CHARACTERIZATION OF THE RANGE
OF THE RADON TRANSFORM
ON HOMOGENEOUS TREES

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(Communicated by Mark Freidlin)

Abstract. This article contains results on the range of the Radon transform $R$ on the set $\mathcal{H}$ of horocycles of a homogeneous tree $T$. Functions of compact support on $\mathcal{H}$ that satisfy two explicit Radon conditions constitute the image under $R$ of functions of finite support on $T$. Replacing functions on $\mathcal{H}$ by distributions, we extend these results to the non-compact case by adding decay criteria.

1. Introduction

We study the Radon transform $R$ on the set $\mathcal{H}$ of horocycles of a homogeneous tree $T$, and describe its image on various function spaces. We show that the functions of compact support on $\mathcal{H}$ that satisfy two explicit Radon conditions constitute the image under $R$ of functions of finite support on $T$. Larger domains and ranges are described by adding decay criteria to the domain and range, although we show that functions on $\mathcal{H}$ need to be replaced by distributions.

The Radon transform (RT for short), in its original formulation by Radon [R], associates to each (sufficiently nice) function on $\mathbb{R}^2$ its one-dimensional Lebesgue integrals along all affine straight lines. This transform has been receiving considerable attention for its highly applicable nature and intrinsic interest, leading to a variety of generalizations.

In $\mathbb{H}^2$ lines correspond to two essentially different kinds of one-dimensional submanifolds: geodesics and horocycles, giving rise to two different RTs (cf. [H]).

Homogeneous trees are widely regarded as discrete counterparts of $\mathbb{H}^2$, as well as objects of thorough study in harmonic analysis in their own right. Exactly like $\mathbb{H}^2$, they feature two distinct kinds of RTs, namely the geodesic RT (a.k.a. the X-ray transform, since it is reminiscent of the CAT-scan procedure (cf. [BC])), and the horocyclic RT. Several of the standard RT issues in this setting have been investigated over time by various authors: e.g., [BCCP], [A] for injectivity and inversion, [CCP2] for range characterization, and [CC] for function space setting for the geodesic RT; [BP], [BFP], [CCP1] for injectivity and inversion of the horocyclic RT.
RT (part of the results therein are rewritten in [CMS] for the Abel transform, which is a multiple of the RT). Another related transform has been studied recently by Cowling and Setti.

In this work, we pursue a description of the range of the horocyclic RT $R$ on a homogeneous tree $T$ of degree $q + 1$ with $q \geq 2$. We first state two natural explicit relations (one of which had already been observed in [BFPp] and [BFP] for radial functions) for functions on the space $\mathcal{H}$ of horocycles of $T$. We then show that among compactly supported functions on $\mathcal{H}$, these conditions completely characterize the range of $R$ on finitely supported functions on (the set of vertices of) $T$. Similar descriptions are valid for the range of $R$ on larger function spaces, although distributions on $\mathcal{H}$ need then to be taken into account. All the results with complete proofs can be found in [CCC].

We thank Hillel Furstenberg for many useful conversations and for his insights into the problem.

2. Preliminaries

The boundary $\Omega$ of $T$ is the set of equivalence classes of infinite paths under the relation $[v_0, v_1, \ldots] \simeq [v_1, v_2, \ldots]$. For any vertex $u$, we denote by $[u, \omega]$ the (unique) path starting at $u$ in the class $\omega$. Then $\Omega$ can be identified with the set of paths starting at $u$. Each $\omega \in \Omega$ induces an orientation on the edges of $T$: $[u, v]$ is positively oriented if $v \in [u, \omega]$.

For $\omega \in \Omega$, and $u, v \in T$, define the horocycle index $\kappa_{\omega}(u, v)$ as the number of positively oriented edges minus the number of negatively oriented edges in the path from $u$ to $v$. Given $u \in T$ and $\omega \in \Omega$, the horocycle through $u$ touching $\omega$ is the set $\{v : \kappa_{\omega}(u, v) = 0\}$. More generally, for any $n \in \mathbb{Z}$, the horocycle of index $n$ touching $\omega$ with respect to $u$ is $h_{\omega,n}^{u} = \{w \in T : \kappa_{\omega}(u, w) = n\}$. Then the set of vertices may be decomposed as $\bigsqcup_{n \in \mathbb{Z}} h_{\omega,n}^{u}$.

For $u$ fixed, the map $(n, \omega) \mapsto h_{\omega,n}^{u}$ is a one-to-one correspondence between $\mathbb{Z} \times \Omega$ and the set $\mathcal{H}$ of horocycles.

**Definition 1.** The $L^1$-horocyclic Radon transform $R$ on $T$ is given by $Rf(h) = \sum_{v \in h} f(v)$ for $f \in L^1 T$, and $h \in \mathcal{H}$.

For $u, v \in T$, set $S(u, v) = \{h \in \mathcal{H} : \exists \omega \in \Omega \text{ s.t. } h = h_{\omega,0}^{u}, v \in [u, \omega]\}$. The topology generated by the sets $S(u, v)$ makes $\mathcal{H}$ totally disconnected. Then $\mathcal{H}$ is homeomorphic to $\mathbb{Z} \times \Omega$, where $\Omega$ is endowed with the compact topology generated by $I_v^u = \{\omega \in \Omega : v \in [u, \omega]\}$. For any $u \in T$, there is a measure $\mu^u$ on $\Omega$:

$$\mu^u(I_v^u) = 1/c_d(u, v).$$

The family of horocycles through a fixed $\omega$ does not depend on the choice of the reference vertex $u$, but indices do: $h_{\omega,n}^v = h_{\omega,n+\kappa_{\omega}(u,v)}^u$.

For simplicity of notation, we fix a root $e$ throughout, and set $h_{\omega,n}^e = h_{\omega,n}^e, \mu = \mu^e, dw = d\mu^e(\omega), k(v, \omega) = \kappa_{\omega}(e, v)$, and $I_v^e = I_v^e$. Notice that $d\mu^e(\omega) = q^{k(v, \omega)} d\omega$.

For $\omega \in \Omega$, let $\omega_n \in [e, \omega]$ be the vertex of length $n$. For $v \in T$, and $0 \leq n \leq |v|$, let $v_n \in [e, v]$ be the vertex of length $n$. For $v \in T$ and $n \geq |v|$, the set $D_n(v) = \{u : |u| = n \text{ and } u_{|v|} = v\}$ is the set of descendants of $v$ of length $n$.

**Definition 2.** For a function $\varphi$ on $\mathcal{H}$, we define the Radon conditions as follows:

$$(R_1) \sum_n \varphi(h_{\omega,n}^v) \text{ is independent of } v \text{ and } \omega.$$
(R2) For any \( v \in T, n \in \mathbb{Z} \),
\[
\int_{\Omega} \varphi(h^v_{\omega,n})d\mu^v_\omega = q^{-n} \int_{\Omega} \varphi(h^v_{\omega,-n})d\mu^v_\omega.
\]

**Proposition 1.** If \( f \in L^1T \), then \( Rf \) is a continuous function satisfying the Radon conditions.

There are, however, continuous functions satisfying the Radon conditions that are of the form \( Rf \) for \( f \notin L^1T \).

Proposition 1 is proved by showing first that the Radon conditions are satisfied for the function \( \varphi = R\chi_u \), where \( \chi_u \) is the characteristic function of \{\( v \}\}, and then extending linearly.

Fix \( v \in T \). For \( 0 \leq t \leq |v| \), let \( I^t_v = \{\omega \in \Omega : k(v, \omega) = 2t - |v|\} \). Then for \( t \neq |v| \), \( I^t_v = I_v - I_{v+1} \), \( I^{|v|}_v = I_v \), and \( \Omega = \bigcup_{t=0}^{|v|} I^t_v \). Using the relations \( h^v_{\omega,n} = h_{\omega,n+k(v,\omega)} \) and \( d\mu^v_\omega = q^{k(v,\omega)}d\omega \), condition (R2) may be rewritten as
\[
(R_2) \quad \sum_{t=0}^{|v|} q^{2t-|v|} \int_{I^t_v} \varphi(h^v_{\omega,n+2t-|v|}) = q^{-n} \sum_{t=0}^{|v|} q^{2t-|v|} \int_{I^t_v} \varphi(h^v_{\omega,-n+2t-|v|}) d\omega.
\]

In §3, we characterize the range of the RT on the set of functions on \( T \) of finite support, and then in §4, after defining \( Rf \) as a distribution on \( \mathcal{H} \), we obtain a similar characterization for the case of \( f \) of infinite support.

3. Functions of compact support

**Theorem 1.** The image of \( R \) on the space of functions on \( T \) of finite (i.e. compact) support is the space of functions on \( \mathcal{H} \) of compact support satisfying the Radon conditions.

The proof is based on the use of a generalization of radiality:

**Definition 3.** Let \( N \) be a non-negative integer.

1. A function \( f \) on \( T \) is \( N \)-radial if for all \( v \in T \) with \( |v| \geq N \), \( f(v) \) depends only on \( v_N \) and \( |v| \).

2. \( f \) has radius \( N \) if \( \{v : |v| \leq N\} \) is the smallest disk centered at \( e \) containing the support of \( f \) (so \( f(v) = 0 \) for \( |v| > N \)).

3. A function \( \varphi \) on \( \mathcal{H} \) is \( N \)-radial if \( \varphi(h_{\omega,n}) \) depends only on \( \omega_N \) and \( n \).

4. \( \varphi \) has radius \( N \) if \([-N, \ldots, N] \times \Omega \) is the smallest such set containing the support of \( \varphi \) (so \( \varphi(h_{\omega,n}) = 0 \) for \( |n| > N \)).

In particular, a 0-radial function on \( T \) is what is generally called radial.

We actually prove a more precise version of Theorem 1, specifically that the image under \( R \) of the set of functions on \( T \) of radius less than or equal to \( N \) is the set of continuous functions on \( \mathcal{H} \) of radius less than or equal to \( N \) satisfying the Radon conditions. This result is established by means of Propositions 2 and 3, whose proofs are outlined below.

For \( N \geq 0 \), let \( E^N \) be the set of \( N \)-radial functions on \( \mathcal{H} \) of radius less than or equal to \( N \) satisfying (R1) and (R2).

**Proposition 2.** \( E^N = E^{N-1} \oplus \bigoplus_{|v|=N} \mathcal{C} R\chi_v \).

It follows by induction that \( E^N \) is the image under \( R \) of the set of functions of radius less than or equal to \( N \).
Proposition 3. If $\varphi$ is a function on $\mathcal{H}$ of compact support satisfying the Radon conditions, then there exists $N$ such that $\varphi \in E^N$.

Let $\{v^1, \ldots, v^N\}$ be an enumeration of the vertices of length $N$. If $v \in T$, $|v| \leq N$, let $A_i^v = \{j : I_{ij} \subset I_v\}$. Thus $I_v^i = \bigcup_{j \in A_i^v} I_{ij}$. If $j_0$ is the index such that $v = v^{j_0}$, then $A_v^{N} = \{j_0\}$. Observe that $\{1, 2, \ldots, c_N\} = \bigcup_{i=0}^{v} A_i^v$ and recall that $\Omega = \bigcup_{i=0}^{v} I_v^i$. Let $\varphi \in E^N$, and set $a_{n,j} = \varphi(h_{\omega,n})$ for $\omega_N = v^j$. Then $(R_2)$ becomes

$$ (R_2^v) \quad \sum_{i=0}^{M} q^{2t} \sum_{j \in A_i^v} a_{n+2t-M,j} = q^n \sum_{i=0}^{M} q^{2t} \sum_{j \in A_i^v} a_{n+2t-M,j}, $$

for $|v| = M \leq N$.

The proof of Propositions 2 and 3 is based on repeated applications of $(R_2^v)$ for various values of $n$ and $M$. For instance, if we set $M = N$ and $n = 2N$, the left-hand side of $(R_2^v)$ reduces to $\sum_{j \in A_0^v} a_{N,j}$, since $n + 2t - M > N$ except for $t = 0$. On the right-hand side, $a_{-n+2t-M} = 0$, except for $t = M = N$, leaving just $\sum_{j \in A_N} a_{-N,j}$, which is $a_{-N,j_0}$, where $v = v^{j_0}$. Thus $\sum_{j \in A_0^v} a_{N,j} = a_{-N,j_0}q^N$. In particular, if $a_{N,j} = 0$ for all $j$, then $a_{-N,j} = 0$ for all $j$.

If $\varphi \in E^N$, then the function $\hat{\varphi} = \varphi - \sum_{j=1}^{c_N} a_{M,j} R(X_{j,v})$ has the property that $\hat{\varphi}(h_{\omega,n}) = 0$ for $n = N$ as well as for $|n| > N$. Hence $\hat{a}_{N,j} = 0$ for all $j$, and so, by what we just proved, $\hat{a}_{-N,j} = 0$ for all $j$. Thus $\varphi \in E^{N-1}$, proving Proposition 2.

Now let $\varphi$ be a function with compact support satisfying the Radon conditions. Since topologically $\mathcal{H} \simeq \mathbb{Z} \times \Omega$ with $\Omega$ compact, there is some positive integer $N$ such that the support of $\varphi$ is contained in $[-N,N] \times \Omega$, i.e. $\varphi(h_{\omega,n}) = 0$ for $|n| > N$. Then $\varphi$ has radius less than or equal to $N$. Again using $(R_2^v)$, it is possible to show that $\varphi$ is $N$-radial. Thus $\varphi \in E^N$, proving Proposition 3, and hence Theorem 1.

4. Non-compact support

In this section we develop a parallel theory for distributions on $\mathcal{H}$ and define certain decay conditions for functions on $T$ and distributions on $\mathcal{H}$.

For $r > 0$, define $A_r$ as the class of all functions $f : T \rightarrow \mathbb{C}$ satisfying the decay condition:

$$ \sum_{n=|v|}^{\infty} t^n \left| \sum_{u \in D_n(v)} f(u) \right| < \infty \quad \forall t \in [0, r), \forall v \in T. $$

Observe that $L^1T \subset A_1$, since for $f \in L^1T$ and $0 \leq t < 1$,

$$ \sum_{n=|v|}^{\infty} t^n \left| \sum_{u \in D_n(v)} f(u) \right| \leq \sum_{n=|v|}^{\infty} \sum_{u \in T} \left| f(u) \right| \leq \sum_{u \in T} \left| f(u) \right| = \|f\|_1. $$

The elementary measurable sets in $\mathcal{H}$ can be generated by all sets of the form $\{h_{\omega, n} \in \mathcal{H} : \omega \in I_v\}$, which may be identified with $\{n\} \times I_v$. A distribution on $\mathcal{H}$ is an element of the dual of the vector space generated by the characteristic functions of the elementary measurable sets of $\mathcal{H}$. Thus, since $I_v = \bigcup_{\omega \in I_v} I_{\omega}$, we may think of a distribution on $\mathcal{H}$ as a function $\varphi$ on the sets $\{n\} \times I_v$ satisfying the
property
\[ \varphi\{n\} \times I_v = \sum_{u^{-}=v} \varphi\{n\} \times I_u. \]

If \( f \in L^1 T \), then \( Rf \) is defined on each horocycle and is bounded. By abuse of notation, we define \( Rf \) as the distribution given by
\[ Rf\{n\} \times I_u = \int_{I_u} Rf(h_{\omega,n}) \, d\omega. \]

Now for a larger class of functions on \( T \), this leads to the following definition of the Radon transform as a distribution:

**Definition 4.** For a function \( f \) on \( T \), let
\[ Rf\{n\} \times I_u = \sum_{m=0}^{\infty} \sum_{|v|=m} f(v) R\chi_v\{n\} \times I_u, \]
if this is defined for all \( u \in T \), and all \( n \in \mathbb{Z} \).

This definition is consistent with the previous formula, since \( f = \sum_{m=0}^{\infty} \sum_{|v|=m} f(v) \chi_v \).

We extend the Radon conditions to the case of distributions as follows:

1. \( R_1 \) \( \sum_{n \in \mathbb{Z}} \varphi\{n\} \times I_v ) / \mu(I_v) \) is independent of \( v \).
2. \( R_2 \) For all \( v \in T \), \( n \in \mathbb{Z} \),
\[ \sum_{t=0}^{\infty} q^{2t-|v|} \varphi\{n+2t-|v|\} \times I^t_v = q^{-n} \sum_{t=0}^{\infty} q^{2t-|v|} \varphi\{-n+2t-|v|\} \times I^t_v. \]

For \( r > 0 \), define \( B_r \) as the class of all distributions \( \varphi \) on \( H \) satisfying the decay condition:
\[ \sum_{n=|v|}^{\infty} t^n q^n |\varphi\{n\} \times I_v| < \infty \quad \text{for all } t \in [0, r), v \in T. \]

**Theorem 2.** For \( r > 1/\sqrt{q} \), \( R(\mathcal{A}_r) \) is the set of all \( \varphi \in B_r \) satisfying the Radon conditions.

A distribution \( \varphi \) on \( H \) is \( N \)-radial if \( \varphi\{n\} \times I_v \) depends only on \( n \) and \( v_N \).

The proof of Theorem 2 is based on the use of \( N \)-radial functions and \( N \)-radial distributions. Given a positive number \( r \), and a non-negative integer \( N \), let \( \mathcal{A}_r^N \) be the space of \( N \)-radial functions in \( \mathcal{A}_r \), and let \( \mathcal{B}_r^N \) be the space of \( N \)-radial distributions in \( B_r \). The key result in proving Theorem 2 is the following

**Proposition 4.** For \( r > 1/\sqrt{q} \), the image of the Radon transform on \( \mathcal{A}_r^N \) is the set of all \( \varphi \in \mathcal{B}_r^N \) satisfying the Radon conditions.

The following example shows that the use of distributions is necessary:

**Example.** Let \( l_1, \ldots, l_q \) be complex numbers of absolute value one, such that \( \sum_{j=1}^{q} l_j = 2/3 \), and set \( l_{q+1} = l_1 \). Label the vertices as follows: let \( x_1, \ldots, x_{q+1} \) be the vertices of length 1. If \( v \neq e \) has already been labeled, write the immediate descendants of \( v \) as \( v x_1, \ldots, v x_q \). Thus a typical vertex \( v \) of length \( N \) is labeled as
where the \( i_j \) are between 1 and \( q \), except for \( i_1 \) which can also be \( q + 1 \). Then define \( f(v) \) as \( l_{i_1} \cdots l_{i_N} (\frac{2}{3})^N \), \( f(e) = 1 \). Thus

\[
\left| \sum_{u \in D_n(v)} f(u) \right| = |f(v)|(8/9)^{n-N} = \left( \frac{8}{5} \right)^n \left( \frac{3}{2} \right)^N.
\]

If \( 0 < t < 9/8 \), then \( \sum_n t^n \left( \frac{5}{8} \right)^n \left( \frac{3}{2} \right)^N \) converges, and so \( f \in A_{9/8} \). By Theorem 2, \( Rf \in B_{9/8} \).

On the other hand, we now show that \( Rf \) cannot be evaluated at any horocycle. A horocycle \( h_{\omega, n} \) is the disjoint union of the sets \( D_{n+2k}(\omega_n + k) \) over the set of all non-negative integers \( k \), for \( n \geq 0 \). Now

\[
\left| \sum_{v \in D_{n+2k}(\omega_n + k)} f(v) \right| = \left( \frac{4}{3} \right)^n \left( \frac{32}{27} \right)^k
\]

and

\[
\left| \sum_{v \in D_{n+2k}(\omega_n + k + 1)} f(v) \right| = \left( \frac{3}{2} \right)^n \left( \frac{32}{27} \right)^k.
\]

Since the second sum has a larger absolute value, the absolute value of the difference is at least \( \frac{1}{2} \left( \frac{4}{3} \right)^n \left( \frac{32}{27} \right)^k \). Thus the series for defining \( Rf(h_{\omega, n}) \) does not converge, for \( n \geq 0 \). For \( n < 0 \),

\[
h_{\omega, n} = \prod_{k=0}^{\infty} (D_{n+2k}(\omega_k) - D_{n+2k}(\omega_{k+1})),
\]

and the same conclusion holds. Since point evaluation cannot be defined, \( Rf \) cannot be induced by a function.

**References**


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