ELECTRONIC RESEARCH ANNOUNCEMENTS
OF THE AMERICAN MATHEMATICAL SOCIETY
Volume 1, Issue 3, 1995

ON SIMPLE IGUSA LOCAL ZETA FUNCTIONS

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(Communicated by David Kazhdan)

Abstract. The objective of this announcement is the statement of some recent results on the classification of generalized Igusa local zeta functions associated to irreducible matrix groups. The definition of a simple Igusa local zeta function will motivate a complete classification of certain generalized Igusa local zeta functions associated to simply connected simple Chevalley groups. In addition to the novelty of these results are the various methods used in their proof. These methods include use of the concept of canonical basis from quantum group theory and a formula expressing Serre's canonical measure μ_c in terms of a suitably normalized Haar measure μ and density function Φ. The relevance of these results in the general theory of Igusa local zeta functions is also discussed.

1. Introduction

The study of local zeta functions of Igusa type commenced in earnest in 1974 in [3]. Through the work of various people was developed the concept of a generalized Igusa local zeta function. For ease of explication, only the specialization of this concept of interest here is introduced. The reader is referred to [5] or [4] for a more general definition.

Let K be a finite algebraic extension of $\mathbf{Q}_p$. Let O_K, πO_K denote the ring of integers of K, ideal of nonunits of O_K, respectively. Set $\mathrm{card}(O_K/\pi O_K) = q$. For technical reasons assume $2 \not| \, q$ (see [5]). Let $|\ |_K$ be the absolute value on K normalized as $|\pi|_K = q^{-1}$. Let $G^{'}$ be a simply connected simple Chevalley K-group and ρ a finite dimensional K-representation of $G^{'}$. The almost direct product $G = \rho(G^{'})(\mathbf{G_m}\mathbf{1}_{dim\,\rho})$, where $\mathbf{G_m}$ denotes the algebraic group defined by K^* and $\mathbf{1}_{dim\,\rho}$ the $dim\,\rho \times dim\,\rho$ identity matrix, is a K-subgroup of $GL_{dim\,\rho}$, not contained in $SL_{dim\,\rho}$. Set T as the maximal K-split torus in G, f as the positive generator of $\mathrm{Hom}(G, \mathbf{G_m})$ and $G^0 = G_K \bigcap \mathrm{Mat}_{dim\,\rho}(O_K)$ where G_K denotes the K-rational points of G.

Received by the editors September 23, 1995.
1991 *Mathematics Subject Classification.* Primary 11D79.
Key words and phrases. Local zeta functions, canonical bases.

Definition 1.1. The generalized Igusa local zeta function associated to $(G^{'}, \rho)$ is

$$Z_K(s) = \int_{G^0} |f(g)|_K^s \mu_c(g)$$

where μ_c is Serre's canonical measure defined on G_K and $s \in \mathbf{C},\ \mathrm{Re}(s) > 0$.

Typically, one would like to compute a finite form for $Z_K(s)$ and examine this finite form for further relations. For example, the generalized Igusa local zeta function associated to $(SL_1, 1)$ has finite form $Z(q^{-1}, q^{-s}) = (1 - q^{-1})/(1 - q^{-s-1})$ with relation

$$Z(q^{-1}, q^{-s})|_{q \mapsto q^{-1}} = q^{-s} Z(q^{-1}, q^{-s})\,.$$

Note that this generalized Igusa local zeta function is universal in the variables q^{-1}, q^{-s} (see [4]). There is a criterion for answering the typical questions in the case of a generalized Igusa local zeta function associated to $(G^{'}, \rho)$.

Since the measure μ_c is difficult to work with, define a function Φ on G_K by

$$\Phi(g) = \frac{\mu_c(g(G_K \bigcap GL_{dim\,\rho}(O_K)))}{\mu(G_K \bigcap GL_{dim\,\rho}(O_K))}$$

where μ is the Haar measure on G_K normalized to be canonical measure on the group $G_K \bigcap GL_{dim\,\rho}(O_K)$. It follows that $\mu_c(g) = \Phi(g)\mu(g)$ and, thus, that

$$Z_K(s) = \int_{G^0} |f(g)|_K^s \Phi(g)\mu(g)\,.$$

The details are in [5]. Via the p-adic Bruhat decomposition for G_K, the computation of $\Phi(g)$ is reduced to that of $\Phi(\xi(\pi))$ where $\xi \in \mathrm{Hom}(\mathbf{G_m}, T)$ is as in [4] or [5].

Definition 1.2. The generalized Igusa local zeta function $Z_K(s)$ associated to (G', ρ) is *simple* if Φ is independent of ξ.

Remark on Definition 1.2. As just explained above, $Z_K(s) = Z_K(s, \Phi)$. Therefore, the property of $Z_K(s)$ being a simple generalized Igusa local zeta function depends only on $Z_K(s)$ itself.

Criterion ([4])**.** *If $Z_K(s)$ is simple, then $Z_K(s)$ has a finite form that expresses $Z_K(s)$ as a rational function $Z(q^{-1}, q^{-s})$ satisfying the functional equation*

$$Z(q^{-1}, q^{-s})|_{q \mapsto q^{-1}} = q^{-deg(f)s} Z(q^{-1}, q^{-s}). \qquad \bullet$$

Remarks on the Criterion. Note first that $Z_K(s)$ is universal in the variables q^{-1}, q^{-s}. Note also that Φ is not "residual" as defined by J. Denef and D. Meuser because $\Phi(g)$ does not only depend on $g \bmod \pi O_K$. Therefore, it is not immediate that $Z_K(s)$ has either a finite form or the functional equation $\bullet$ (see [1]).

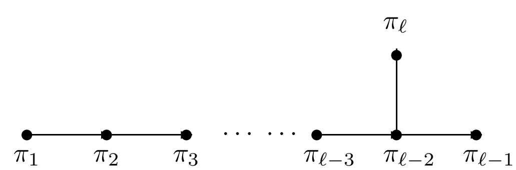
2. Results

There is now motivation for classifying as simple or not the generalized Igusa local zeta functions associated to $(G^{'}, \rho)$. By assumption, $G^{'}$ is in one of the families of nine simply connected simple Chevalley K-groups distinguished by their respective Lie algebras A_ℓ, B_ℓ, C_ℓ, D_ℓ, E_6, E_7, E_8, F_4 and G_2. Denote by $\mathbf{L}G^{'}$ the Lie algebra of $G^{'}$. It is well known that the nodes of the Dynkin diagram of $\mathbf{L}G^{'}$ may be labeled canonically by the fundamental representations of $G^{'}$. We denote these fundamental representations by π_i, $1 \le i \le rk\,\mathbf{L}G^{'}$. Recall that ρ has a unique expression in terms of the fundamental representations π_i as

$$\rho = \mathop{*}_{i=1}^{rk\,\mathbf{L}G'} \pi_i^{n_i}$$

where $*$ denotes the Cartan product. We are able to parametrize the simple generalized Igusa local zeta functions associated to $(G^{'}, \rho)$ by examining the above Cartan product decomposition of ρ.

Therefore, orient the Dynkin diagrams of the simple Lie algebras as in [8] and label their nodes consecutively with the π_i from left to right reserving for a branched node the last index $rk\,\mathbf{L}G^{'}$. For example, the Dynkin diagram for D_ℓ would be labeled



Definition 2.1. Given $(G^{'}, \rho)$, the subgraph of the Dynkin diagram of $G^{'}$ determined by the elimination of any nodes (and their incident edges) associated to fundamental representations π_i of $G^{'}$ not occurring in the Cartan product decomposition of ρ is called the *decomposition diagram.*

Our result is the following.

Theorem 2.1 ([7]). *Let $G^{'}$ be of type $A_\ell\,(\ell \ge 1)$, $B_\ell\,(\ell \ge 2)$, $C_\ell\,(\ell \ge 3)$, $D_\ell\ (\ell \ge 4)$, E_6, E_7, E_8, F_4 or G_2. The generalized Igusa local zeta function $Z_K(s)$ associated to $(G^{'}, \rho)$ is simple if and only if the decomposition diagram of $(G^{'}, \rho)$ is connected.*

Thus, in the case of $G^{'}$ of type D_ℓ and $\rho = \mathop{*}_{i=1}^{\ell} \pi_i^{n_i}$, the corresponding $Z_K(s)$ is simple unless one or more of the following conditions hold:

(1) there exist $n_{i'}$, $n_{j'}$ nonzero, $1 \le i' < j' \le \ell - 2$, such that $j' - i' > 1$ and $n_{i'+1} = \ldots = n_{j'-1} = 0$, or

(2) there exists $n_{i'}$ nonzero, $1 \le i' < \ell - 2$, such that $n_{\ell-1}$, n_ℓ or $n_{\ell-1}$ and n_ℓ are nonzero and $n_{i'+1} = \ldots = n_{\ell-2} = 0$.

The main intermediary result, Theorem 2.2 below, is proved using the concept of canonical basis from quantum group theory. The reader is referred to [2] for background.

Theorem 2.2 ([7]). *Let ρ be a finite dimensional irreducible K-representation of G'. Suppose the $Z_K(s)$ corresponding to (G', ρ) is simple. Then the $Z_K(s)$ corresponding to $\left(G', \overset{n}{\underset{i=1}{*}} \rho\right)$ is simple.*

Remark on the proof of Theorem 2.2. The proof is an induction argument taking advantage of various properties of the canonical basis of the representation space V of ρ.

Use of Theorem 2.2 allows one to prove Theorem 2.1 via case-by-case examination.

One can see that most choices of (G', ρ) yield non-simple $Z_K(s)$. There are some isolated results related to determining the finite form and verifying the functional equation $\bullet$ for $Z_K(s)$ in the non-simple case (see [6]). It is, however, unknown which of these $Z_K(s)$ have finite forms and satisfy $\bullet$.

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