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NONABELIAN SYLOW SUBGROUPS OF FINITE GROUPS OF EVEN ORDER

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ABSTRACT. We have been able to prove that every nonabelian Sylow subgroup of a finite group of even order contains a nontrivial element which commutes with an involution. The proof depends upon the consequences of the classification of finite simple groups.

The purpose of this note is to announce [4]:

Main Theorem. *Every nonabelian Sylow subgroup of a finite group of even order contains a nontrivial element which commutes with an involution.*

Let G be a finite group and $\Gamma(G)$ the prime graph of G. $\Gamma(G)$ is the graph such that the vertex set is the set of prime divisors of $|G|$, and two distinct vertices p and r are joined by an edge if and only if there exists an element of order pr in G. Let $n(\Gamma(G))$ be the number of connected components of $\Gamma(G)$ and $d_G(p, r)$ the distance between two vertices p and r of $\Gamma(G)$. It has been proved that $n(\Gamma(G)) \leq 6$ in [12], [10], [11], [9].

Theorem 1. *Let G be a finite group of even order and p a prime divisor of $|G|$. If $d_G(2, p) \geq 2$, then a Sylow p-subgroup of G is abelian.*

Theorem 1 is a restatement of the Main Theorem in terms of the prime graph $\Gamma(G)$.

Corollary 1. *Let G be a finite group of even order and p a prime divisor of $|G|$. If Δ is a connected component of $\Gamma(G)-\{p\}$ not containing 2, then a Sylow r-subgroup of G is abelian for any $r \in \Delta$.*

There is a certain relation between a subgraph $\Gamma(G) - \{p\}$ of $\Gamma(G)$ and Brauer characters of p-modular representations of G (see [3]).

Theorem 2. *Let G be a finite nonabelian simple group and p an odd prime divisor of $|G|$. Then $d_G(2, p) = 1$ or 2 provided $d_G(2, p) < \infty$.*

The significance of the prime graphs of finite groups can be found in [1], [3], [5], [6], [7], [8], [14], [15]. We apply the classification of finite simple groups (see [1], [2], [4], [7], [10], [13]). It has been proved that a minimal counterexample to Theorem 1

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is a nonabelian simple group. We will give some examples of case by case analysis for finite simple groups. Theorem 1 holds true for the sporadic simple groups by Atlas of Finite Groups although we can find several typos in it. For a positive integer k let $\pi(k)$ be the set of all prime divisors of k. Let $\pi_0 = \{p \in \pi(G) \mid d_G(2,p) = 1\}$. Then we do not have to think about primes in π_0 in order to give the proof of Theorem 1.

Example. Let G be the alternating group on n-letters and $p \in \pi(G)$. It is trivial that Theorem 1 holds true for A_5 and A_6. Assume that $n \geq 7$. If $p \leq n-4$, then $d_G(2,p) = 1$. If $p \geq n-3$, then Sylow p-subgroups of G are cyclic of order p. Thus Theorem 1 holds true for the alternating groups.

Example. Let $G = PSL(n,q)$, $q \equiv 0 \ (2)$. Then $|G| = q^{n(n-1)/2} \prod_{i=1}^{n-1} (q^{i+1} - 1) d^{-1}$, $d = (n, q-1)$. Let I_j be the $j \times j$ identity matrix. Put

$$t'_k = \begin{pmatrix} I_k & 0 & 0 \\ 0 & I_{n-2k} & 0 \\ I_k & 0 & I_k \end{pmatrix}.$$

Then t'_k $(1 \leq k \leq r)$, where $r = [n/2]$, are representatives of the conjugacy classes of involutions in $SL(n,q)$. The centralizer of t'_k in $SL(n,q)$ is the set of all matrices of the form

$$\begin{pmatrix} A & 0 & 0 \\ H & B & 0 \\ K & L & A \end{pmatrix},$$

where $(\det A)^2 \det B = 1$ and A is a $k \times k$ nonsingular matrix. Denote t_k the homomorphic image of t'_k in $PSL(n,q)$. Then t_k $(1 \leq k \leq r)$ are representatives of the conjugacy classes of involutions in $PSL(n,q)$. Let $C_k = C_G(t_k)$. Then

$$\pi(C_k) = \pi(2 \prod_{i=1}^{n-2k} (q^i - 1)/(q-1)d)$$

and

$$\pi_0 = \pi(\prod_{k=1}^{r} |C_k|) = \pi(2 \prod_{i=1}^{n-2} (q^i - 1)).$$

Suppose $n \geq 4$. Then the only factors of $|G|$ to be considered are $(q^{n-1}-1)(q^n-1)$. There are maximal tori $T(A_{n-2})$ of order $(q^{n-1}-1)d^{-1}$ and $T(A_{n-1})$ of order $(q^n-1)/(q-1)d$. Let $p \in \pi(T(X)) - \pi_0$, where $X = A_{n-1}$ or A_{n-2}. Let P be a Sylow p-subgroup of $T(X)$. Then $d_G(2,p) = 1$ or P is a Sylow p-subgroup of G. Since P is abelian, Theorem 1 holds true for $G = PSL(n,q)$, $n \geq 4$.

Suppose that $n = 3$. Then $|G| = q^3(q^2-1)(q^3-1)d^{-1}$ and there are three classes of maximal tori of orders

$$(q-1)^2 d^{-1}, \quad (q^2-1)d^{-1}, \quad (q^2+q+1)d^{-1}.$$

We note that a torus of order $(q^2+q+1)d^{-1}$ is an isolated subgroup. If $q > 4$, then $d_G(2,r) = 2$ for $r \in \pi(q+1)$. Let R be a Sylow r-subgroup of G. Then R is contained in a maximal torus of order $(q^2-1)d^{-1}$. If $q = 4$, then $G = PSL(3,4)$ and $|G| = 2^6 \cdot 3^2 \cdot 5 \cdot 7$. If $q = 2$, then $G = PSL(3,2)$ and $|G| = 2^3 \cdot 3 \cdot 7$. We have verified Theorem 1 for $n = 3$. It is trivial that Theorem 1 holds true for $PSL(2,q)$.

Theorem 3. *Let G be a simple group of Lie type and T a maximal torus. Let $p \in \pi(T) - \pi_0$. Then T contains a Sylow p-subgroup of G.*

Theorem 3 is a corollary of Theorem 1. Actually we prove Theorem 3 for specified tori when we verify Theorem 1 for the simple groups of Lie type.

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