ELECTRONIC RESEARCH ANNOUNCEMENTS
OF THE AMERICAN MATHEMATICAL SOCIETY
Volume 2, Number 1, August 1996

OPTIMAL REGULARITY FOR QUASILINEAR EQUATIONS IN STRATIFIED NILPOTENT LIE GROUPS AND APPLICATIONS

LUCA CAPOGNA

(Communicated by Thomas Wolff)

Abstract. We announce the optimal $C^{1+\alpha}$ regularity of the gradient of weak solutions to a class of quasilinear degenerate elliptic equations in nilpotent stratified Lie groups of step two. As a consequence we also prove a Liouville type theorem for 1-quasiconformal mappings between domains of the Heisenberg group $\mathbb{H}^n$.

Statement of the results

Consider the quasilinear elliptic equation

$$\sum_{i=1}^{n} \partial_{x_i} A_i(x, \nabla u) = 0, \tag{1}$$

where $A_i(x,\xi) : \mathbb{R}^{n+n} \to \mathbb{R}$, $i = 1, \dots, n$, are differentiable functions satisfying $\lambda|\eta|^2 \le \sum_{i,j=1}^{n} \partial_{\xi_j} A_i(x,\xi)\eta_i\eta_j \le \lambda^{-1}|\eta|^2$, and $\sum_{i,j=1}^{n} \partial_{x_j} A_i(x,\xi) \le C(1+|\xi|)$, for every $\eta \in \mathbb{R}^n$, and almost every $x, \xi \in \mathbb{R}^n$. The sharp $C^{1+\alpha}$ regularity of weak solutions to (1) is one of the pillars on which the modern theory of quasilinear partial differential equations rests. The ideas on which its proof is based form a recurring theme in nonlinear analysis: first use difference quotients to prove that the weak solutions admit second (weak) derivatives, then differentiate the equations and observe that the derivatives of the solution are themselves solutions to some linear partial differential equations, whose coefficients are not very regular. At this point the regularity theory for linear equations with nonsmooth coefficients provides the final step in the proof of the Hölder continuity of the gradient of weak solutions to (1). The observation that one could reduce the study of quasilinear equations to studying linear equations with "bad" coefficients goes back to the pioneering work of Morrey, and has been developed by many mathematicians in the last thirty years. Although (1) is a relatively simple elliptic equation, the regularity theorem has far-reaching applications in calculus of variations (see, for instance, [Gi]) and in the theory of quasiconformal mappings in space [G1].

Two natural and inter-related questions arise: Can the ellipticity hypothesis in (1) be somewhat weakened, and still expect regularity of the gradient of the solutions? Is this "new" problem of some geometric relevance? In this announcement we provide positive answers to both questions.

Received by the editors March 15, 1996.
1991 *Mathematics Subject Classification.* Primary 35H05.
Alfred P. Sloan Doctoral Dissertation Fellow.

In the proof of his famous rigidity theorem [Mo], Mostow introduced quasiconformal mappings in the setting of stratified nilpotent Lie groups [F]. With this name one refers to the class of simply connected Lie groups G endowed with a stratification of the Lie algebra $g = V^1 \oplus \cdots \oplus V^r$, with $r \geq 1$ (the *step* of the group), such that $[V^1, V^j] = V^{j+1}$, $j = 1, \ldots, r-1$, and $[V^1, V^r] = 0$. The simplest example of a stratified Lie group is the Euclidean space $\mathbb{R}^n$, with $r = 1$. A less trivial, and genuinely non-Euclidean example is provided by the Heisenberg group $\mathbb{H}^n$, $n \geq 1$, whose Lie algebra is $h^n = \mathbb{R}^{2n} \oplus \mathbb{R}$. The central role played by the Heisenberg group in many problems of complex geometry, representation theory and partial differential equations makes $\mathbb{H}^n$ the prototype *par excellence* of stratified nilpotent Lie groups. The theory of quasiconformal mappings between domains of the Heisenberg group has been developed recently in a series of papers by Korányi and Reimann [KR1]–[KR3] and by Pansu [P]. As in the Euclidean case (see [G1] and [R]), this development led to various questions concerning the regularity of weak solutions to a class of quasilinear equations similar to (1).

The notion of quasiconformality is a metric one, and in this setting it is related to the Carnot-Carathéodory metric associated to a basis $X_1^1, \ldots, X_m^1$, $m = m^1 = \dim(V^1)$ of V^1 (with our notation we do not distinguish between elements of g and left invariant vector fields). Since this metric is nonisotropic, it is natural to expect some nonisotropic structure in the relevant equations. In order to be more precise we need to recall that the exponential mapping $\exp : g \to G$ is a diffeomorphism, and so we can use exponential coordinates $p = (p_1^1, \ldots, p_m^1, p_1^2, \ldots, p_{\dim(V^2)}^2, \ldots)$ on G. Let $Xu = (X_1^1 u, \ldots, X_m^1 u)$ denote the *horizontal* gradient of the function u. Consider the equation

$$\sum_{i=1}^{m} X_i^1 A_i(p, Xu) = f(p), \tag{2}$$

where $A_i(p,\xi) : G \times \mathbb{R}^m \to \mathbb{R}$, $i = 1, \ldots, m$, are differentiable functions satisfying

$$\lambda|\eta|^2 \leq \sum_{i,j=1}^{m} \partial_{\xi_j} A_i(p,\xi)\eta_i\eta_j \leq \lambda^{-1}|\eta|^2,$$

and

$$\sum_{i=1}^{m} \sum_{j=1}^{\dim(g)} \partial_{p_j} A_i(p,\xi) \leq C(1+|\xi|),$$

for any $\eta \in \mathbb{R}^m$, for almost every $p \in G$ and $\xi \in \mathbb{R}^m$, and for some positive λ, C. The models we have in mind are

$$A_i(p,\xi) = \sum_{j=1}^{m^1} a_{ij}(p)\xi_j + a_i(p),$$

or

$$A_i(p,\xi) = |\xi|^{p-2}\xi_i,$$

for $0 < M < |\xi| < M^{-1}$ and $p > 2$. Let $\Omega \subset G$ be an open set, and let $S^{k,p}_{\text{loc}}(\Omega)$ denote the space of $L^p_{\text{loc}}(\Omega)$ functions that are weakly differentiable k times in the horizontal directions, and such that their horizontal derivatives up to order k lie in

$L^p_{\text{loc}}(\Omega)$. A function $u \in S^{1,2}_{\text{loc}}(\Omega)$ is a weak solution to (2) if and only if

$$\int_\Omega \sum_{i=1}^m A_i(p, Xu) X_i^1 \phi \, dp = \int_\Omega f(p)\phi(p) dp,$$

for every $\phi \in C_0^\infty(\Omega)$. The regularity of the weak solutions to (2) is measured in terms of the Hölder continuity with respect to the Carnot-Carathéodory distance $d(x, y)$ associated to $X_1^1, \dots, X_m^1$. More precisely, for $0 < \alpha < 1$ we define the Folland-Stein class

$$\Gamma^\alpha(\Omega) = \left\{ u \;\Big|\; \sup_{x,y\in\Omega} \frac{|u(x) - u(y)|}{d(x, y)^\alpha} < \infty \right\},$$

and its local version $\Gamma^\alpha_{\text{loc}}(\Omega) = \{u \mid \eta u \in \Gamma^\alpha(\Omega) \text{ for some } \eta \in C_0^\infty(\Omega)\}$. If $k \in \mathbb{N}$, the symbol $\Gamma^{k,\alpha}_{\text{loc}}(\Omega)$, will denote the set of functions having horizontal derivatives up to order k in $\Gamma^\alpha_{\text{loc}}(\Omega)$.

The local Hölder regularity of weak solutions to (2) has been established independently in [X1] and [CDG1]. The boundary regularity was studied in [D]. One would like to have an analogue of the Euclidean $C^{1+\alpha}$ regularity of weak solutions in this setting also, namely the $\Gamma^{1,\alpha}$ regularity. However, the classical argument breaks down at its first step and a new method is required. In fact, one should observe that if one differentiates (even formally) the equation along a vector field, terms containing derivatives of the solution along the commutators arise. There is no a priori control over these derivatives, in fact the notion of weak solution implies only differentiability in the horizontal directions $X_1^1, \dots, X_m^1$.

In the special case of stratified nilpotent Lie groups of step two, we have been able to overcome these difficulties. We prove

Theorem A. *Let $\Omega \subset \mathbb{H}^n$ be an open set, and $u \in S^{1,2}_{\text{loc}}(\Omega)$ a weak solution of*

$$\sum_{i=1}^m X_i^1 A_i(p, Xu) = 0$$

in Ω. There exists $\lambda > 0$ such that

$$u \in \Gamma^{1,\lambda}_{\text{loc}}(\Omega).$$

This result is only a particular case of the following

Theorem B. *Let G be a stratified nilpotent Lie group of step two, $\Omega \subset G$ be an open set, and $u \in S^{1,2}_{\text{loc}}(\Omega)$ be a weak solution to (2) in Ω. Denote by X^2_ν, $\nu = 1, \dots, m^2 = \dim(V^2)$, a basis of V^2. If we let $Q = m + 2m^2$ denote the homogeneous dimension of G, and $\sum_{\nu=1}^{m^2} |X^2_\nu f| \in L^s_{\text{loc}}(\Omega)$ with $s > Q/2$, then $X^2_\nu u \in \Gamma^\lambda_{\text{loc}}(\Omega)$ for every $\nu = 1, \dots, m^2$. If we also assume $\sum_{i=1}^{m^1} |X^1_i f| \in L^s_{\text{loc}}(\Omega)$, then*

$$u \in \Gamma^{1,\lambda}_{\text{loc}}(\Omega)$$

for every $i = 1, \dots, m^1$.

Theorem B rests on the following crucial result:

Theorem C. *Using the notation of Theorem B, if $f, \sum_{\nu=1}^{m^2} |X^2_\nu f| \in L^2_{\text{loc}}(\Omega)$, then*

$$X^2_\nu u \in S^{1,2}_{\text{loc}}(\Omega)$$

and

$$u \in S^{2,2}_{\mathrm{loc}}(\Omega), \quad i = 1, \dots, m^1.$$

The strategy for the proof of Theorem C is described in the next section.

In view of Theorem B one can see that (surprisingly!) the sharp regularity of the weak solutions is not $\Gamma^{1,\alpha}$ but the Hölder continuity of the full gradient of u.

Corollary. *If u is a weak solution to (2) in some open set $\Omega \subset G$, and the hypotheses in Theorem B above are satisfied, then there exists $\alpha > 0$ such that*

$$u \in C^{1+\alpha}_{\mathrm{loc}}(\Omega).$$

Several remarks are in order: (1) The results in Theorems B and C are new even for linear equations of the form

$$\sum_{i,j=1}^{m^1} X^1_i \left(a_{ij} X^1_j u + a_i \right) = f,$$

with a_{ij} a positive definite $m^1 \times m^1$ matrix with Lipschitz coefficients, and a_i Lipschitz functions. However, in this case, the local Γ^α regularity of the horizontal gradient is much easier to prove; in fact it can be done by using the freezing techniques, without differentiating the equation. This approach does not work when nonlinear dependence from the horizontal gradient is allowed on the left-hand side of the equation.

(2) In the setting of nilpotent stratified Lie groups of step two, the higher differentiability of Theorem C gives a "nonlinear" analogue of the L^2 estimates proved in [K], [FS], and [F] for the sub-Laplacian $\mathcal{L} = \sum_{i=1}^{m^1} {X^1_i}^2$, but it does not generalize them. In fact, here we need the extra differentiability assumption $X^2_\nu f \in L^2_{\mathrm{loc}}(\Omega)$ for every $\nu = 1, \dots, m^2$. However, differentiability of f in the commutators direction is necessary to obtain $X^2_\nu u \in S^{1,2}_{\mathrm{loc}}(\Omega)$, which is, in turn, a crucial ingredient of the proof of Theorem B. It seems worthwhile to observe also that in the Euclidean case $m^2 = 0$, and so Theorems B and C do not require more hypotheses on f than the classical L^2 estimates.

(3) Theorem B is sharp, in the sense that for equations of the form

$$\sum_{i=1}^{m^1} X^1_i A_i(p, Xu) = 0,$$

the local Hölder regularity of the gradient is the best one can expect unless more differentiability is required on the on the A_i's.

(4) If one assumes that the weak solution to (2) is $\Gamma^\alpha_{\mathrm{loc}}(\Omega)$, then the conditions on the Lebesgue norm of $X^k_i f$ in Theorem B can be weakened to conditions on the norm of $X^k_i f$ in some *ad hoc* Morrey spaces.

In view of the applications to the theory of quasiconformal mappings, it is important to develop a higher regularity theory for (2). The cornerstone of such theory is given by Theorem B, but two other ingredients from the linear theory are needed. The first is the subelliptic analogue of the interior Schauder theory, developed in [X2]; the second is the important L^2-estimates established in [K], [FS], and [F].

Theorem D. *Let u be a weak solution to (2), and λ as in Theorem B. Assume $X^2_\nu f \in L^\infty_{\mathrm{loc}}(\Omega)$, for $\nu = 1, \dots, m^2$. If $k > 0$, and $\partial_\xi A_i, f, \partial_{p^l_\nu} A_i(p, Xu) \in \Gamma^{k,\lambda}_{\mathrm{loc}}(\Omega)$,* *for $l = 1,2$ and $\nu = 1, ..., m^l$, then $u \in \Gamma^{k+2,\lambda}_{\text{loc}}(\Omega)$. In particular, if $A_i(p,\xi), f(p)$ are smooth functions of p and ξ, then u is smooth.*

Theorem D has the following important

Corollary. *For $p \geq 2$, consider a weak solution, u, to*

$$\sum_{j=1}^{m^1} X_j^1 \left(|Xu|^{p-2} X_j^1 u \right) = 0$$

in an open set $\Omega \subset G$. If there exists $M > 0$ such that $M \leq |Xu| \leq M^{-1}$ in Ω, then $u \in C^\infty(\Omega)$.

The previous corollary may be applied (in the case $p = Q$) to the theory of quasiconformal mappings in the Heisenberg group.

In [G1], Gehring showed that the only 1-quasiconformal mappings in $\mathbb{R}^3$ are the Möbius transformations. In his famous argument, a regularity theorem, analogous to Theorem D, plays a crucial role. In [KR1], Korányi and Reimann show that C^4 1-quasiconformal mappings between domains of $\mathbb{H}^1$ are obtained by taking the restriction of $SU(1,2)$ group actions. They conjecture that the same should be true without the regularity assumption. Using the corollary, we can prove the conjecture and identify all the 1-quasiconformal mappings of the Heisenberg group.

Theorem E. *If $f : D \to f(D)$ is a 1-quasiconformal mapping defined on an open connected set $D \subset \mathbb{H}^n$, then f is the action of a group element in $SU(1, n+1)$, restricted to D.*

Once one has Theorem D, the proof of Theorem E is similar to that in [G1], but here one has to take into account another "pathology" of the Heisenberg group: the horizontal gradient of the gauge distance vanishes on the center of the group. The deep results of Pansu [P], and Korányi and Reimann [KR1]–[KR3], as well as some theorem from [CDG2], play a fundamental role in the adaptation of Gehring's argument.

This Liouville type theorem has been recently and independently proved by Tang [T] in the case of three-dimensional strongly pseudoconvex CR manifolds. However, his proof is completely different from ours as it does not rest on the regularity of solutions to degenerate elliptic quasilinear equations.

Since the commutators of step two commute with the horizontal vector fields, it should be clear that the equations satisfied by the derivatives of the solution along the commutators must be considerably simpler than the ones satisfied by the derivatives of the solution along the horizontal directions. This observation would induce one to think that the nonhorizontal derivatives of weak solutions should enjoy extra regularity properties. Indeed, this is the case. We present an argument based on a weak reverse Hölder inequality (analogous to the one in [G2]) that will lead to the following higher integrability result.

Theorem F. *Let u be a weak solution to (2) in an open set $\Omega \subset G$. If $f = 0$ and $A_i(p,\xi)$ does not depend on V^2 (when expressed in exponential coordinates), then there exists $\varepsilon > 0$ such that*

$$Tu \in S^{1,2+\varepsilon}_{\text{loc}}(\Omega),$$

for any $T \in V^2$.

Let us observe that, in the case of lowest dimension, $Q = 4$, Theorem F provides enough regularity in the coefficients of one of the pde's solved by $X_i^1 u$ so that the non-Euclidean version of the De Giorgi, Nash, Moser, Stampacchia theorem in [CDG1] can be used. This would give a proof of Theorem B that does not involve the use of Morrey's ideas.

Sketch of the proofs

The classical argument for the proof of the optimal regularity of weak solutions to (1) can be divided into two parts: the first deals with the higher differentiability (in the weak L^2 sense) of the solutions; the second consists in proving regularity results for second order elliptic linear partial differential equations with nonsmooth coefficients. The first part of the program is carried by using difference quotients.

Since we are interested in horizontal derivatives, the approach in the stratified nilpotent Lie groups setting should, arguably, start by considering difference quotients of the solutions along the horizontal directions X_i^1, $i = 1, \ldots, m$. This is not possible. In fact, in contrast with the Euclidean case, the difference quotient along any nonzero vector $X \in V^1$ of a function $u \in S^{1,2}_{\mathrm{loc}}(\Omega)$, that is,

$$u_{(1,X)}(p) = \left(\frac{u(pe^{sX}) - u(p)}{|s|} \right)$$

in general is not in $S^{1,2}_{\mathrm{loc}}(\Omega)$, for any value of s. This is due to the noncommutative nature of the group, and can be easily seen by using exponential coordinates and the Baker-Campbell-Hausdorff formula

$$e^X e^Y = e^{X+Y+\frac{1}{2}[X,Y]},$$

where $X, Y \in g$. In fact one has, for any $\theta \in G$, that

$$\begin{aligned} X_i^1 u(p\theta) &= \frac{d}{dt} u(pe^{tX_i^1}\theta) = \frac{d}{dt} u(p\theta e^{tX_i^1 + t\sum_{l=1}^{m^1} \theta_l^1 [X_i^1, X_l^1]}) \\ &= \left(X_i^1 u + \sum_{l=1}^{m^1} \theta_l^1 [X_i^1, X_l^1] u \right)(p\theta), \end{aligned} \tag{3}$$

for any $p \in G$. Equation (3) determines, with its structure, the strategy to be followed in the proof of the sharp regularity theorem: show that the weak solutions are differentiable (in the weak sense) along the commutators direction, and then deal with the horizontal derivatives.

This approach presents some immediate advantages but also new difficulties. On the one hand, the difference quotient along any vector $T \in V^2$ of a function $u \in S^{1,2}_{\mathrm{loc}}(\Omega)$ is always in $S^{1,2}_{\mathrm{loc}}(\Omega)$ (this is a simple consequence of (3)); on the other hand, however, the fact that a priori a weak solution is only differentiable along horizontal vector fields is an obstruction to the use of such difference quotients.

In order to remove this considerable obstruction we introduce difference quotients of fractional order along the directions $X_1^2, \ldots, X_{m^2}^2$. In exponential coordinates, for $1 \le i_0 \le m^2$, we write these difference quotients as

$$u_{(\alpha, X_{i_0}^2)}(p) = \left(\frac{u(p_1^1, \ldots, p_m^1, p_1^2, \ldots, p_{i_0}^2 + s^2, \ldots, p_{m^2}^2) - u(p_1^1, \ldots, p_m^1, p_1^2, \ldots, p_{i_0}^2, \ldots, p_{m^2}^2)}{|s|^{2\alpha}} \right),$$

where $0 < \alpha < 1$, and $s \neq 0$. The estimates on the fractional derivatives of the solution will be carried through by using the following two norms on the space $C_0^\infty(g)$. The first is

$$|w|_{\frac{\partial}{\partial p^2_{i_0}},\alpha} = \sup_{\varepsilon > s > 0} \int_{\mathbb{R}^{\dim(g)}} |w_{(\alpha, X^2_{i_0})}|^2 dp. \tag{4}$$

The second is the usual Sobolev norm

$$\begin{aligned}\Big\|\frac{\partial^\alpha w}{\partial p^2_\nu}\Big\|_{L^2(g)} = \int_{\mathbb{R}^{\dim(g)}} &|h|^{2\alpha} |\hat{w}(p^1_1, \ldots, p^2_{\nu-1}, h, p^2_{\nu+1}, \ldots, p^2_{m^2})|^2 \\ &\times dp^1_1 \cdots dp^2_{\nu-1} dh\, dp^2_{\nu+1} \cdots dp^2_{m^2},\end{aligned} \tag{5}$$

where we have denoted by $\hat{w}$ the partial Fourier transform in the variable p^2_ν. The two norms (4) and (5) are related by the following theorem of Peetre [Pe] (see also [S]):

Theorem G. *Let $0 < \beta < \alpha < 1$, and $w \in C_0^\infty(G)$. There exists a positive constant depending only on α and β such that*

$$C\Big\|\frac{\partial^\beta w}{\partial p^2_\nu}\Big\|_{L^2(g)} \leq |w|_{\frac{\partial}{\partial p^2_\nu},\alpha} \leq C^{-1}\Big\|\frac{\partial^\alpha w}{\partial p^2_\nu}\Big\|_{L^2(g)}.$$

The differentiability of the weak solutions to (2) along the commutator directions is achieved by an iteration argument, where the order of differentiability is increased step by step, from one half to one. The first step in this iteration is given by the following proposition, that in turn is just a consequence of the Baker-Campbell-Hausdorff formula.

Proposition H. *Let $\Omega \subset G$ be an open set, and $B(x_0, r) \subset \Omega$ a homogeneous gauge ball. If $\eta \in C_0^\infty(B(x_0, r))$ and $w \in C^\infty(\Omega)$, then there exists a positive constant C such that*

$$|w\eta|_{\frac{\partial}{\partial p^2_\nu},\frac{1}{2}} \leq C \sum_{l=1}^{m^1} ||X(w\eta)||_{L^2(\Omega)}.$$

Let us describe the iteration without going into the details. From Proposition H one has that the one-half derivative of u along any commutator direction exists in the L^2-sense. Through the use of fractional difference quotients one can prove a Caccioppoli type inequality for derivatives of solutions to (2); namely, for $\eta \in C_0^\infty(B(0, 2R))$ we have

$$\int_\Omega |(\eta X u)_{(X^2_{i_0}, \frac{1}{2})}|^2 dp \leq C\bigg[\int_{B(0,2R)} |Xu|^2 + |f|^2 dp\bigg].$$

This inequality (almost) proves that the L^2 norm of the horizontal gradient of the weak derivative of order one half along the commutators of the solution is controlled by the L^2 norm of the horizontal gradient of u. Using Proposition H once more (with $w = \frac{\partial^{\frac{1}{2}}}{\partial p^2_{i_0}} u$) one obtains that the full derivative along the commutator is in L^2.

This is the rough idea behind the proof of the first part of Theorem C: an interplay between the Baker-Campbell-Hausdorff formula and a Caccioppoli type inequality. However, the argument we have just described is not precise. Things are more complicated because, in view of Theorem G, the two fractional norms

(4) and (5) are not equivalent, so a longer, and more difficult, iteration has to be carried through in order to reach the (nonfractional) differentiability.

Once one has differentiability of the weak solution along the commutators, the difference quotients along horizontal directions can be used and the classical approach may be pursued, yielding $u \in S^{2,2}_{\text{loc}}(\Omega)$. There are more difficulties, of course, due to the noncommutative structure of the vector fields, but these can be overcome since a certain amount of regularity of the derivatives along the commutators directions is now known.

Remark. It seems worthwhile to observe that the iteration argument works also for any nilpotent stratified Lie group, without the restriction to the step two case. It is always true that the derivatives of weak solutions to (2) along the directions of the highest component of g, namely V^r, are in $S^{1,2}_{\text{loc}}(\Omega)$. In this case the iteration on the order of differentiability goes from $\frac{1}{r}$ to one with step $\frac{1}{r}$. However, the next level offers some new difficulties. In fact one needs to estimate the commutators between horizontal vector fields and fractional derivatives in the direction V^s, with $1 < s < r$, in terms of derivatives (maybe fractionary) along the directions V^l, with $l > s$. This problem is the object of work in progress, and is the only obstruction to the extension of Theorems B and C to any nilpotent stratified Lie group.

Let us go back to the step two case and describe briefly the ideas behind the proof of Theorem B.

Theorem C allows us to differentiate the equation (2) along the horizontal and the V^2 directions. In this way one obtains two sets of equations. Since the group is of step two, it is clear that the equations obtained by differentiating along the V^2 directions must be considerably simpler than the other one (all the commutators are missing). Following the classical approach we would like to treat such equations as linear equations. The presence of terms with mixed derivatives, horizontal and nonhorizontal, creates a problem: these terms are not regular enough to allow the use of the various regularity theorems for subelliptic linear equations with nonsmooth coefficients [CDG1], [X1]. This problem can be solved by recurring to the same strategy used in Theorem C. First, one deals with the equations obtained by differentiating (2) along the commutator directions. These are simpler and can be dealt with by developing, to some extent, a non-Euclidean analogue of some of Morrey's techniques (Chapter 5 in [M]). Once one has obtained the sharp regularity for the "mixed" derivatives, the other set of equations may be studied, the terms corresponding to the commutators being now regular enough.

Acknowledgement

The results announced here are part of my Ph.D. dissertation at Purdue University. I am very grateful to my thesis advisor, Professor Nicola Garofalo, for having introduced me to the problem addressed in this paper.

References

[CDG1] L. Capogna, D. Danielli, and N. Garofalo, *An embedding theorem and the Harnack inequality for nonlinear subelliptic equations*, Comm. P.D.E. **18** (1993), 1765–1794. MR **94j:**35038

[CDG2] ———, *Capacitary estimates and the local behavior of solutions of nonlinear subelliptic equations*, to appear in the Amer. J. of Math.

[D] D. Danielli, *Regularity at the boundary for solutions of nonlinear subelliptic equations*, Indiana Univ. Math. Jour. **44** (1995), 269–286. CMP 95:14

[F] G. B. Folland, *Subelliptic estimates and function spaces on nilpotent Lie groups*, Arkiv. für Math. **13** (1975), 161–207. MR **58:**13215

[FS] G. B. Folland and E. M. Stein, *Estimates for the $\bar{\partial}_b$ complex and analysis on the Heisenberg group*, Comm Pure and Appl. Math. **27** (1974), 459–522. MR **51:**3719

[G1] F. W. Gehring, *Rings and quasiconformal mappings in space*, Trans. of the American Mathematical Society **103** (1962), 353–393. MR **25:**3166

[G2] ———, *The L^p-integrability of the partial derivatives of a quasiconformal map*, Acta Math. **130** (1973), 265–277. MR **53:**5861

[Gi] E. Giusti, *Minimal surfaces and functions of bounded variation*, vol. 80, Boston, Birkhäuser, Monographs in Math., 1984. MR **87a:**58041

[K] J. J. Kohn, *Pseudo-differential operators and hypoellipticity*, Proceedings of Symposia in Pure Mathematics **XXIII** (1973), 61–69. MR **49:**3356

[KR1] A. Korányi and H. M. Reimann, *Quasiconformal mappings on the Heisenberg group*, Inv. Math. **80** (1985), 309–338. MR **86m:**32035

[KR2] ———, *Horizontal normal vectors and conformal capacity of spherical rings in the Heisenberg group*, Bull. Sci. Math. (2) **111** (1987), 3–21. MR **88e:**22013

[KR3] ———, *Foundations for the theory of quasiconformal mappings on the Heisenberg group*, Adv. in Math. **111** (1995), 1–87. MR **96c:**30021

[M] C. B. Morrey, Jr., *Multiple integrals in the calculus of variations*, Springer-Verlag, New York, 1966. MR **34:**2380

[Mo] G. D. Mostow, *Strong rigidity of locally symmetric spaces*, Princeton University Press, Princeton, NJ, 1973. MR **52:**5874

[P] P. Pansu, *Métriques de Carnot-Carathéodory et quasiisométries des espaces symétriques de rang un*, Ann. Math. **129** (1989), 1–60. MR **90e:**53058

[Pe] J. Peetre, *A theory of interpolation of normed spaces*, Notas de Matématica **39** (1968). MR **39:**4662

[R] Y. G. Reshetnyak, *Space mappings with bounded distortion*, Transl. of Math. Mon., vol 73, American Mathematical Society, 1989. MR **90d:**30067

[S] E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Univ. Press, 1970. MR **44:**7280

[T] P. Tang, *Quasiconformal homeomorphisms on CR 3-manifolds: regularity and extremality*, to appear in Ann. Acad. Sci. Fenn. Ser. A. CMP 95:15

[X1] C.-J. Xu, *Subelliptic variational problems*, Bull. Soc. Math. France **118** (1990), 147–169. MR **92b:**49008

[X2] ———, *Regularity for quasilinear second order subelliptic equations*, Comm. Pure and Appl. Math. **XLV** (1992), 77–96. MR **93b:**35042

Department of Mathematics, Purdue University, West Lafayette, IN 47907
E-mail address: `capogna@math.purdue.edu`