1 topology such that*

$$\|\phi\|_{s+2} \le c\|\alpha\|_s.$$

Remark. If $s < 1$ then, by saying that α is orthogonal in the W^1-sense, we mean that α is the W^s-limit of smooth forms that are orthogonal in W^1 to $\mathcal{H}_q$.

Finally we have

THEOREM 1.5. *Let Ω be a smoothly bounded domain in $\mathbb{R}^{N+1}$. Let $W^1_q(\Omega)$ denote the 1-Sobolev space of q-forms. Then we have the strong orthogonal decomposition*

$$W^1_q = dd^*(W^1_q) \oplus d^*d(W^1_q) \oplus \mathcal{H}_q\,,$$

where $\mathcal{H}_q$ is a finite dimensional subspace.

What is novel in our treatment of the Hodge theory for the de Rham operator in the Sobolev topology is the presence of the operator $\mathcal{K}$ when mediating between d' and d^*. The contribution of $\mathcal{K}$ to the boundary value problem is non-trivial and its study occupies a large portion of our work.

The analysis of d^* in the Sobolev topology fits rather naturally into an abstract theory of pseudodifferential operators of transmission type as developed by Boutet de Monvel in [BDM1]–[BDM3]. Details are provided in our paper [FKP1].

2. The $\overline{\partial}$-Neumann Problem

As previously indicated, the main goal of this work is to develop a new approach to the $\overline{\partial}$-Neumann problem. We have used our work on the Hodge theory of the de Rham operator in the Sobolev topology as a stepping stone in this program. We now outline the steps of our theory that have been developed thus far for the $\overline{\partial}$-Neumann problem. Full details will appear in [FKP2].

In analogy with what occurred above, we let Ω be a smoothly bounded domain in $\mathbb{C}^n$. We consider the complex

$$\bigwedge^{0,0} \overset{\overline{\partial}}{\to} \bigwedge^{0,1} \overset{\overline{\partial}}{\to} \bigwedge^{0,2} \overset{\overline{\partial}}{\to} \cdots$$

All notation is analogous to that stated for the d-complex; we shall not repeat all the definitions. When we calculate adjoints, it shall be in the $W^1(\Omega)$ topology.

Let ρ be a defining function for Ω and set

$$N = \sum_{j=1}^{n} \frac{\partial \rho}{\partial \bar{z}_j} \frac{\partial}{\partial z_j}.$$

Then we set

$$\frac{\partial}{\partial n} \equiv N + \overline{N}.$$

Proposition 2.1. *Let $\phi \in \bigwedge^{0,1}(\overline{\Omega})$. Then $\phi \in dom\,\overline{\partial}^*$ if and only if*

$$\frac{\partial}{\partial n}\big(\phi \llcorner \overline{N}\big)\Big|_{b\Omega} = 0,$$

that is, if and only if

$$\sum_j \frac{\partial}{\partial n}\left(\phi_j \frac{\partial \rho}{\partial z_j}\right)\Big|_{b\Omega} \equiv \sum_j \frac{\partial \rho}{\partial z_j}\frac{\partial \phi_j}{\partial n}\Big|_{b\Omega} = 0.$$

Proposition 2.2. *Let $\phi \in \bigwedge^{0,1}(\overline{\Omega}) \cap dom\,\overline{\partial}^*$. Then*

$$\overline{\partial}^*\phi = \vartheta\phi + \mathcal{K}\phi,$$

where ϑ is the formal adjoint of the $\overline{\partial}$ operator and $\mathcal{K}$ is the function which is the solution of the following boundary value problem:

$$\begin{cases} \big(-\triangle + I\big)(\mathcal{K}\phi) & = 0 \qquad\qquad\qquad\qquad \text{on } \Omega, \\ \dfrac{\partial}{\partial n}\big(\mathcal{K}\phi\big) & = \displaystyle\sum_{k=1}^{n}\Bigg[2\cdot\sum_{p=1}^{n}\bigg(Y_p^*\big[(Y_p\phi_k)\frac{\partial\rho}{\partial z_k}\big] \\ & \qquad + \overline{Y}_p^*\big[(\overline{Y}_p\phi_k)\tfrac{\partial\rho}{\partial z_k}\big]\bigg) + \phi_k\tfrac{\partial\rho}{\partial z_k}\Bigg] \quad \text{on } b\Omega. \end{cases}$$

Here Y_p denotes the tangential component of $\partial/\partial z_p$, and Y_p^* is its formal adjoint.

Now recall that, in the classical L^2 theory (see [FOK]), a major point of the analysis is that the boundary value problem is not elliptic: the classical coercive estimate does not hold. One obtains a substitute estimate from below for the quadratic form Q that is defined to be essentially the polarization of $\Box$. A similar, but more complicated, circumstance now obtains when we work in the Sobolev W^1 topology.

If $\phi, \psi \in \bigwedge^{0,1}(\overline{\Omega}) \cap \mathrm{dom}\,\overline{\partial}^*$, then we set

$$Q(\phi,\psi) = \langle\overline{\partial}\phi, \overline{\partial}\psi\rangle_1 + \langle\overline{\partial}^*\phi, \overline{\partial}^*\psi\rangle_1 + \langle\phi,\psi\rangle_1.$$

We also set

$$E(\phi)^2 \equiv \sum_{j,k}\left\|\frac{\partial\phi_j}{\partial\bar{z}_k}\right\|_1^2 + \|\phi\|_1^2 + \|\phi\|^2_{W^1(b\Omega)} + \|\phi \llcorner \bar{N}\|^2_{W^{3/2}(b\Omega)}.$$

We let $\mathcal{L}$ denote the Levi form. Then our basic estimate is this:

THEOREM 2.3. *There is a constant* $C_0 > 0$ *such that, for all* $\phi \in \bigwedge^{0,1}(\overline{\Omega}) \cap$ *dom* $\overline{\partial}^*$*, we have*

$$Q(\phi, \phi) \geq C_0 \Bigg\{ \sum_{j,k} \left\| \frac{\partial \phi_j}{\partial \bar{z}_k} \right\|_1^2 + \|\mathcal{K}\phi\|_1^2 + \|\phi\|_1^2 + \int_{b\Omega} \mathcal{L}(\phi, \phi) + 2 \sum_p \int_{b\Omega} \mathcal{L}\left(\frac{\partial \phi}{\partial z_p}, \frac{\partial \phi}{\partial z_p} \right) + 2 \sum_p \int_{b\Omega} \mathcal{L}\left(\frac{\partial \phi}{\partial \bar{z}_p}, \frac{\partial \phi}{\partial \bar{z}_p} \right) \Bigg\}.$$

As a consequence, if Ω *is a smoothly bounded strongly pseudoconvex domain, then there exists* $C_1 > 0$ *such that*

$$\frac{1}{C_1} E(\phi) \leq Q(\phi, \phi) \leq C_1 E(\phi).$$

Of course this theorem makes it clear (especially in view of the well known calculations in [FOK]) how the Levi form—in particular how the property of strong pseudoconvexity—will come into play in further developments of the theory. The inequality in Theorem 2.3 will enable us to prove subelliptic estimates for the canonical solution to the $\overline{\partial}$ problem in the Sobolev inner product. In particular, our work produces a complete existence and regularity theory for a new canonical solution to the $\overline{\partial}$-Neumann problem. Other canonical solutions, besides that of Kohn, have been explored in [PHO].

A complete explication of the $\overline{\partial}$-Neumann problem in the W^s topology, together with applications, will appear in [FKP2].

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